

Greedy Algorithms

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

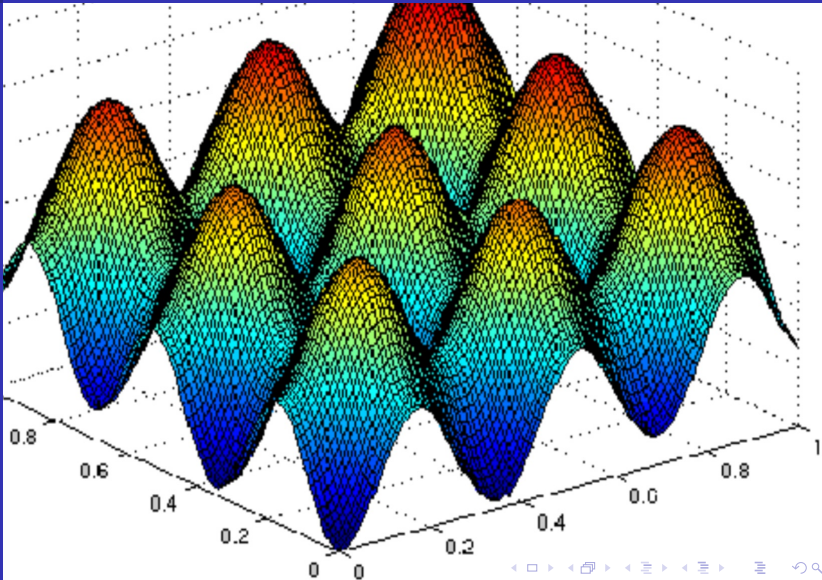
Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms



Greedy Algorithms

- Greedy algorithms are mainly designed to solve **combinatorial optimization problems**:

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Greedy Algorithms

- Greedy algorithms are mainly designed to solve **combinatorial optimization problems**:

Given an input, we want to compute an optimal solution according to some **objective function**.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- Greedy algorithms are mainly designed to solve **combinatorial optimization problems**:

Given an input, we want to compute an optimal solution according to some **objective function**.

- The solutions are formed by a sequence of elements.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Greedy Algorithms

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

- Greedy algorithms are mainly designed to solve **combinatorial optimization problems**:
Given an input, we want to compute an optimal solution according to some **objective function**.
- The solutions are formed by a sequence of elements.
- For example: Given a graph $G = (V, E)$ and two vertices $u, v \in V$, we want to find a path from u to v having the minimum number of edges.

Greedy Algorithms

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

- Greedy algorithms are mainly designed to solve **combinatorial optimization problems**:

Given an input, we want to compute an optimal solution according to some **objective function**.

- The solutions are formed by a sequence of elements.
- For example: Given a graph $G = (V, E)$ and two vertices $u, v \in V$, we want to find a path from u to v having the minimum number of edges.

The solution is a sequence of vertices or edges.

Greedy Algorithms

A greedy algorithm obtains an optimal solution to a combinatorial optimization problem by making a sequence of choices (without backtracking).

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Greedy Algorithms

A greedy algorithm obtains an optimal solution to a combinatorial optimization problem by making a sequence of choices (without backtracking).

- Greedy algorithms make locally optimal **myopic choices** to construct incrementally a global solution.

Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection
Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

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Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection
Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

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- In some cases this will lead to a globally optimal solution.
- Often **easy** greedy algorithms are used to obtain **quickly** solutions to optimization problems, even though they do not always yield optimal solutions.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- In some cases this will lead to a globally optimal solution.
- Often **easy** greedy algorithms are used to obtain **quickly** solutions to optimization problems, even though they do not always yield optimal solutions.
- For many problems the greedy technique yields good heuristics, or even good approximation algorithms.

Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection
Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

Greedy Algorithms

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

- At each step we choose the best (myopic) choice at the moment for the corresponding component of the solution, and then solve the subproblem that arise by taking this decision.
- The choice may depend on previous choices, but not on future choices.
- At each choice, the algorithm reduces the problem into a smaller one, and obtains one component of the solution.
- **A greedy algorithm never backtracks.**

Greedy Algorithms

For the greedy strategy to work correctly, it is necessary that the problem under consideration has two characteristics:

- **Greedy choice property:** We can arrive to the global optimum by selecting a local optimums.
- **Optimal substructure:** After making some local decision, it must be the case that there is an optimal solution to the problem that contains the partial solution constructed so far.

In many cases, the local criteria for selecting a part of the solution allow us to define a global order that directs the greedy algorithm.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

The FRACTIONAL KNAPSACK problem

FRACTIONAL KNAPSACK: Given as input a set of n items, where item i has weight w_i and value v_i , together with a maximum total weight W . We want to select a set of fractions of items, to maximize the profit, within the allowed total weight.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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Observe that, from each item, we can select any arbitrary fraction of its weight keeping the same fraction of their value.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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Observe that, from each item, we can select any arbitrary fraction of its weight keeping the same fraction of their value.

Example. $n = 5$ and $W = 100$

Item	1	2	3	4	5
w	10	20	30	40	50
v	20	30	66	40	60



Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

FRACTIONAL KNAPSACK: GREEDY SCHEMA

GreedyFKnapsack (n, v, w, W)

$O = \{1, \dots, n\}; S = \emptyset; Val = 0; i = 0;$

while $W > 0$ **do**

Let $i \in O$ be the item with **property P**

if $w[i] \leq W$ **then**

$S = S \cup \{(i, 1)\}; W = W - w[i]; Val = Val + v[i];$

else

$S = S \cup \{(i, W/w[i])\}; Val = Val + v[i] * W/w[i];$

$W = 0$

end if

Remove i from O .

end while

return S

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: most valuable object

Example. $n = 5$ and $W = 100$

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Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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Item	1	2	3	4	5
w	10	20	30	40	50
v	20	30	66	40	60

Item	1	2	3	4	5
Selected	0	0	1	0.5	1

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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Example. $n = 5$ and $W = 100$

Item	1	2	3	4	5
w	10	20	30	40	50
v	20	30	66	40	60

Item	1	2	3	4	5
Selected	0	0	1	0.5	1

Total selected weight **100** and total value **146**

Selecting the most valuable object is a correct greedy rule?

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: the lighter object

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Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: the lighter object

Example. $n = 5$ and $W = 100$

Item	1	2	3	4	5
w	10	20	30	40	50
v	20	30	66	40	60

Item	1	2	3	4	5
Selected	1	1	1	1	0

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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v	20	30	66	40	60

Item	1	2	3	4	5
Selected	1	1	1	1	0

Total selected weight **100** and total value **156**

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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v	20	30	66	40	60

Item	1	2	3	4	5
Selected	1	1	1	1	0

Total selected weight **100** and total value **156**

Selecting the **most valuable object** does not provide a correct solution.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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w	10	20	30	40	50
v	20	30	66	40	60

Item	1	2	3	4	5
Selected	1	1	1	1	0

Total selected weight **100** and total value **156**

Selecting the **most valuable object** does not provide a correct solution.

Selecting the **lighter object** is a correct greedy rule?

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: the highest ratio value/weight

Example. $n = 5$ and $W = 100$

Item	1	2	3	4	5
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Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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Example. $n = 5$ and $W = 100$

Item	1	2	3	4	5
w	10	20	30	40	50
v	20	30	66	40	60

Item	1	2	3	4	5
ratio	2.0	1.5	2.2	1.0	1.2

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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v	20	30	66	40	60

Item	1	2	3	4	5
ratio	2.0	1.5	2.2	1.0	1.2
Selected	1	1	1	0	0.8

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: the highest ratio value/weight

Example. $n = 5$ and $W = 100$

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w	10	20	30	40	50
v	20	30	66	40	60

Item	1	2	3	4	5
ratio	2.0	1.5	2.2	1.0	1.2
Selected	1	1	1	0	0.8

Total selected weight **100** and total value **164**

Selecting the **lighter object** does not provide a correct solution.

Highest ratio value/weight is a correct greedy rule?

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: highest ratio value/weight

Theorem

The GreedyFKnapsack selecting the item with the best ratio value/weight always finds an optimal solution to the FRACTIONAL KNAPSACK problem

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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Theorem

The GreedyFKnapsack selecting the item with the best ratio value/weight always finds an optimal solution to the FRACTIONAL KNAPSACK problem

Proof.

Assume that the n items are sorted so that

$$\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \dots \geq \frac{v_n}{w_n}$$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: highest ratio value/weight

Let $X = (x_1, \dots, x_n)$, $x_i \in [0, 1]$, be the portions of items selected by the algorithm.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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Let $X = (x_1, \dots, x_n)$, $x_i \in [0, 1]$, be the portions of items selected by the algorithm.

- If $x_i = 1$, for all i , the computed solution is optimal.
We take all!

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- If $x_i = 1$, for all i , the computed solution is optimal.
We take all!
- Otherwise, let j be the smallest value for which $x_j < 1$.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: highest ratio value/weight

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- If $x_i = 1$, for all i , the computed solution is optimal.
We take all!
- Otherwise, let j be the smallest value for which $x_j < 1$.
- According with the algorithm,
 $x_i = 1$, for $i < j$, and
 $x_i = 0$, for $i > j$.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: highest ratio value/weight

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- If $x_i = 1$, for all i , the computed solution is optimal.
We take all!
- Otherwise, let j be the smallest value for which $x_j < 1$.
- According with the algorithm,
 $x_i = 1$, for $i < j$, and
 $x_i = 0$, for $i > j$.
- Furthermore, $\sum_{i=1}^n x_i w_i = W$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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Let $Y = (y_1, \dots, y_n)$, $y_i \in [0, 1]$, be the portions of items selected in a **feasible** solution, i.e.,

$$\sum_{i=1}^n y_i w_i \leq W$$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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$$\sum_{i=1}^n y_i w_i \leq W$$

- We have, $\sum_{i=1}^n y_i w_i \leq W = \sum_{i=1}^n x_i w_i$
- So, $0 \leq \sum_{i=1}^n x_i w_i - \sum_{i=1}^n y_i w_i = \sum_{i=1}^n (x_i - y_i) w_i$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: highest ratio value/weight

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$$\sum_{i=1}^n y_i w_i \leq W$$

- We have, $\sum_{i=1}^n y_i w_i \leq W = \sum_{i=1}^n x_i w_i$
- So, $0 \leq \sum_{i=1}^n x_i w_i - \sum_{i=1}^n y_i w_i = \sum_{i=1}^n (x_i - y_i) w_i$
- Then, the value difference can be expressed as

$$\begin{aligned} v(X) - v(Y) &= \sum_{i=1}^n x_i v_i - \sum_{i=1}^n y_i v_i = \sum_{i=1}^n (x_i - y_i) v_i \\ &= \sum_{i=1}^n (x_i - y_i) w_i \frac{v_i}{w_i} \end{aligned}$$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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We want to bound $v(x) - v(y) = \sum_{i=1}^n (x_i - y_i) w_i \frac{v_i}{w_i}$.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- If $i < j$, $x_i = 1$, so $x_i - y_i \geq 0$ but, as $\frac{v_i}{w_i} \geq \frac{v_j}{w_j}$,

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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$$(x_i - y_i) \frac{v_i}{w_i} \geq (x_i - y_i) \frac{v_j}{w_j}$$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- If $i < j$, $x_i = 1$, so $x_i - y_i \geq 0$ but, as $\frac{v_i}{w_i} \geq \frac{v_j}{w_j}$,

$$(x_i - y_i) \frac{v_i}{w_i} \geq (x_i - y_i) \frac{v_j}{w_j}$$

- If $i > j$, $x_i = 0$, so $x_i - y_i \leq 0$ but, as $\frac{v_i}{w_i} \leq \frac{v_j}{w_j}$,

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- If $i < j$, $x_i = 1$, so $x_i - y_i \geq 0$ but, as $\frac{v_i}{w_i} \geq \frac{v_j}{w_j}$,

$$(x_i - y_i) \frac{v_i}{w_i} \geq (x_i - y_i) \frac{v_j}{w_j}$$

- If $i > j$, $x_i = 0$, so $x_i - y_i \leq 0$ but, as $\frac{v_i}{w_i} \leq \frac{v_j}{w_j}$,

$$(x_i - y_i) \frac{v_i}{w_i} \geq (x_i - y_i) \frac{v_j}{w_j}$$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: highest ratio value/weight

We want to bound $v(x) - v(y) = \sum_{i=1}^n (x_i - y_i) w_i \frac{v_i}{w_i}$.

- If $i < j$, $x_i = 1$, so $x_i - y_i \geq 0$ but, as $\frac{v_i}{w_i} \geq \frac{v_j}{w_j}$,

$$(x_i - y_i) \frac{v_i}{w_i} \geq (x_i - y_i) \frac{v_j}{w_j}$$

- If $i > j$, $x_i = 0$, so $x_i - y_i \leq 0$ but, as $\frac{v_i}{w_i} \leq \frac{v_j}{w_j}$,

$$(x_i - y_i) \frac{v_i}{w_i} \geq (x_i - y_i) \frac{v_j}{w_j}$$

- The same inequality in both cases.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: highest ratio value/weight

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: highest ratio value/weight

- Using the derived inequalities, we have

$$\begin{aligned}v(x) - v(y) &= \sum_{i=1}^n (x_i - y_i) w_i \frac{v_i}{w_i} \\ &\geq \sum_{i=1}^n (x_i - y_i) w_i \frac{v_j}{w_j} \geq \frac{v_j}{w_j} \sum_{i=1}^n (x_i - y_i) w_i \geq 0\end{aligned}$$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: highest ratio value/weight

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- So, $v(X) - v(Y) \geq 0$, and **x is an optimal solution.**

End Proof

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

GreedyFKnapsack: highest ratio value/weight

GreedyFKnapsack (n, v, w, W)

$O = \{1, \dots, n\}; S = \emptyset; Val = 0; i = 0;$

while $W > 0$ **do**

Let $i \in O$ be an item with highest value/weight

if $w[i] < W$ **then**

$S = S \cup \{(i, 1)\}; W = W - w[i]; Val = Val + v[i];$

else

$S = S \cup \{(i, W/w[i])\}; V = Val + v[i] * W/w[i];$

$W = 0$

end if

Remove i from O

end while

return S

Cost?

Definitions

Fractional
Knapsack

Some selection
criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity
selection

Minimizing lateness

Optimal prefix
codes

data compression

prefix codes

Huffman code

Approximation
algorithms

GreedyFKnapsack: highest ratio value/weight

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$W = 0$

end if

Remove i from O

end while

return S

Cost? $O(n^2)$

Definitions

Fractional
Knapsack

Some selection
criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity
selection

Minimizing lateness

Optimal prefix
codes

data compression

prefix codes

Huffman code

Approximation
algorithms

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else

$S = S \cup \{(i, W/w[i])\}; V = Val + v[i] * W/w[i];$

$W = 0$

end if

Remove i from O

end while

return S

Cost? $O(n^2)$ a better implementation?

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

FRACTIONAL KNAPSACK

GreedyFKnapsack (n, v, w, W)

Sort the items in decreasing value of v_i/w_i

$S = \emptyset$; $Val = 0$; $i = 0$;

while $W > 0$ and $i < n$ **do**

if $w[i] < W$ **then**

$S = S \cup \{(i, 1)\}$; $W = W - w[i]$; $Val = Val + v[i]$;

else

$S = S \cup \{(i, W/w[i])\}$; $Val = Val + v[i] * W/w[i]$;

$W = 0$;

end if

$++i$;

end while

return S

This algorithm has cost of $T(n) = O(n \log n)$.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

FRACTIONAL KNAPSACK

Theorem

The FRACTIONAL KNAPSACK problem can be solved in time $O(n \log n)$.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

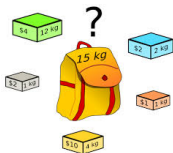
prefix codes

Huffman code

Approximation algorithms

0-1 KNAPSACK

0-1 KNAPSACK Given as input a set of n items, where item i has weight w_i and value v_i , together with a maximum total weight W permissible. We want to select a set of items to maximize the profit, within allowed weight W .



Items cannot be fractioned, you have to take all or nothing.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

The greedy algorithm for the fractional version does not work for 0-1 KNAPSACK

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

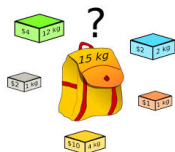
Huffman code

Approximation algorithms

The greedy algorithm for the fractional version does not work for 0-1 KNAPSACK

Example: $n = 3$ and $W = 50$

Item	1	2	3
w	10	20	30
v	60	100	120
v/w	6	5	4



The algorithm will select item 1, with value 60. This is not an optimal solution, as 2 and 3 form a better solution, with value 220.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

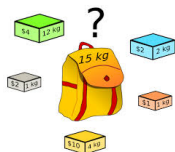
Huffman code

Approximation algorithms

The greedy algorithm for the fractional version does not work for 0-1 KNAPSACK

Example: $n = 3$ and $W = 50$

Item	1	2	3
w	10	20	30
v	60	100	120
v/w	6	5	4



The algorithm will select item 1, with value 60. This is not an optimal solution, as 2 and 3 form a better solution, with value 220.

But, 0-1 KNAPSACK is known to be NP-hard.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Tasks or Activities Scheduling problems

General Setting:

- Given: A set of n tasks (with different characteristics) to be processed by a single/multiple processor system (according to different constraints).
- Provide a schedule, (when and where a (each) task must be executed), so as to optimize some objective criteria.

Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection
Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

Some mono processor scheduling problems

- 1 INTERVAL SCHEDULING problem:** Tasks have **start** and **finish** times. The objective is to make an executable selection with **maximum** size.
- 2 WEIGHTED INTERVAL SCHEDULING problem:** Tasks have **start** and **finish** times and its execution produce **profits**. The objective is to make an executable selection giving **maximum** profit.
- 3 JOB SCHEDULING problem (Lateness minimization):** Tasks have **processing time** (could start at any time) and a **deadline**, define the lateness of a task as the time from its deadline to its starting time. **Find an executable schedule, including all the tasks, that minimizes the total lateness.**

Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection
Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

The INTERVAL SCHEDULING problem

The INTERVAL SCHEDULING (aka ACTIVITY SELECTION problem)

- Given a set of n tasks where, for $i \in [n]$, task i has a **start time** s_i and a **finish time** f_i , with $s_i < f_i$.
- The processor is a single machine, that can process only one task at a time.
- A task must be processed completely from its starting time to its finish time.
- We want to find a set of **mutually compatible tasks**, where activities i and j are compatible if $[s_i, f_i) \cap (s_j, f_j] = \emptyset$, with **maximum size**.

A solution is a set of mutually compatible activities, and the objective function to maximize is the cardinality of the solution set.

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

Optimal prefix codes

- data compression
- prefix codes
- Huffman code

Approximation algorithms

Example: one input

Task :	1	2	3	4	5	6	7	8
Start (s):	3	2	2	1	8	6	4	7
Finish (f):	5	5	3	5	9	9	5	8

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

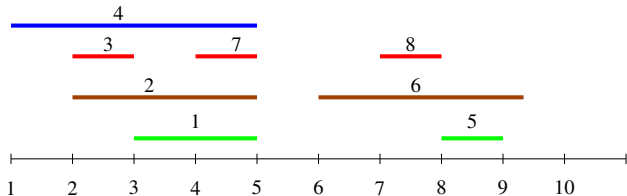
prefix codes

Huffman code

Approximation algorithms

Example: one input

Task :	1	2	3	4	5	6	7	8
Start (s):	3	2	2	1	8	6	4	7
Finish (f):	5	5	3	5	9	9	5	8



Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Designing a greedy algorithm

To apply the greedy technique to a problem, we must take into consideration the following,

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Designing a greedy algorithm

To apply the greedy technique to a problem, we must take into consideration the following,

- A **local criteria** to allow the selection,
- having in mind a **property** ensuring that a partial solution can be completed to an optimal solution.

Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection
Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

Designing a greedy algorithm

To apply the greedy technique to a problem, we must take into consideration the following,

- A **local criteria** to allow the selection,
- having in mind a **property** ensuring that a partial solution can be completed to an optimal solution.

As for the `FRACTIONALKNAPSACK` problem, the selection criteria might lead to a sorting criteria. In such a case, greedy processes the input in this particular order.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

The Interval Scheduling problem: Earlier finish time

IntervalScheduling(A)

$S = \emptyset$; $T = \{1, \dots, n\}$;

while $T \neq \emptyset$ **do**

 Let i be the task that finishes earlier among those in T

$S = S \cup \{i\}$;

 Remove from T , i and all tasks $j \in T$ with $s_j \leq t_i$

end while

return S .

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

The Interval Scheduling problem: Earlier finish time

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end while

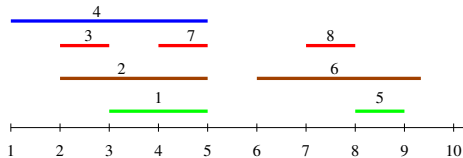
return S .

task : 3 4 2 7 8 5 6

s : 3 1 2 4 8 5 6

f : 3 5 5 5 8 9 9

SOL: 3 1 8 5



IntervalScheduling: correctness

Theorem

*The **IntervalScheduling** algorithm produces an optimal solution to the INTERVAL SCHEDULING problem.*

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

IntervalScheduling: correctness

Theorem

*The **IntervalScheduling** algorithm produces an optimal solution to the INTERVAL SCHEDULING problem.*

Proof.

We want to prove that:

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

IntervalScheduling: correctness

Theorem

*The **IntervalScheduling** algorithm produces an optimal solution to the INTERVAL SCHEDULING problem.*

Proof.

We want to prove that:

There is an optimal solution that includes the task with the earlier finishing time.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

IntervalScheduling: correctness

Theorem

*The **IntervalScheduling** algorithm produces an optimal solution to the INTERVAL SCHEDULING problem.*

Proof.

We want to prove that:

There is an optimal solution that includes the task with the earlier finishing time.

We will assume that this is not the case and reach contradiction.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

IntervalScheduling: correctness

- Let i be a task that finishes at the earliest finish time.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

IntervalScheduling: correctness

- Let i be a task that finishes at the earliest finish time.
- Let S be an optimal solution with $i \notin S$.
Let $k \in S$ be the task with the earlier finish time among those in S .

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

IntervalScheduling: correctness

- Let i be a task that finishes at the earliest finish time.
- Let S be an optimal solution with $i \notin S$.
Let $k \in S$ be the task with the earlier finish time among those in S .
- Any task in S finishes after time $A[k].f$, so they start also after $A[k].f$. As $A[i].f \leq A[k].f$, $S' = (S - \{k\}) \cup \{i\}$ is a set of mutually compatible tasks.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

IntervalScheduling: correctness

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- Any task in S finishes after time $A[k].f$, so they start also after $A[k].f$. As $A[i].f \leq A[k].f$, $S' = (S - \{k\}) \cup \{i\}$ is a set of mutually compatible tasks.
- As $|S'| = |S|$, S' is an optimal solution that includes i .

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

IntervalScheduling: correctness

Optimal substructure

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

IntervalScheduling: correctness

Optimal substructure

After each greedy choice, we are left with an optimization subproblem, of the same form as the original. In the subproblem we removed the selected task and all tasks that overlap with the selected one.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

IntervalScheduling: correctness

Optimal substructure

After each greedy choice, we are left with an optimization subproblem, of the same form as the original. In the subproblem we removed the selected task and all tasks that overlap with the selected one.

An optimal solution to the original problem is formed by the selected task (one that finishes earliest possible) and an optimal solution to the corresponding subproblem.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

IntervalScheduling: correctness

Optimal substructure

After each greedy choice, we are left with an optimization subproblem, of the same form as the original. In the subproblem we removed the selected task and all tasks that overlap with the selected one.

An optimal solution to the original problem is formed by the selected task (one that finishes earliest possible) and an optimal solution to the corresponding subproblem.

End Proof

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Interval Scheduling: cost

IntervalScheduling(A)

$S = \emptyset$; $T = [n]$; $O(n)$

while $T \neq \emptyset$ **do**

Let i be the task that finishes earlier among those in T

$O(n)$

$S = S \cup \{i\}$;

Remove i and all tasks overlapping i from T $O(n)$

end while

return S .

It takes $O(n^2)$

Definitions

Fractional
Knapsack

Some selection
criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity
selection

Minimizing lateness

Optimal prefix
codes

data compression

prefix codes

Huffman code

Approximation
algorithms

Interval Scheduling: cost

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$O(n)$

$S = S \cup \{i\}$;

Remove i and all tasks overlapping i from T $O(n)$

end while

return S .

It takes $O(n^2)$ Too slow, a better implementation?

Definitions

Fractional
Knapsack

Some selection
criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity
selection

Minimizing lateness

Optimal prefix
codes

data compression

prefix codes

Huffman code

Approximation
algorithms

Interval Scheduling: cost

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Let i be the task that finishes earlier among those in T

$O(n)$

$S = S \cup \{i\}$;

Remove i and all tasks overlapping i from T $O(n)$

end while

return S .

It takes $O(n^2)$ Too slow, a better implementation?

We have to find a fastest way to select i and discard i and the overlapping tasks.

Definitions

Fractional
Knapsack

Some selection
criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity
selection

Minimizing lateness

Optimal prefix
codes

data compression

prefix codes

Huffman code

Approximation
algorithms

The Interval Scheduling problem: algorithm 2

IntervalScheduling2(A)

Sort A in increasing order of $A.f$

$S = \{0\}$

$j = 0$ {pointer to last task in solution}

for $i = 1$ **to** $n - 1$ **do**

if $A[i].s \geq A[j].f$ **then**

$S = S \cup \{i\}; j = i;$

end if

end for

return S .

Definitions

Fractional
Knapsack

Some selection
criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity
selection

Minimizing lateness

Optimal prefix
codes

data compression

prefix codes

Huffman code

Approximation
algorithms

IntervalScheduling2: correctness

Theorem

*The **IntervalScheduling2** algorithm produces an optimal solution to the INTERVAL SCHEDULING problem in time $O(n \log n)$*

Proof.

- A task that does not verify $A[i].s \geq A[j].f$ overlaps with task $j \in S$. It starts before j and finishes after j finishes. Therefore, it cannot be part of a solution together with j .
- As the tasks are sorted by finish time at each step, we select, among those tasks that start later than j , the one that finishes earlier.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

IntervalScheduling2: correctness

- **IntervalScheduling2** makes the same greedy choice as **IntervalScheduling**, therefore it computes an optimal solution.
- The most costly step in **IntervalScheduling2** is the sorting, which can be done in $O(n \log n)$ time using Merge sort.

End Proof

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

Interval scheduling

- Weighted activity selection
- Minimizing lateness

Optimal prefix codes

- data compression
- prefix codes
- Huffman code

Approximation algorithms

IntervalScheduling2: particular case

If we know that the tasks start and finish time are given in seconds within a day (24 hours),

IntervalScheduling2 can be implemented with cost

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

IntervalScheduling2: particular case

If we know that the tasks start and finish time are given in seconds within a day (24 hours),

IntervalScheduling2 can be implemented with cost $O(n)$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Adding weights: greedy choice does not always work.

WEIGHTED ACTIVITY SELECTION problem:

Given a set of n activities to be processed by a single machine, where each activity i has a **start time** s_i and a **finish time** f_i , with $s_i < f_i$, and a **weight** w_i .

We want to find a set S of mutually compatible activities so that $\sum_{i \in S} w_i$ is **maximum** among all such sets.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

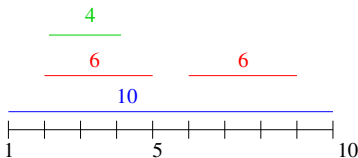
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Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

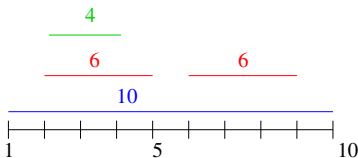
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We want to find a set S of mutually compatible activities so that $\sum_{i \in S} w_i$ is **maximum** among all such sets.



IntervalScheduling2 selects the green and the second red activity with weight 10 which is not an optimal solution.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

What about maximizing locally the selected weight?

WeightedAS-max-weight (A)

$S = \emptyset; T = [n];$

while $T \neq \emptyset$ **do**

 Let i be the task with highest weight among those in T .

$S = S \cup \{i\}$

 Remove i and all tasks overlapping i from T

end while

return S

Definitions

Fractional
Knapsack

Some selection
criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity
selection

Minimizing lateness

Optimal prefix
codes

data compression

prefix codes

Huffman code

Approximation
algorithms

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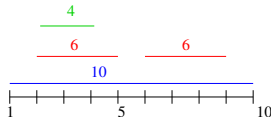
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Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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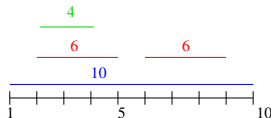
 Let i be the task with highest weight among those in T .

$S = S \cup \{i\}$

 Remove i and all tasks overlapping i from T

end while

return S



The algorithm chooses the blue task with weight 10, and the optimal solution is formed by the two red intervals with total weight of 12.

Greedy approach

- Easy to come up with one or more greedy algorithms
- Easy to analyze the running time.
- Hard to establish correctness.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Greedy approach

- Easy to come up with one or more greedy algorithms
- Easy to analyze the running time.
- Hard to establish correctness.
- Most greedy algorithms we came up are not correct on all inputs.

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

Optimal prefix codes

- data compression
- prefix codes
- Huffman code

Approximation algorithms

A Job Scheduling problem

LATENESS MINIMIZATION problem.

- We have a **single processor** and n tasks (or jobs) to be processed.
- Once a task starts to be processed it continues using the processor until its completion.

Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection
Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

A Job Scheduling problem

LATENESS MINIMIZATION problem.

- We have a **single processor** and n tasks (or jobs) to be processed.
- Once a task starts to be processed it continues using the processor until its completion.
- Processing task i takes **time t_i** . Furthermore, task i has a **deadline d_i** .

Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

A Job Scheduling problem

LATENESS MINIMIZATION problem.

- We have a **single processor** and n tasks (or jobs) to be processed.
- Once a task starts to be processed it continues using the processor until its completion.
- Processing task i takes **time t_i** . Furthermore, task i has a **deadline d_i** .
- The goal is to schedule **all the tasks**, i.e., determine the time at which to start processing each tasks.
- We want to minimize, over all the tasks, the maximum amount of time that the finish time of a tasks exceeds its deadline.

Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

Minimize Lateness: a more formal formulation

- We have a single processor
- We have n jobs such that job i :
 - requires $t_i > 0$ units of processing time,
 - it has to be finished by time d_i ,
 - A schedule will determine a finish time f_i

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

Optimal prefix codes

- data compression
- prefix codes
- Huffman code

Approximation algorithms

Minimize Lateness: a more formal formulation

- We have a single processor
- We have n jobs such that job i :
 - requires $t_i > 0$ units of processing time,
 - it has to be finished by time d_i ,
 - A schedule will determine a finish time f_i
- Under this schedule lateness of i is:

$$L_i = \begin{cases} 0 & \text{if } f_i \leq d_i, \\ f_i - d_i & \text{otherwise.} \end{cases}$$

- The **lateness** of a valid schedule is $\max_i L_i$.

Goal: find a schedule with minimum lateness

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

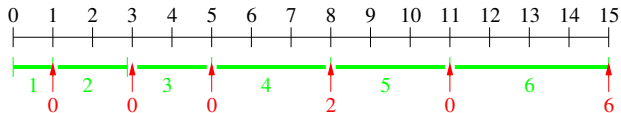
Huffman code

Approximation algorithms

Minimize Lateness: an example

We must assign starting time s_i to each i , making sure that the processor only processes a job at a time, in such a way that $\max_i L_i$ is minimum.

6 tasks: t: 1 2 2 3 3 4 d: 9 8 15 6 14 9



Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Minimize Lateness

We can try different **task selection criteria** to schedule the jobs following a generic greedy algorithm.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Minimize Lateness

We can try different **task selection criteria** to schedule the jobs following a generic greedy algorithm.

LatenessXX (A)

Sort A according to XX

$S[0] = 0; t = A[0].t; L = \max(0, t - A[0].d);$

for $i = 1$ **to** $n - 1$ **do**

$S[i] = t$

$t = t + A[i].t$

$L = \max(L, \max(0, t - A[i].d))$

end for

return (S, L)

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Minimize Lateness: selection criteria

Process jobs with short time first

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Minimize Lateness: selection criteria

Process jobs with short time first

i	t_i	d_i
1	1	6
2	5	5

1 at time 0 and 2 at time 1 lateness 1, but
2 at time 0 and 1 at time 5 has lateness 0.
It does not work.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Minimize Lateness: selection criteria

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i	t_i	d_i
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1 at time 0 and 2 at time 1 lateness 1, but
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It does not work.

Process first jobs with smaller $d_i - t_i$ time

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Minimize Lateness: selection criteria

Process jobs with short time first

i	t_i	d_i
1	1	6
2	5	5

1 at time 0 and 2 at time 1 lateness 1, but
2 at time 0 and 1 at time 5 has lateness 0.
It does not work.

Process first jobs with smaller $d_i - t_i$ time

i	t_i	d_i	$d_i - t_i$
1	1	2	1
2	10	10	0

2 should start at time 0, that
does not minimize lateness.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Process urgent jobs first

Sort in increasing order of d_i .

LatenessUrgent (A)

Sort A by increasing order of $A.d$

$S[0] = 0$; $t = A[0].t$;

$L = \max(0, t - A[0].d)$;

for $i = 1$ **to** $n - 1$ **do**

$S[i] = t$

$t = t + A[i].t$

$L = \max(L, \max(0, t - A[i].d))$

end for

return (S, L)

Definitions

Fractional
Knapsack

Some selection
criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity
selection

Minimizing lateness

Optimal prefix
codes

data compression

prefix codes

Huffman code

Approximation
algorithms

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Sort A by increasing order of $A.d$

$S[0] = 0$; $t = A[0].t$;

$L = \max(0, t - A[0].d)$;

for $i = 1$ **to** $n - 1$ **do**

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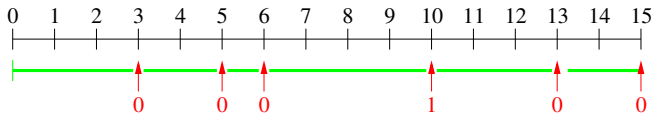
$t = t + A[i].t$

$L = \max(L, \max(0, t - A[i].d))$

end for

return (S, L)

i	t	d	pos. sorted by d
1	1	9	3
2	2	8	2
3	2	15	6
4	3	6	1
5	3	14	5
6	4	9	4



Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Process urgent jobs first: Complexity

LatenessUrgent (A)

Sort A by increasing order of $A.d$

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for $i = 1$ **to** $n - 1$ **do**

$S[i] = t$

$t = t + A[i].t$

$L = \max(L, \max(0, t - A[i].d))$

end for

return (S, L)

Time complexity

Running-time of the algorithm without sorting $O(n)$

Total running-time: $O(n \lg n)$

Definitions

Fractional
Knapsack

Some selection
criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity
selection

Minimizing lateness

Optimal prefix
codes

data compression

prefix codes

Huffman code

Approximation
algorithms

Process urgent jobs first: Correctness

Lemma

There is an optimal schedule minimizing lateness that does not have idle steps.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

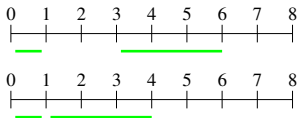
Approximation algorithms

Process urgent jobs first: Correctness

Lemma

There is an optimal schedule minimizing lateness that does not have idle steps.

From a schedule with idle steps, we always can eliminate gaps to obtain another schedule with the same or better lateness:



Definitions

Fractional
Knapsack

Some selection
criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity
selection

Minimizing lateness

Optimal prefix
codes

data compression

prefix codes

Huffman code

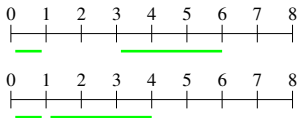
Approximation
algorithms

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LatenessUrgent has no idle steps.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Inversions and the exchange argument

A schedule S has **an inversion** if $S(i) < S(j)$ and $d_j < d_i$.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- If there is an inversion, there must be one in consecutive positions.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

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- If there is an inversion, there must be one in consecutive positions.
- Exchanging two adjacent inverted jobs reduces the number of inversions by 1

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- If there is an inversion, there must be one in consecutive positions.
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- If we can show that the change does not increase the max lateness. We can apply the exchanging process as many times as needed until we get the jobs scheduled according to our criteria without increasing the max lateness.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

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- Exchanging two adjacent inverted jobs reduces the number of inversions by 1
- If we can show that the change does not increase the max lateness. We can apply the exchanging process as many times as needed until we get the jobs scheduled according to our criteria without increasing the max lateness.
- The above will show that our ordering provides an optimal solution

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Inversions and exchange argument

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Exchanging two adjacent inverted jobs reduces the number of inversions by 1 and does not increase the max lateness.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Inversions and exchange argument

Lemma

Exchanging two adjacent inverted jobs reduces the number of inversions by 1 and does not increase the max lateness.

Proof.

Assume that in schedule S , i is scheduled just before j and that they form an inversion.

Let S' be the schedule obtained from S interchanging i with j .

- $S[k] = S'[k]$ for $k \neq i$ and $k \neq j$.
- Thus, only i and j can change lateness.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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Let S' be the schedule obtained from S interchanging i with j .

- $S[k] = S'[k]$ for $k \neq i$ and $k \neq j$.
- Thus, only i and j can change lateness.
- Job j is scheduled earlier in S' than in S , so its lateness cannot increase.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Inversions and exchange argument: Cont.

- Let L_i, L_j and L'_i, L'_j be the lateness of jobs i and j in S and S' , respectively. Recall $d_j < d_i$.
- Let f_i, f_j and f'_i, f'_j be the finish times of jobs i and j in S and S' , respectively.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- We have $f_i < f_j$, $f'_i < f'_j$, $f'_i = f_j$, and $f'_j < f_j$.

Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection
Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

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- We have $f_i < f_j, f'_i < f'_j, f'_i = f_j$, and $f'_j < f_j$.
- Furthermore, $L'_j \leq L_j$.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- Furthermore, $L'_j \leq L_j$.
- If $f_j \leq d_i$,

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

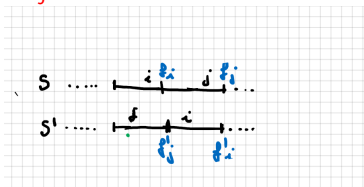
prefix codes

Huffman code

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- Let f_i, f_j and f'_i, f'_j be the finish times of jobs i and j in S and S' , respectively.
- We have $f_i < f_j, f'_i < f'_j, f'_i = f_j$, and $f'_j < f_j$.
- Furthermore, $L'_j \leq L_j$.
- If $f_j \leq d_i$,



$$L'_j = L_i = 0$$

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

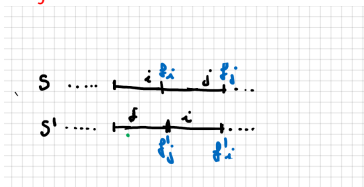
Optimal prefix codes

- data compression
- prefix codes
- Huffman code

Approximation algorithms

Inversions and exchange argument: Cont.

- Let L_i, L_j and L'_i, L'_j be the lateness of jobs i and j in S and S' , respectively. Recall $d_j < d_i$.
- Let f_i, f_j and f'_i, f'_j be the finish times of jobs i and j in S and S' , respectively.
- We have $f_i < f_j, f'_j < f'_i, f'_i = f_j$, and $f'_j < f_j$.
- Furthermore, $L'_j \leq L_j$.
- If $f_j \leq d_i$,



$$L'_i = L_i = 0$$

Both schedules have the same latency.

Inversions and exchange argument: Cont.

- Let L_i, L_j and L'_i, L'_j be the lateness of jobs i and j in S and S' , respectively. Recall $d_j < d_i$.
- Let f_i, f_j and f'_i, f'_j be the finish times of jobs i and j in S and S' , respectively.
- We have $f_i < f_j, f'_i < f'_j, f'_i = f_j$, and $f'_j < f_j$.
- If $d_i < f_j$,

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

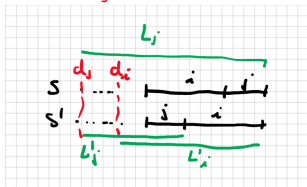
prefix codes

Huffman code

Approximation algorithms

Inversions and exchange argument: Cont.

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- We have $f_i < f_j, f'_j < f'_i, f'_i = f_j$, and $f'_j < f_j$.
- If $d_i < f_j$,



$$L'_i \leq L_j$$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

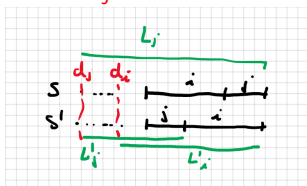
prefix codes

Huffman code

Approximation algorithms

Inversions and exchange argument: Cont.

- Let L_i, L_j and L'_i, L'_j be the lateness of jobs i and j in S and S' , respectively. Recall $d_j < d_i$.
- Let f_i, f_j and f'_i, f'_j be the finish times of jobs i and j in S and S' , respectively.
- We have $f_i < f_j, f'_j < f'_i, f'_i = f_j$, and $f'_j < f_j$.
- If $d_i < f_j$,



$$L'_i \leq L_j$$

So, S' has the same or better lateness than S .

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Inversions and exchange argument: Cont.

Therefore, in both cases, the swapping does not increase the maximum lateness of the schedule.

End Proof

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Correctness of LatenessUrgent

Theorem

Algorithm LatenessUrgent solves correctly the LATENESS MINIMIZATION problem. in $O(n \log n)$ time

Proof.

According to the design, the schedule S produced by **LatenessUrgent** has no inversions and no idle steps.

Assume \hat{S} is an optimal schedule. We can assume that it has no idle steps.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Correctness of Lateness Urgent

- If \hat{S} has 0 inversions, S sorts jobs by deadlines and $\hat{S} = S$.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Correctness of Lateness Urgent

- If \hat{S} has 0 inversions, S sorts jobs by deadlines and $\hat{S} = S$.
- Otherwise, \hat{S} has an inversion on two adjacent jobs.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Correctness of LatenessUrgent

- If \hat{S} has 0 inversions, S sorts jobs by deadlines and $\hat{S} = S$.
- Otherwise, \hat{S} has an inversion on two adjacent jobs.
Let i, j be an adjacent inversion.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- If \hat{S} has 0 inversions, S sorts jobs by deadlines and $\hat{S} = S$.
- Otherwise, \hat{S} has an inversion on two adjacent jobs.
Let i, j be an adjacent inversion.

As we have seen, exchanging i and j does not increase lateness but it decreases the number of inversions.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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Let i, j be an adjacent inversion.

As we have seen, exchanging i and j does not increase lateness but it decreases the number of inversions.

As \hat{S} is optimal, the new schedule is also optimal but has one inversion less.

Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

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- If \hat{S} has 0 inversions, S sorts jobs by deadlines and $\hat{S} = S$.
- Otherwise, \hat{S} has an inversion on two adjacent jobs.
Let i, j be an adjacent inversion.
As we have seen, exchanging i and j does not increase lateness but it decreases the number of inversions.
As \hat{S} is optimal, the new schedule is also optimal but has one inversion less.
- Repeating, if needed the interchange of adjacent inversions, we will reach an optimal schedule with no inversions. Therefore, S is optimal.

End Proof

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

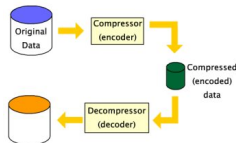
Optimal prefix codes

- data compression
- prefix codes
- Huffman code

Approximation algorithms

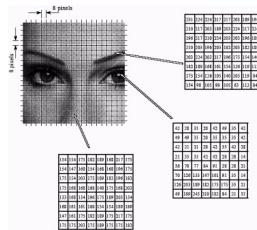
DATA COMPRESSION

Given as input a text \mathcal{T} over a finite alphabet Σ . We want to represent \mathcal{T} with as few bits as possible.



The goal of data compression is to reduce the time to transmit large files, and to reduce the space to store them.

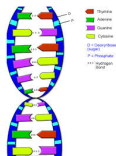
If we are using variable-length encoding we need a system easy to encode and decode.



- Definitions
- Fractional Knapsack
 - Some selection criteria
 - Highest v/w
 - 0-1 Knapsack
- Scheduling
 - Interval scheduling
 - Weighted activity selection
 - Minimizing lateness
- Optimal prefix codes
 - data compression
 - prefix codes
 - Huffman code
- Approximation algorithms

Example.

AAACAGTTGCAT . . . GGTCCCTAGG
130.000.000



- *Fixed-length encoding*: $A = 00$, $C = 01$, $G = 10$ and $T = 11$. Needs 260Mbytes to store.
- *Variable-length encoding*: If A appears 7×10^8 times, C appears 3×10^6 times, G 2×10^8 and T 37×10^7 , better to assign a shorter string to A and longer to C

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Prefix codes

Given a set of symbols Σ , a **prefix code**, is $\phi : \Sigma \rightarrow \{0, 1\}^+$ (symbols to chain of bits) where for distinct $x, y \in \Sigma$, $\phi(x)$ is not a prefix of $\phi(y)$.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- $\phi(A) = 1$ and $\phi(C) = 101$ then ϕ is **not** a prefix code.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- $\phi(A) = 1, \phi(T) = 01, \phi(G) = 000, \phi(C) = 001$ is a prefix code.

Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection
Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

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- $\phi(A) = 1, \phi(T) = 01, \phi(G) = 000, \phi(C) = 001$ is a prefix code.
- Prefix codes easy to decode (left-to-right):

$\underbrace{000}_G \underbrace{1}_A \underbrace{01}_T \underbrace{1}_A \underbrace{001}_C \underbrace{1}_A \underbrace{01}_T \underbrace{000}_G \underbrace{001}_C \underbrace{01}_T$

Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection
Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

Prefix tree

We can identify an encoding with prefix property with a labeled binary tree.

A **prefix tree** T is a binary tree with the following properties:

- One leaf for symbol,
- Left edge labeled 0 and right edge labeled 1,
- Labels on the path from the root to a leaf specify the code for the symbol in that leaf.

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

Optimal prefix codes

- data compression
- prefix codes**
- Huffman code

Approximation algorithms

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Σ	code
A	1
T	01
G	000
C	001

Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection
Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

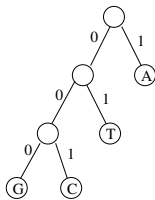
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Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection
Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

The encoding length

- Given a text S on Σ , with $|S| = n$, and a prefix code ϕ , $B(S)$ is the length of the encoded text.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

The encoding length

- Given a text S on Σ , with $|S| = n$, and a prefix code ϕ , $B(S)$ is the length of the encoded text.
- For $x \in \Sigma$, define the **frequency** of x as

$$f(x) = \frac{\text{number occurrences of } x \in S}{n}$$

Note: $\sum_{x \in \Sigma} f(x) = 1.$

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

Optimal prefix codes

- data compression
- prefix codes**
- Huffman code

Approximation algorithms

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- We get the formula,

$$B(S) = \sum_{x \in \Sigma} n f(x) |\phi(x)| = n \sum_{x \in \Sigma} f(x) |\phi(x)|.$$

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

Optimal prefix codes

- data compression
- prefix codes
- Huffman code

Approximation algorithms

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$$B(S) = \sum_{x \in \Sigma} n f(x) |\phi(x)| = n \sum_{x \in \Sigma} f(x) |\phi(x)|.$$

- $\alpha(S) = \sum_{x \in \Sigma} f(x) |\phi(x)|$ is the **average number of bits per symbol** or **compression factor**.

Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection
Minimizing lateness

Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

The encoding length

- In terms of the prefix tree of ϕ , the length of a codeword $|\phi(x)|$ is the depth of the leaf labeled x in T ($d_T(x)$).
- Thus, $\alpha(T) = \sum_{x \in \Sigma} f(x)d_T(x)$.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

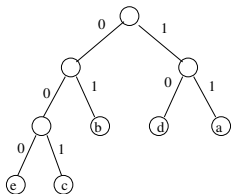
prefix codes

Huffman code

Approximation algorithms

Fixed versus variable length codes: Example.

- Let $\Sigma = \{a, b, c, d, e\}$ and let S be a text over Σ with frequencies:
 $f(a) = .32, f(b) = .25, f(c) = .20, f(d) = .18, f(e) = .05$
- If we use a fixed length ϕ code, we need $\lceil \lg 5 \rceil = 3$ bits, we get compression 3.
- Consider the prefix-code ϕ_1 :



$$\alpha = .32 \cdot 2 + .25 \cdot 2 + .20 \cdot 3 + .18 \cdot 2 + .05 \cdot 3 = 2.25$$

- In average, ϕ_1 reduces the bits per symbol over the fixed-length code from 3 to 2.25, about 25%

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Fixed versus variable length codes: Example.

Is 2.25 the maximum compression?

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

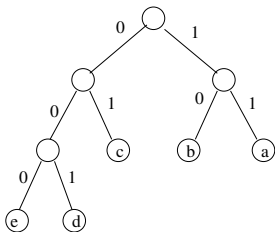
prefix codes

Huffman code

Approximation algorithms

Fixed versus variable length codes: Example.

Is 2.25 the maximum compression? Consider the prefix-code ϕ_2 :



$$\alpha = .32 \cdot 2 + .25 \cdot 2 + .20 \cdot 2 + .18 \cdot 3 + .05 \cdot 3 = 2.23$$

is that the best? (the maximum compression using a prefix code)

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Optimal prefix code.

Given a text, an **optimal prefix code** is a prefix code that minimizes the total number of bits needed to encode the text, i.e., α .

Intuitively, in the prefix tree of an optimal prefix code, symbols with high frequencies should have small depth and symbols with low frequency should have large depth.

Before describing the algorithm we analyze some properties of optimal prefix trees.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

A property of optimal prefix trees.

A binary tree T is **full** if every interior node has two sons.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

A property of optimal prefix trees.

A binary tree T is **full** if every interior node has two sons.

Lemma

The prefix tree describing an optimal prefix code is full.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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The prefix tree describing an optimal prefix code is full.

Proof.

- Let T be the prefix tree of an optimal code, and suppose it contains a u with a unique son v .

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

Optimal prefix codes

- data compression
- prefix codes**
- Huffman code

Approximation algorithms

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Lemma

The prefix tree describing an optimal prefix code is full.

Proof.

- Let T be the prefix tree of an optimal code, and suppose it contains a u with a unique son v .
- If u is the root, construct T' by deleting u and using v as root. Otherwise, let w be the father of u . Construct T' by deleting u and connecting directly v to w .
- In both cases T' is a prefix tree and all the leaves in the subtree rooted at v reduce its height by 1 in T' .
- T' yields a code with less bits, so T is not optimal.

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

Optimal prefix codes

- data compression
- prefix codes
- Huffman code

Approximation algorithms



Greedy approach: Huffman code

Greedy approach due to David Huffman (1925-99) in 1952, while he was a PhD student at MIT



Wish to produce a labeled binary full tree, in which the leaves are as close to the root as possible. Moreover symbols with low frequency will be placed deeper than the symbol with high frequency.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Greedy approach: Huffman code

- Given the frequencies $f(x)$ for every $x \in \Sigma$
- The algorithm keeps a dynamic sorted list in a priority queue Q .
- Construct a tree in bottom-up fashion
 - Insert symbols as *leaves* with key f .
 - Extract the two first elements of Q and join them by a new *virtual node* with key the sum of the f 's of its children. Insert the new node in Q .
- When Q has size 1, the resulting tree will be the prefix tree of an optimal prefix code.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Huffman Coding: Construction of the tree.

Huffman Σ, S

Given Σ and S {compute the frequencies $\{f\}$ }

Construct priority queue Q of leaves for Σ , ordered by increasing f

```
while  $Q.size() > 1$  do  
    create a new node  $z$   
     $x = \text{Extract-Min}(Q)$   
     $y = \text{Extract-Min}(Q)$   
    make  $x, y$  the sons of  $z$   
     $f(z) = f(x) + f(y)$   
    Insert  $(Q, z, f(z))$   
end while  
 $\phi = \text{Extract-Min}(Q)$ 
```

If Q is implemented with a Heap, takes time $O(n \lg n)$.

Definitions

Fractional
Knapsack

Some selection
criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity
selection

Minimizing lateness

Optimal prefix
codes

data compression

prefix codes

Huffman code

Approximation
algorithms

Example

Consider the text: *for each rose, a rose is a rose, the rose*
with $\Sigma = \{\text{for/ each/ rose/ a/ is/ the/ ./ b}\}$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Example

Consider the text: *for each rose, a rose is a rose, the rose*
with $\Sigma = \{\text{for/ each/ rose/ a/ is/ the/ ./ b}\}$

Frequencies:

$$\begin{aligned}f(\text{for}) &= 1/21, & f(\text{rose}) &= 4/21, & f(\text{is}) &= 1/21, \\f(\text{a}) &= 2/21, & f(\text{each}) &= 1/21, & f(\text{.}) &= 2/21, \\f(\text{the}) &= 1/21, & f(\text{b}) &= 9/21.\end{aligned}$$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Example

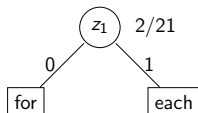
Consider the text: *for each rose, a rose is a rose, the rose*
with $\Sigma = \{ \text{for/ each/ rose/ a/ is/ the/ ,/ b} \}$

Frequencies:

$$f(\text{for}) = 1/21, f(\text{rose}) = 4/21, f(\text{is}) = 1/21, \\ f(\text{a}) = 2/21, f(\text{each}) = 1/21, f(,) = 2/21, \\ f(\text{the}) = 1/21, f(\text{b}) = 9/21.$$

Priority Queue:

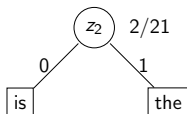
$$Q = ((\text{for}:1/21), (\text{each}:1/21), (\text{is}:1/21), (\text{the}:1/21), (\text{a}:2/21), (, :2/21), , \\ (\text{rose}:4/21), (\text{b}: 9/21))$$



Then, $Q = ((\text{is}:1/21), (\text{the}:1/21), (\text{a}:2/21), (, :2/21), (z_1:2/21), (\text{rose}:4/21), \\ (\text{b}:9/21))$

Example.

$Q = ((\text{is}:1/21), (\text{the}:1/21), (\text{a}:2/21), (, :2/21), (z_1:2/21), (\text{rose}:4/21), (\text{b}:9/21))$



Then, $Q = ((\text{a}:2/21), (, :2/21), (z_1:2/21), (z_2:2/21), (\text{rose}:4/21), (\text{b}:9/21))$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

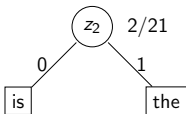
prefix codes

Huffman code

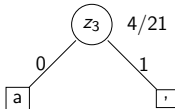
Approximation algorithms

Example.

$Q = ((\text{is}:1/21), (\text{the}:1/21), (\text{a}:2/21), (, :2/21), (z_1:2/21), (\text{rose}:4/21), (\text{b}:9/21))$



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Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

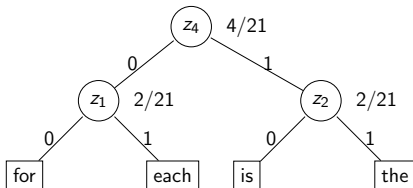
prefix codes

Huffman code

Approximation algorithms

Example.

$Q = ((z_1:2/21), (z_2:2/21), (rose:4/21), (z_3:4/21), (b:9/21))$



Then, $Q = ((rose:4/21), (z_3:4/21), (z_4:4/21), (b:9/21))$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

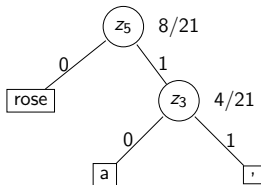
prefix codes

Huffman code

Approximation algorithms

Example.

$$Q = ((\text{rose}:4/21), (z_3:4/21), (z_4:4/21), (b:9/21))$$



$$\text{Then, } Q = ((z_4:4/21), (z_5:8/21), (b:9/21))$$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

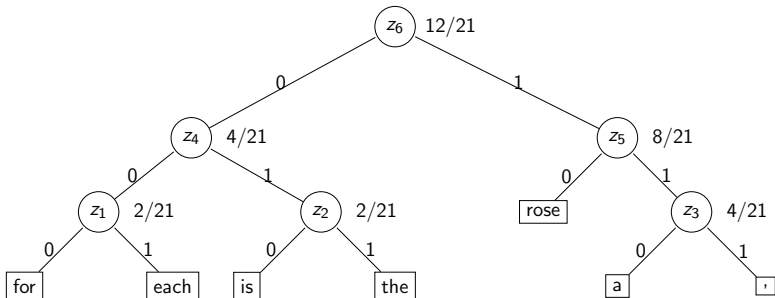
prefix codes

Huffman code

Approximation algorithms

Example.

$$Q = ((z_4:4/21), (z_5:8/21), (b:9/21))$$



$$\text{Then, } Q = ((b:9/21), (z_6:12/21))$$

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

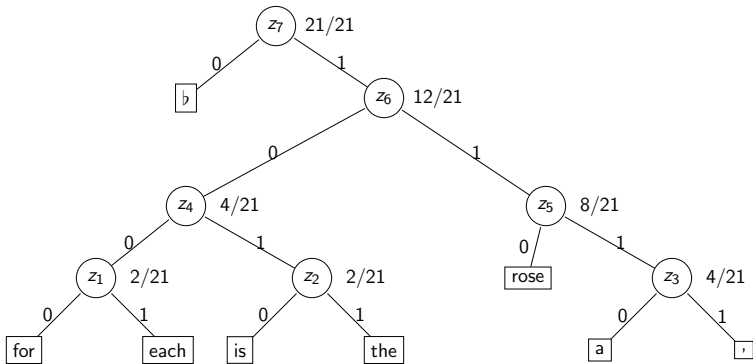
Optimal prefix codes

- data compression
- prefix codes
- Huffman code

Approximation algorithms

Example.

$$Q = ((b:9/21), (z_6:12/21))$$



$$\text{Then, } Q = ((z_7:21/21))$$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Example

- Therefore *for each rose, a rose is a rose, the rose* is Huffman coded as

100001001011101111110011001010011100110111110110110

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- Therefore *for each rose, a rose is a rose, the rose* is Huffman coded as

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- The solution is not unique!

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Example

- Therefore *for each rose, a rose is a rose, the rose* is Huffman coded as

100001001011101111110011001010011100110111110110110

- The solution is not unique!
- The encoded length is 51, and compression is $51/21 = 2.428\dots$

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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- With a fixed size code, we need 4 bits per symbol, length 84 bits instead of 51.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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10000100101101111110011001010011100110111110110110

- The solution is not unique!
- The encoded length is 51, and compression is $51/21 = 2.428\dots$
- With a fixed size code, we need 4 bits per symbol, length 84 bits instead of 51.
- Why does the Huffman's algorithm produce an optimal prefix code?

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Correctness

Theorem (Greedy property)

Let Σ be an alphabet, and let x, y be two symbols with the lowest frequency. There is an optimal prefix code ϕ in which $|\phi(x)| = |\phi(y)|$ and both codes differ only in the last bit.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

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Proof.

Assume that T is optimal but that x and y have not the same code length. In T there must be two symbols a and b siblings at max. depth. Assume $f(a) \leq f(b)$ and $f(x) \leq f(y)$, otherwise sort them accordingly.

We construct T' by exchanging x with a and y with b . As $f(x) \leq f(a)$ and $f(y) \leq f(b)$ then $B(T') \leq B(T)$. So T' is optimal and verifies the property. □

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Correctness

Theorem (Optimal substructure)

Assume T' is an optimal prefix tree for $(\Sigma - \{x, y\}) \cup \{z\}$ where x, y are two symbols with the lowest frequencies, and z has frequency $f(x) + f(y)$. The T obtained from T' by making x and y children of z is an optimal prefix tree for Σ .

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Correctness

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Proof.

Let T_0 be any prefix tree for Σ . We must show $B(T) \leq B(T_0)$.

By the previous result, we only need to consider T_0 where x and y are siblings, their parent has frequency $f(x) + f(y)$.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Correctness

- Let T'_0 be obtained by removing x, y from T_0 . As T'_0 is a prefix tree for $(\Sigma - \{x, y\}) \cup \{z\}$, then $B(T'_0) \geq B(T')$.
- Comparing T_0 with T'_0 we get,

$$B(T_0) = B(T'_0) + f(x) + f(y),$$

$$B(T) = B(T') + f(x) + f(y) = B(T).$$

- Putting together the three identities, we get $B(T) \leq B(T_0)$.

End Proof

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

More on Huffman codes

Huffman is optimal under assumptions:

- The compression is **lossless**, i.e. *uncompressing the compressed file yield the original file.*
- We must know the alphabet beforehand (characters, words, etc.),
- We must pre-compute the frequencies of symbols, i.e. read the data twice, **which make it very slow for many real applications.**
- A good source for extensions of Huffman encoding compression is the Wikipedia article on it:
[https://en.wikipedia.org/wiki/Huffman_coding.](https://en.wikipedia.org/wiki/Huffman_coding)

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Approximation algorithms

- Many times the Greedy strategy yields a **feasible solution** with value which is **near** to the optimum solution.
- In many practical cases, when finding the global optimum is hard, the greedy may yield a *good enough* feasible solution: **An approximation to the optimal solution.**
- An **approximation algorithm** for the problem always computes a close valid output. Heuristics also could yield good solutions, but they do not have a theoretical guarantee of closeness.
- Greedy is one of the algorithmic techniques used to design approximations algorithms.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

Greedy and approximation algorithms

- For any optimization problem, let $c(*)$ be the value of the optimization function, let $\mathcal{A}px$ be an algorithm, that for each input x produces a valid solution $\mathcal{A}px(x)$ to x . Let $\text{opt}(x)$ be the cost of an optimal solution to x .
- We want to design a **fast** algorithm that produce solutions **close** to the optimal.
- For a NP-hard scheduling problem, we don't know if it has polynomial time algorithms, we want to design algorithms that are fast (polynomial) and that outputs **good** solutions always.

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

Optimal prefix codes

- data compression
- prefix codes
- Huffman code

Approximation algorithms

Approximation algorithm: Formal definition

- For a given optimization problem, let $\mathcal{A}p_x$ be an algorithm, that for each input x produces a valid solution with cost $\mathcal{A}p_x(x)$ to x . Let $\text{opt}(x)$ be the cost of an optimal solution to x .
- For $r > 1$, $\mathcal{A}p_x$ is an **r -approximation algorithm** if, for any input x :

$$\frac{1}{r} \leq \frac{\mathcal{A}p_x(x)}{\text{opt}(x)} \leq r.$$

- r is called **the approximation ratio**.
- Given an optimization problem, for any input x , we require
 - in a MAX problem, $\mathcal{A}p_x(x) \leq \text{opt}(x) \leq r\mathcal{A}p_x(x)$.
 - in a MIN problem, $\text{opt}(x) \leq \mathcal{A}p_x(x) \leq r\text{opt}(x)$.

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

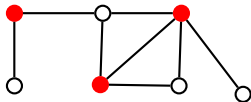
Optimal prefix codes

- data compression
- prefix codes
- Huffman code

Approximation algorithms

An easy example: VERTEX COVER problem

Recall the problem of Vertex cover: Given a graph $G = (V, E)$ with $|V| = n, |E| = m$ find the minimum set of vertices $S \subseteq V$ such that it covers every edge of G .



Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

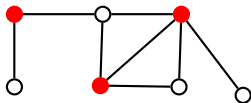
Optimal prefix codes

- data compression
- prefix codes
- Huffman code

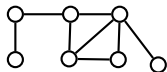
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GreedyVC for $I: G = (V, E)$
 $E' = E, S = \emptyset,$
while $E' \neq \emptyset$ **do**
 Pick $e \in E',$ say $e = (u, v)$
 $S = S \cup \{u, v\},$
 $E' = E' - \{(u, v) \cup \{\text{edges incident to } u, v\}\}$
end while
return $S.$



Definitions

Fractional Knapsack

Some selection criteria
Highest v/w
0-1 Knapsack

Scheduling

Interval scheduling
Weighted activity selection
Minimizing lateness

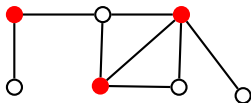
Optimal prefix codes

data compression
prefix codes
Huffman code

Approximation algorithms

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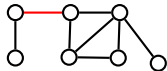
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Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

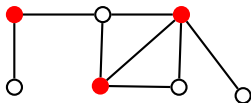
prefix codes

Huffman code

Approximation algorithms

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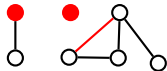
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Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

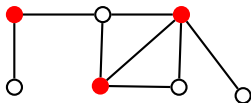
prefix codes

Huffman code

Approximation algorithms

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Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

An easy example: Vertex cover

Theorem

GreedyVC runs in $O(m + n)$ steps. Moreover, if S is solution computed on input G , $|S| \leq 2\text{opt}(G)$.

Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

An easy example: Vertex cover

Theorem

GreedyVC runs in $O(m + n)$ steps. Moreover, if S is solution computed on input G , $|S| \leq 2\text{opt}(G)$.

Proof.

- The edges selected among by **GreedyVC** do not share any vertex.
- Therefore, an optimal solution must have at least one of the two endpoints of each edge while **GreedyVC** takes both.
- So, $|S| \leq 2\text{opt}(G)$.



Definitions

Fractional Knapsack

Some selection criteria

Highest v/w

0-1 Knapsack

Scheduling

Interval scheduling

Weighted activity selection

Minimizing lateness

Optimal prefix codes

data compression

prefix codes

Huffman code

Approximation algorithms

An easy example: Vertex cover

- The decision problem for Vertex Cover: given G and k , does G have a vertex cover with k or less vertices?, is NP-complete.
- Moreover, unless $P=NP$, vertex cover can not be approximated within a factor $r \leq 1.36$
- No approximate algorithm with $r < 2$ is known.

Definitions

Fractional Knapsack

- Some selection criteria
- Highest v/w
- 0-1 Knapsack

Scheduling

- Interval scheduling
- Weighted activity selection
- Minimizing lateness

Optimal prefix codes

- data compression
- prefix codes
- Huffman code

Approximation algorithms