

Shortest Paths in Digraphs

Distances and shortest paths

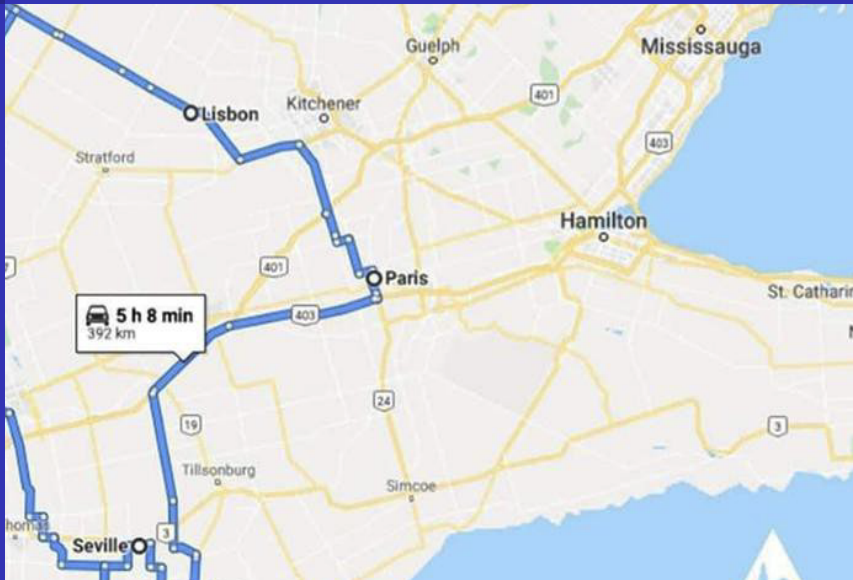
- Applications
- Definitions
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- SP problems

Single source

- Dijkstra's
- Bellman-Ford
- DAGs

All pairs

- Floyd-Warshall
- Johnson's



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Myriad of applications

- Finding the shortest paths between 2 locations (Google maps, etc.)
- Internet router protocols: OSPF (Open Shortest Path First) is used to find a shortest path to interchange packages between servers (IP)
- Traffic information systems
- Routing in VSLI
- etc ...

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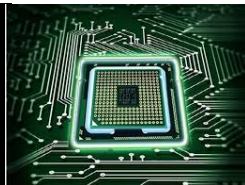
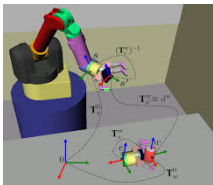
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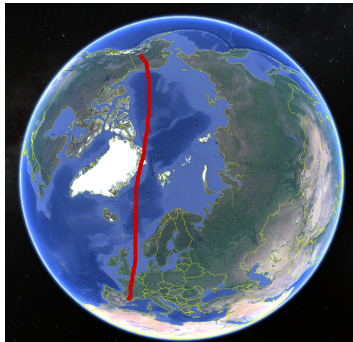
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Distance between two points

Distance is usually thought of as a pure geometric notion, often the **Euclidean distance** L_2

We use measures of distance that are not geometric: energy consumption, traveling time, payments, costs, etc..



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Paths and weights

Given a digraph $G = (V, E)$ with edge's weights $w : E \rightarrow \mathbb{R}$.

- A **path** is a sequence of vertices $p = (v_0, \dots, v_k)$ so that $(v_i, v_{i+1}) \in E$, for $0 \leq i < k$.
- A path $p = (v_0, \dots, v_k)$ has **length** $\ell(p) = k$ and **weight** $w(p) = \sum_{i=0}^{k-1} w(v_i, v_{i+1})$.



This path has length 4 and weight -1.

- For a path $p = \{u, \dots, v\}$, we write $u \rightsquigarrow^p v$ to say that it starts at u and ends at v .
- Note that the definition of path allows repeated vertices

Distance

- We want to associate a distance value $\delta(u, v)$ to each pair of vertices u, v in a weighted digraph (G, w) , measuring the **minimum weight** over the weights of the paths going from u to v .
- We have two cases:
 - $\{p | u \rightsquigarrow^p v\} = \emptyset$, i.e., there is no path from u to v , in such a case we define $\delta(u, v) = +\infty$.
 - $\{p | u \rightsquigarrow^p v\} \neq \emptyset$. In this case, if $\min\{w(p) | u \rightsquigarrow^p v\}$ exists, we define the distance as

$$\delta(u, v) = \min_p \{w(p) | u \rightsquigarrow^p v\}$$

otherwise, the distance cannot be defined.

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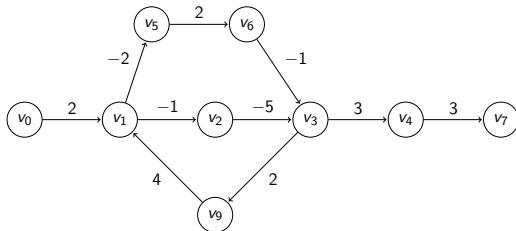
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Distances: examples



$$\delta(v_4, v_7) = 3 \quad \delta(v_4, v_3) = +\infty \quad \delta(v_3, v_2) = 5 \quad \delta(v_0, v_4) = -1$$

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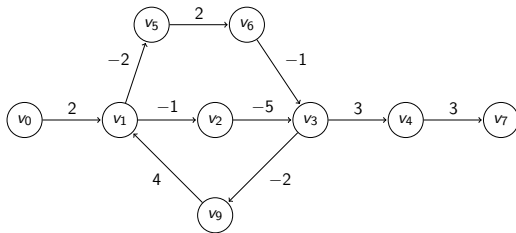
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Distances: examples



$\delta(v_4, v_7) = 3$ $\delta(v_4, v_3) = +\infty$ $\delta(v_3, v_2)$ cannot be defined

$w(v_3, v_9, v_1, v_2) = 1$ $w(v_3, v_9, v_1, v_2, v_3, v_9, v_1, v_2) = -3$

$w(v_3, v_9, v_1, v_2, v_3, v_9, v_1, v_2, v_3, v_9, v_1, v_2) = -7$

$w(v_3, v_9, v_1, v_2, v_3, v_9, v_1, v_2, v_3, v_9, v_1, v_2, v_3, v_9, v_1, v_2) = -11$

...

The cycle v_1, v_2, v_3, v_9, v_1 has weight -4!

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When the distance cannot be defined?

A **cycle** is a path that starts and ends at the same vertex.

A **negative weight cycle** is a cycle c having $w(c) < 0$

Theorem

Let $G = (V, E, w)$ be a weighted digraph.

A distance among all pairs of vertices $u, v \in V(G)$ can be defined iff G has no negative weight cycles.

Proof

- If $\delta(u, v)$ can be defined, for every $u \in V$, $\delta(u, u) \geq 0$, so any cycle has non negative weight.
- If G has a negative weight cycle C , the distance among pairs of vertices in C cannot be defined.

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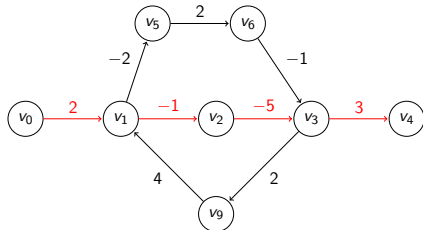
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- The previous theorem states conditions under which a distance measure for all pairs cannot be defined.
- It might be possible to have a digraph with a negative weight cycle, but that distances among some pairs of vertices can be defined, even if not for all pairs.

Shortest paths

- For $u, v \in V$, such that $\delta(u, v)$ is defined and $\delta(u, v) < +\infty$,
- a **shortest path** from u to v is a path p , starting at u and ending at v , having $w(p) = \delta(u, v)$.



$$\delta(v_0, v_4) = -1$$

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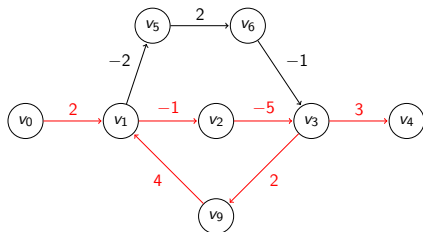
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Shortest paths

- For $u, v \in V$, such that $\delta(u, v)$ is defined and $\delta(u, v) < +\infty$,
- a **shortest path** from u to v is a path p , starting at u and ending at v , having $w(p) = \delta(u, v)$.



$$\delta(v_0, v_4) = -1$$

There are infinite shortest paths $v_0 \rightsquigarrow v_4$

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Undirected graphs and unweighted graphs

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- If G is undirected, we consider every edge as doubly directed and assign the same weight to both directions.
- If the graph or digraph is **unweighted**, we assign to **each edge a weight of 1**.
In this case the **weight of a path coincides with its length**.

Optimal substructure of shortest path

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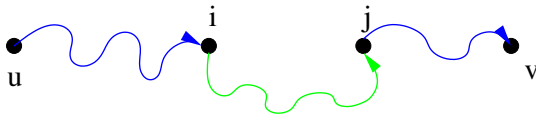
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Given $G = (V, E, w)$, for any shortest path $p : u \rightsquigarrow v$ and any pair of vertices i, j in p , the sub-path $p' = i \rightsquigarrow j$ of p is a shortest path, i.e., $w(p') = \delta(i, j)$.

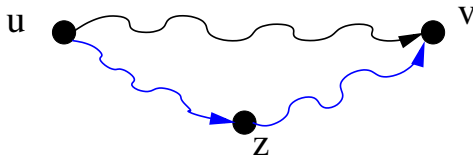


Triangle Inequality

$\delta(u, v)$ is the shortest distance from u to v , i.e., the shortest path $u \rightsquigarrow v$ has weight \leq that the weight of any other path from u and v .

Theorem

Let $G = (V, E, w)$ be such that, for each $u, v \in V$, $\delta(u, v)$ can be defined. For $u, v, z \in V(G)$, $\delta(u, v) \leq \delta(u, z) + \delta(z, v)$.



$u \rightsquigarrow z \rightsquigarrow v$ is a path from u to v .

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Shortest Path Tree

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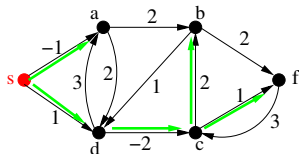
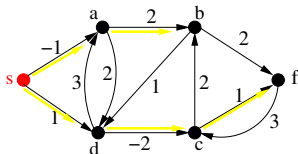
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Given $G = (V, E, w)$ and a distinguished $s \in V$, a **shortest path tree** is a directed sub-tree, $T_s = (V', E')$, of G , s.t.

- T_s is rooted at s ,
- V' is the set of vertices in G reachable from s ,
- For $v \in V'$ the path $s \rightsquigarrow v$ in T_s has weight $\delta(s, v)$.



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Single source shortest path: Given $G = (V, E, w)$ and $s \in V$, find a shortest path from s to each other vertex in G , if it exists.

To solve this problem we present two algorithms strategies,

- **Dijkstra's algorithm:** a very efficient greedy algorithm which only works for **positive weights**. You should know it.
- **Bellman-Ford algorithm**, devised by several independent teams **Bellman, Ford, Moore, Shimbel**. It works for general weights and detects whether the distance can be defined.

Both algorithms assume that the input graph is given by adjacency lists.

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All pairs shortest paths: Given $G = (V, E, w)$ without negative weight cycles, for each $u, v \in V(G)$, find a shortest path from u to v if it exists.

To solve this problem we present two algorithms strategies,

- **Floyd-Warshall algorithm**, devised by several independent teams **Roy, Floyd, Warshall**. Uses dynamic programming and takes as input the weighted adjacency matrix of G .
- **Johnson's algorithm**: an efficient algorithm for sparse graphs.

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Single source shortest path (SSSP)

Given $G = (V, E, w)$ and $s \in V$, compute $\delta(s, v)$, for $v \in V - \{s\}$.

- The algorithm maintains, for $v \in V$, an overestimate $d[v]$ of $\delta(s, v)$ and a candidate predecessor $p[v]$ on a shortest path from s to v .
- Initially, $d[v] = +\infty$, for $v \in V - \{s\}$, $d[s] = 0$ and $p[v] = v$.
- Repeatedly improve estimates towards the goal $d[v] = \delta(s, v)$
- On selected $(u, v) \in E$ apply the **Relax** operation

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Relaxing and edge

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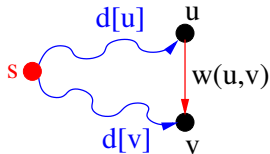
Relax(u, v)

if $d[v] > d[u] + w(u, v)$

then

$d[v] = d[u] + w(u, v)$

$p[v] = u$



Relaxing and edge

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Relax: invariant

$d[v] \geq \delta(s, v)$ and, if $d[v] < +\infty$, $p[v]$ is the predecessor of v in a path from s to v with weight $d[v]$, .

Let d be the values before executing Relax and d' the ones after executing it.

$$\delta(s, v) \leq \delta(s, u) + w(u, v) \leq d[u] + w(u, v)$$

$$\delta(s, v) \leq d[v]$$

$$d'[v] = \min\{d[v], d[u] + w(u, v)\} \geq \delta(s, v).$$

The second part also follows from this formula.

SSSP: Dijkstra

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Edsger .W.Dijkstra, "*A note on two problems in connexion with graphs*". Num. Mathematik 1, (1959)



- Works only when $w(e) \geq 0$.
- Greedy algorithm, at each step for a vertex v , $d[v]$ becomes $\delta(s, v)$ with correct distance
- Relax edges out of the actual vertex.
- Uses a priority queue Q

Recall: Dijkstra SSSP

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Dijkstra(G, w, s)

Set $d[u] = +\infty$ and $p[u] = u$, $u \in V$.

$d[s] = 0$

$S = \emptyset$, Insert all the vertices in Q with key d

while $Q \neq \emptyset$ **do**

$u = \text{EXT-MIN}(Q)$

$S = S \cup \{u\}$

for all $v \in \text{Adj}[u]$ and $v \notin S$ **do**

Relax(u, v)

 change, if needed, the key of v in Q

Recall: Dijkstra SSSP

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Theorem

Consider the set S at any point in the algorithm execution. For each $u \in S$, $d[u] = \delta(s, u)$

Proof

The proof is by induction on the size of $|S|$.

Recall: Dijkstra SSSP (correctness)

- For $|S| = 1$, $S = \{s\}$ and $d[s] = 0 = \delta(s, s)$.
- Assume that the statement is true for $|S| = k$ and that the next vertex selected by the algorithm in the ExtractMin is v .
 - Consider a s, v shortest path P , let y be the first vertex in P that does not belong to S and let $x \in S$ be the node just before y in P .
 - By induction hypothesis $d[x] = \delta(s, x)$
 - As P is a shortest path, the edge (x, y) has been relaxed with $d[x] = \delta(s, x)$, and $w \geq 0$, we get $\delta(s, y) = d[y] = d[x] + w(x, y) \leq \delta(s, v)$.
 - As the algorithm selected v , $d[v] \leq d[y]$, therefore $d[v] = \delta(s, v)$.

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Recall: Dijkstra SSSP

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Theorem

Using a priority queue Dijkstra's algorithm can be implemented on a graph with n nodes and m edges to run in $O(m)$ time plus the time for n ExtractMin and m ChangeKey operations.

Q implementation	Worst-time complexity
Heap	$O(m \lg n)$
Fibonacci heap	$O(m + n \lg n)$

SSSP: Bellman-Ford

Richard E. Bellman
(1958)

Lester R. Ford Jr.
(1956)

Edward F. Moore
(1957)

Alfonso Shimbel (1955)
(Shimbel matrices)



- The BF algorithm works for graphs with general weights.
- It detects the existence of negative cycles.

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Bellman Ford Algorithm (BF)

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BF (G, w, s)

For $v \in V$, $d[v] = +\infty$, $p[v] = v$

$d[s] = 0$

for $i = 1$ to $n - 1$ **do**

for all $(u, v) \in E$ **do**

Relax(u, v)

for all $(u, v) \in E$ **do**

if $d[v] > d[u] + w(u, v)$ **then**

return Negative-weight cycle

return d, p

BF Algorithm: Example

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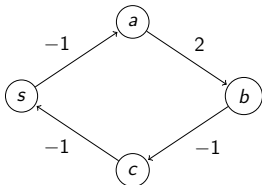
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	<i>s</i>	<i>a</i>	<i>b</i>	<i>c</i>
0	0	$+\infty$	$+\infty$	$+\infty$
1	0	-1	$+\infty$	$+\infty$
2	0	-1	1	$+\infty$
3	0	-1	1	0

$d[s] = 0$ but $d[c] + w(c, s) = -1$

BF reports **Negative cycle**

BF Algorithm: Example

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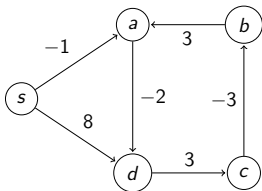
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	<i>s</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
0	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$
1	0	-1	$+\infty$	$+\infty$	8
2	0	-1	$+\infty$	11	-3
3	0	-1	8	0	-3
4	0	-1	-3	0	-3

d verifies the triangle inequality

Complexity of BF

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```
BF ( $G, w, s$ )  
Initialize  $\forall v \neq s, d[v] = \infty, p[v] = v$   
Initialize  $d[s] = 0$   
for  $i = 1$  to  $n - 1$  do  
    for all  $(u, v) \in E$  do  
        Relax $(u, v)$   
for all  $(u, v) \in E$  do  
    if  $d[v] > d[u] + w(u, v)$  then  
        return Negative-weight cycle  
return  $d, p$ 
```

$O(nm)$

Correctness of BF

Lemma

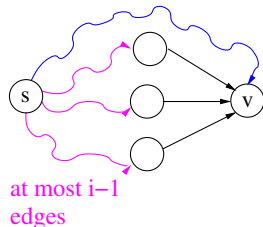
Consider the vector d computed by BF at the end of the i -th iteration. For $v \in V$, $d[v] \leq w(P)$ for every path P such that $s \rightsquigarrow^P v$ and $\ell(P) \leq i$.

Proof (Induction on i)

Before the i -th iteration, $d[v] \leq \min\{w(p)\}$ over all paths p with at most $i - 1$ edges.

The i -th iteration considers all paths with $\leq i$ edges reaching v , when relaxing the last edge in such paths.

□



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Correctness of BF

Theorem

If (G, w) has no negative weight cycles, BF computes correctly $\delta(s, v)$.

Proof

- Without negative-weight cycles, shortest paths are always simple (no repeated vertices), i.e., at most n vertices and $n - 1$ edges.
- By the previous lemma, the $n - 1$ iterations yield $d[v] \leq \delta(s, v)$.
- By the invariant of the relaxation algorithm $d[v] \geq \delta(s, v)$.

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Theorem

BF reports "negative-weight cycle" iff there exists a negative weight cycle in G reachable from s .

Proof

- Without negative-weight cycles in G , the previous theorem implies $d[v] = \delta(s, v)$, and by triangle inequality $d[v] \leq \delta(s, u) + w(u, v)$, so BF won't report a negative cycle if it doesn't exist.
- If there is a negative-weight cycle, then one of its edges can be relaxed, so BF will report correctly.

SSSP in a direct acyclic graphs (dags).

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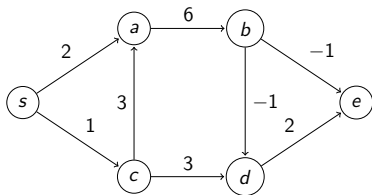
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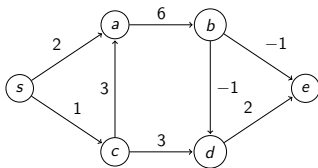
SSSP in DAG

Given an edge weighted dag $G = (V, E, w)$ and $s \in V$, find a shortest path from s to each other vertex in G , if it exists.



SSSP in a direct acyclic graphs (dags).

- A DAG has no cycles, so a distance can be defined among any pair of vertices.
- In particular there are shortest paths from s to any vertex v reachable from s .
- To obtain a faster algorithm we look for a good ordering of the edges: **topological sort**.



s, c, a, b, d, e

SSSP in a direct acyclic graphs (dags).

Let $Pre(v) = \{u \in V \mid (u, v) \in E\}$

SSSP-DAG(G, w)

Sort V in topologica order

For $v \in V$ set $d[v] = \infty$ and $p[v] = v$
 $d[s] = 0$.

for all $v \in V - \{s\}$ **in order do**

$d[v] = \min_{u \in Pre(v)} \{d[u] + w_{uv}\}$

$p[v] = \arg \min_{u \in Pre[v]} \{d[u] + w_{uv}\}$

Complexity? $T(n) = O(n + m)$

Correctness? $d[u] = \delta(s, u)$, for $u \in Pre(v)$

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All pairs shortest paths (APSP)

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- Given $G = (V, E, w)$, $|V| = n, |E| = m$, we want to determine $\forall u, v \in V, \delta(u, v)$.
- We assume that G does not contain negative cycles.
- **Naive idea:** We apply n times BF or Dijkstra (if there are not negative weights)
- Repetition of BF: $O(n^2m)$
- Repetition of Dijkstra: $O(nm \lg n)$ (if Q is implemented by a heap)

All pairs shortest paths: APSP

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- Unlike in the SSSP algorithm that assumed adjacency-list representation of G , for the APSP algorithm we consider the **adjacency matrix representation** of G .
- For convenience $V = \{1, 2, \dots, n\}$. The $n \times n$ adjacency matrix $W = (w(i, j))$ of G , w :

$$w_{ij} = \begin{cases} 0 & \text{if } i = j \\ w_{ij} & \text{if } (i, j) \in E \\ +\infty & \text{if } i \neq j \text{ and } (i, j) \notin E \end{cases}$$

All pairs shortest paths: APSP

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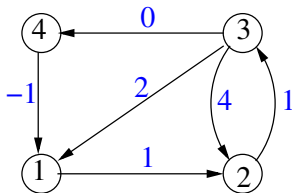
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- The input is a $n \times n$ adjacency matrix $W = (w_{ij})$



$$W = \begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & 1 & \infty \\ 2 & 4 & 0 & 0 \\ -1 & \infty & \infty & 0 \end{pmatrix}$$

- The output is a $n \times n$ matrix D , where $D[i,j] = \delta(i,j)$ and a $n \times n$ matrix P where $P[i,j]$ is the predecessor of j in a shortest path from i to j

Floyd-Warshall Algorithm

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Bernard Roy: *Transitivité et connexité* C.R.Aca. Sci. 1959

Robert Floyd: *Algorithm 97: Shortest Path*. CACM 1962

Stephen Warshall: *A theorem on Boolean matrices*. JACM, 1962

The FW Algorithm is a **dynamic programming** algorithm that exploits the recursive structure of shortest paths.

Optimal substructure of APSP

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- Recall: any subpath of a shortest path is a shortest path
- Let $p = p_1, \underbrace{p_2, \dots, p_{r-1}}_{\text{intermediate } v}, p_r$ and
- Let $d_{ij}^{(k)}$ be the minimum weight of a path $i \rightsquigarrow j$ s.t. the intermediate vertices are in $\{1, \dots, k\}$.
- When $k = 0$, $d_{ij}^{(0)} = w_{ij}$ (no intermediate vertices).

The recurrence

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Let p a path $i \rightsquigarrow j$ with intermediate vertices in $\{1, \dots, k\}$ and weight $d_{ij}^{(k)}$

- If k is not an intermediate vertex of p , then $d_{ij}^{(k)} = d_{ij}^{(k-1)}$.
- If k is an intermediate vertex of p , then $p = i \rightsquigarrow^{p_1} k \rightsquigarrow^{p_2} j$
- p_1 and p_2 are shortest paths with intermediate vertices in $\{1, \dots, k-1\}$.

$$\text{Therefore } d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \\ \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\} & \text{if } k \geq 1 \end{cases}$$

FW-algorithm

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BFW (W)

$d^{(0)} = W$

for $k = 1$ to n **do**

for $i = 1$ to n **do**

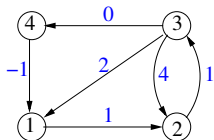
for $j = 1$ to n **do**

$d_{ij}^{(k)} = \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$

return $d^{(n)}$

- Time complexity: $T(n) = O(n^3), S(n) = O(n^3)$
- Correctness follows from the recurrence argument.

FW: Example



$$D^{(0)} = \begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & 1 & \infty \\ 2 & 4 & 0 & 0 \\ -1 & \infty & \infty & 0 \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & 1 & \infty \\ 2 & \textcolor{red}{3} & 0 & 0 \\ -1 & \textcolor{red}{0} & \infty & 0 \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 1 & \textcolor{red}{2} & \infty \\ \infty & 0 & 1 & \infty \\ 2 & 3 & 0 & 0 \\ -1 & 0 & \textcolor{red}{1} & 0 \end{pmatrix}$$

$$D^{(3)} = \begin{pmatrix} 0 & 1 & 2 & \textcolor{red}{2} \\ \textcolor{red}{3} & 0 & 1 & \textcolor{red}{1} \\ 2 & 3 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 0 & 1 & 2 & 2 \\ \textcolor{red}{0} & 0 & 1 & 1 \\ \textcolor{red}{-1} & \textcolor{red}{0} & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

$d_{3,2}^2 = 3, 3 \rightarrow 1 \rightarrow 2$ (interm vertices in $\{1, 2\}$)

$d_{3,2}^4 = 0, 3 \rightarrow 4 \rightarrow 1 \rightarrow 2$ (interm vertices in $\{1, 2, 3, 4\}$)

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FW: Constructing shortest paths

- To construct the matrix P , where $p_{i,j}$ is the predecessor of j in a shortest path $i \rightsquigarrow j$,
- we define a sequence of matrices $P^{(0)}, \dots, P^{(n)}$.
 $p_{i,j}^k$ is the predecessor in a shortest path $i \rightsquigarrow j$, which uses only vertices in $\{1, \dots, k\}$.
- $p_{i,j}^{(0)} = \begin{cases} \text{NIL} & \text{if } i = j \text{ or } w_{ij} = +\infty, \\ i & \text{if } i \neq j \text{ and } w_{ij} \neq +\infty. \end{cases}$
- For $k \geq 1$ we get the recurrence:

$$p_{i,j}^{(k)} = \begin{cases} p_{i,j}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}, \\ p_{k,j}^{(k-1)} & \text{otherwise.} \end{cases}$$

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BFW with paths

BFW W

$d^{(0)} = W$

Initialize $p^{(0)}$

for $k = 1$ to n **do**

for $i = 1$ to n **do**

for $j = 1$ to n **do**

if $d_{ij}^{(k)} \leq d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$ **then**

$d_{ij}^{(k)} = d_{ij}^{(k-1)}$

$p_{ij}^{(k)} = p_{ij}^{(k-1)}$

else

$d_{ij}^{(k)} = d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

$p_{ij}^{(k)} = p_{kj}^{(k-1)}$

return $d^{(n)}$

Complexity: $T(n) = O(n^3)$

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APSP: Johnson's algorithm

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- A faster algorithm for sparse graphs, i.e., $m = o(n^2)$
- The graph is given by adjacency list and we assume that it has no negative weight cycles. In fact the algorithm detects its existence.

Johnson's algorithm

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Donald B. Johnson: *Efficient algorithms for shortest paths in sparse networks*, JACM 1977



- The algorithm uses BF to reduce the problems to one with positive weights.
- Then it runs n times Dijkstra's algorithm.

Weight modification that preserve path weight

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Lemma

Let $G = (V, E, w)$ be a weighted digraph. Let $f : V \rightarrow \mathbb{R}$ and, for $(u, v) \in E$, let $w'(u, v) = w(u, v) + f(u) - f(v)$. Let p be a path $u \rightsquigarrow^p v$ in G . Then $w'(p) = w(p) + f(u) - f(v)$.

Proof

As an intermediate vertex w in the path is the end of one edge and the start of another the contribution of $f(w)$ cancels.

The weight modification

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- Let $G = (V, E, w)$ be a weighted digraph with no negative weight cycle.
- Construct a graph $G' = (V', E', w')$ by adding to G a new vertex s and edges (s, u) , for $u \in V$. Define $w'(e) = w(e)$ if $e \in E' \cap E$ and 0 otherwise.
- Let d be the output of the BF algorithm on input (V', E', w', s) .
- As G has no negative weight cycles, G' has no negative weight cycles, so BF computes $d : V \rightarrow \mathbb{R}$. Furthermore, for $u \in V$, $d(u) = \delta_{G'}(s, u)$.

The weight modification

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Lemma

Let $G = (V, E, w)$ be a weighted digraph with no negative weight cycles. Let $d : V \rightarrow \mathbb{R}$ be the function computed by the BF algorithm on G' described before. Let $G_d = (V, E, w')$ where $w'(u, v) = w(u, v) + d(u) - d(v)$.

If p is a shortest path $u \rightsquigarrow^P v$ in G , p is a shortest path in G_d . Furthermore, $\delta_{G_d}(u, v) = \delta_G(u, v) + d(u) - d(v)$.

Proof

For any path p , $u \rightsquigarrow^P v$, $w'(p) = w(p) + d(u) - d(v)$. As the last term depends only on u and v , the claim follows.

The weight modification

Lemma

*Let $G = (V, E, w)$ be a weighted digraph with no negative weight cycles. Let $d : V \rightarrow \mathbb{R}$ be the function computed by the BF algorithm on G' described before. Let $G_d = (V, E, w')$ where $w'(u, v) = w(u, v) + d(u) - d(v)$.
For $(u, v) \in E$, $w'(u, v) \geq 0$.*

Proof

- By triangle inequality, for a path p , $u \rightsquigarrow^p v$,
 $\delta_{G'}(s, v) \leq \delta_{G'}(s, u) + w(p)$,
- i.e., $0 \leq w(p) + \delta_{G'}(s, u) - \delta_{G'}(s, v)$
- Therefore $w'(p) = w(p) + d(u) - d(v) \geq 0$.

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```
Johnson ( $V, E, W$ )  
  Compute  $G'$   
   $f = BF(G', s)$   
  Compute  $G_f$   
  for all  $v \in V$  do  
     $d[v] = \text{Dijkstra}(G_f, v)$   
  for all  $u, v \in V$  do  
     $d[u][v] = d[u][v] + f[v] - f[u]$   
  return  $d$ 
```

- Time complexity: $O(nm)$ + the cost of n calls to Dijkstra
- Correctness follows from the previous lemmas.

Conclusions

SSSP no negative weight cycles accessible from s .

	Dijkstra	BF
$w \geq 0$	$O(m + n \lg n)$	$O(nm)$
$w \in \mathbb{Z}$	NO	$O(nm)$

APSP no negative weight cycles.

	Dijkstra	BF	FW	Johnson
$w \geq 0$	$O(nm + n^2 \lg n)$	$O(n^2 m)$	$O(n^3)$	$O(nm + n^2 \lg n)$
$w \in \mathbb{R}$	NO	$O(n^2 m)$	$O(n^3)$	$O(nm + n^2 \lg n)$

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Conclusions: Remarks for APSP algorithms

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- For sparse graphs with $m = \omega(n)$ $m = o(n^2)$, Johnson is the most efficient.
- For dense graphs with $m = \Theta(n^2)$, FW has the best complexity.
- For unweighted and undirected graphs, there is an algorithm by R.Seidel that works in $O(n^\omega \lg n)$, where n^ω is the complexity of multiplying two $n \times n$ matrices, which of as today is $\omega \sim 2.3$.
- For further reading on shortest paths, see chapters 24 and 25 of CLRS or 4.4 and 6.8–6.10 of KT.

Conclusions: Remarks for SSSP algorithms in graphs with non-negative weights

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- Dijkstra's algorithm provides in addition the vertices sorted by distance.
It is optimal when this is required ([HHRTT FOCS2024]).
- Without this condition:
 - randomized $O(m\sqrt{\log n \log \log n})$ ([DMSY FOCS2023])
 - $O(m(\log n)^{2/3})$ ([DMMSY STOC2025])