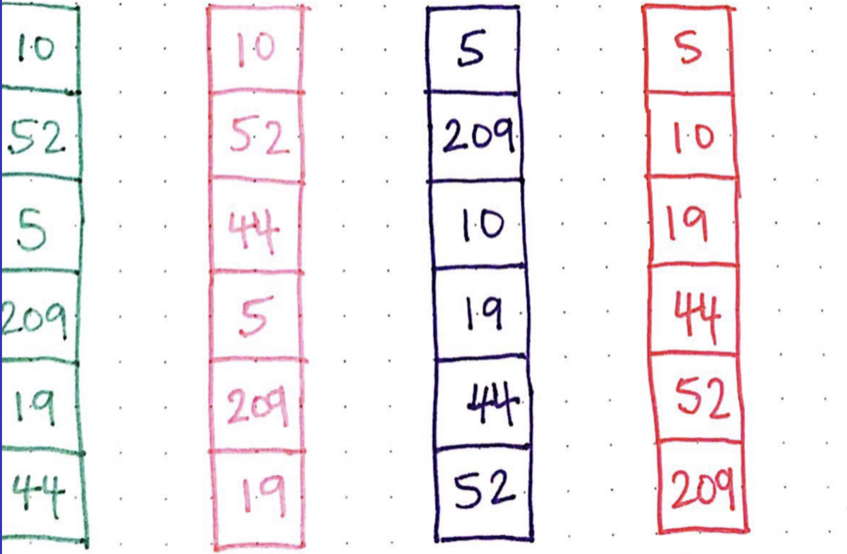
Fast Sorting Algorithms



Sorting algorithms on values in a known range

CLRS Ch.8

- Counting sort
- Radix sort
- Bucket sort
- Lower bounds for general sorting

- The algorithms will sort an array $A[n]$ of non-negative integers in the range $[0, r]$.

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- The algorithms will sort an array $A[n]$ of non-negative integers in the range $[0, r]$.
- The complexity of the algorithms depends on both n and r.
- For some values of r, the algorithms have cost $O(n)$ or $o(n \log n)$.

Counting sort

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- The input $A[n]$, is an array of integers in the range $[0, r]$.
- Uses: $B[n]$ (output) and $C[r + 1]$ (internal).

Counting sort: Algorithm

CountingSort (A, r)

for $i = 0$ to r **do**
  $C[i] = 0$
for $i = 0$ to $n-1$ **do**
  $C[A[i]] = C[A[i]] + 1$ $\{C[j] = |\{i \mid A[i] = j\}|\}$
for $i = 1$ to r **do**
  $C[i] = C[i] + C[i-1]$ $\{C[j] = |\{i \mid A[i] \leq j\}|\}$
for $i = n-1$ downto 0 **do**
  $B[C[A[i]]] = A[i]$;
  $C[A[i]] = C[A[i]] - 1$ $\{C$ holds the sorted elements$\}$

Counting sort: Cost

CountingSort (A, r)
for $i = 0$ to r **do**
$C[i] = 0$ $\{O(r)\}$
for $i = 0$ to $n - 1$ **do**
$C[A[i]] = C[A[i]] + 1$ $\{O(n)\}$
for $i = 0$ to r **do**
do $C[i] = C[i] + C[i - 1]$ $\{O(r)\}$
for $i = n - 1$ downto 0 **do**
$B[C[A[i]] - 1] = A[i];$
$C[A[i]] = C[A[i]] - 1$ $\{O(n)\}$

$T(n) = O(n + r),$

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$T(n) = O(n + r)$, for $r = O(n)$, $T(n) = O(n)$.

Counting sort: stability

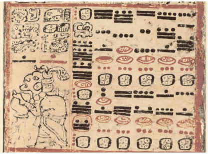
An important property of counting sort is that it is stable: numbers with the same value appear in the output in the same order as they do in the input.

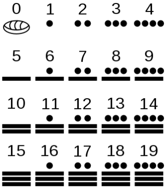
Radix sort: What does radix mean?

Radix means the base in which we express an integer

Radix 10=Decimal; Radix 2= Binary; Radix 16=Hexadecimal;

Radix 20 (The Maya numerical system)





Binary	Hex	Decimal
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	8
1001	9	9
1010	A	10
1011	B	11
1100	C	12
1101	D	13
1110	E	14
1111	F	15

Radix Change: Example

- To convert an integer from binary to decimal:
  $1011 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 11$
- To convert an integer from decimal to binary:
  Repeatedly dividing the enter by 2, will give a result plus a remainder:
  $19 \Rightarrow \underbrace{19/2}_{1}\ \underbrace{9/2}_{1}\ \underbrace{4/2}_{0}\ \underbrace{2/2}_{01} = 10011$
- To transform an integer radix 16 to decimal:
  $(4CF5)_{16} = (4 \times 16^3 + 12 \times 16^2 + 15 \times 16^1 + 5 \times 16^0) = 19701$

- To convert $(4CF5)_{16}$ into binary you have to expand each digit to its binary representation.
  In the above example, $(4CF5)_{16}$ in binary is
  0011110011110101
- To convert an integer in binary to radix 16:
  Make groups of 4 from left to right and replace by the corresponding digit
  1101010010100010000101101111110100 in HEX is
  1A9442DF4

RADIX LSD algorithm

Given an array A with n numbers, each one with d digits in base b the Radix Least Significant Digit, algorithm is

RADIX LSD (A, d, b)
for $i = 1$ to d **do**
  Use a stable sorting algorithm to sort A according to the i-th digit values.

The values to sort are in the range $[0, b^d)$.

Example: $b = 10$ and $d = 3$

329
475
657
839
436
720
355

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329		720
475		475
657		355
839	$\Rightarrow$	436
436		657
720		329
355		839

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329		720		720
475		475		329
657		355		436
839	$\Rightarrow$	436	$\Rightarrow$	839
436		657		355
720		329		657
355		839		475

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329		720		720		329
475		475		329		355
657		355		436		436
839	$\Rightarrow$	436	$\Rightarrow$	839	$\Rightarrow$	475
436		657		355		657
720		329		657		720
355		839		475		839

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Theorem

RADIX LSD sorts correctly the n given numbers.

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IH: Assume that it is true for $d - 1$ digits.
Looking at the the d-th digit, we have

- if $a_d < b_d$, $a < b$ and the algorithm places a before b,
- if $a_d = b_d$, as we are using a stable sorting, a and b remain in the same order as in the previous step. By IH, they are already the correct one.

□

Time complexity

Given n numbers, each number with at most d digits, and each digit in the range 0 to b, if we use counting sorting at each round of RADIX LSD:

$$T(n, d, b) = \Theta(d(n + b)).$$

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- Consider that each number has a value up to $f(n)$.
- Then the number of digits in base b is $d \leq \lceil \log_b f(n) \rceil$, so $T(n, b) = \Theta(\log_b f(n)(n + b))$.
- If $\log_b f(n) = \omega(1)$, $T(n) = \omega(n)$ and RADIX is not linear.

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- If $\log_b f(n) = \omega(1)$, $T(n) = \omega(n)$ and RADIX is not linear.
- Note that we could select a basis $b = b(n)$ such that $b(n) = O(n)$.

RADIX: selecting the base

Can we tune the parameters?

- Yes, in some cases, we can select the best radix to express the input values.
- For numbers in binary, we can select as new radix $\hat{b}$ a power of 2. This simplifies the computation as we have only to look to pieces of bits to change from one representation to anoter.
- For ex., if we have numbers of $d = 64$ bits ($b = 2$), and take the new radix to be $\hat{b} = 2^8$, we have $\hat{d} = 4$ new digits per number.

1 1 0 0 1 0 1 0 0 0 1 1 0 1 0 0 1 1 1 0 1 0 0 1 1 1 0 0 1 0 0 0

RADIX: selecting the base

Given n, $d(n)$-bits integers, we want to choose $c(n)$, $1 < c(n) < d(n)$ to use as new radix $\hat{b} = 2^{c(n)}$.

- In the new radix, the number of digits is $\hat{d}(n) = \lceil d(n)/c(n) \rceil$ digits,
- Running RADIX LSD with base $2^{c(n)}$ has cost

$$T(n) = \Theta(\hat{d}(n)(n + 2^{c(n)})) = \Theta((d(n)/c(n)(n + 2^{c(n)})).$$

- The highest choice for c is roughly $\lceil \lg n \rceil$.
  - Then, $2^c(n) = O(n)$.
  - So, the cost is, $O(\frac{d}{\lg n} n)$.
  - Which provides, linear cost if $\frac{d(n)}{\lg n} = O(1)$.

Bucket sort

- Suppose the values to sort are in the range $[0 \ldots m-1]$.
- The algorithm starts with an array of m empty buckets numbered 0 to $m-1$.
- Scan the list and place element $A[i]$ in bucket $A[i]$.
- Output the buckets in order.

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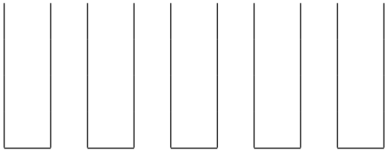
- It needs an array of buckets.
- The values in the list to be sorted are the indexes to the buckets.
- No comparisons are done.

4	2	1	2	0	4	1	2	2	3	4	1	0

4	2	1	2	0	4	1	2	2	3	4	1	0



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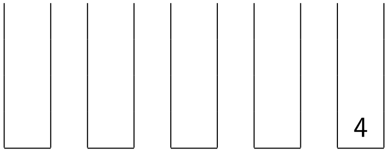

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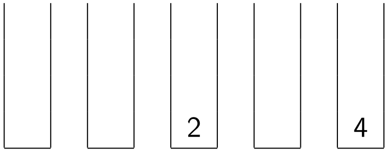

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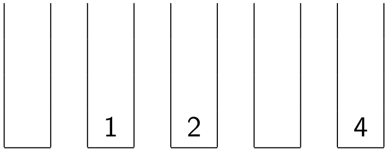

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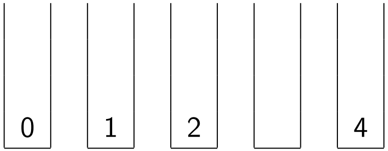

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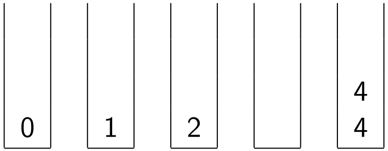

4	2	1	2	0	4	1	2	2	3	4	1	0

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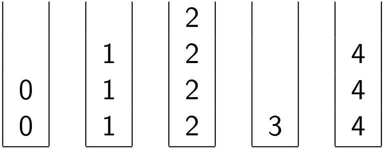

4	2	1	2	0	4	1	2	2	3	4	1	0

		2		
	1	2		4
0	1	2		4
0	1	2	3	4

0	0	1	1	1	2	2	2	2	3	4	4	4

Bucket sort: values or keys?

- When sorting values, each bucket can be just a counter.
- When sorting entries according to keys, a bucket is a queue.

Bucket sort: complexity

- Bucket initialization: $O(m)$
- From array to buckets: $O(n)$
- From buckets to array: $O(m + n)$
- Total cost is $O(n + m)$

When m is small compared to n, Bucket sort is $O(n)$

Bucket sort: extensions

- In the presented algorithm each bucket contains elements with the same key.
- The algorithm can be implemented in such a way that buckets hold elements with different keys.
- In such a case we have to take care of the additional cost of sorting the elements in each bucket.

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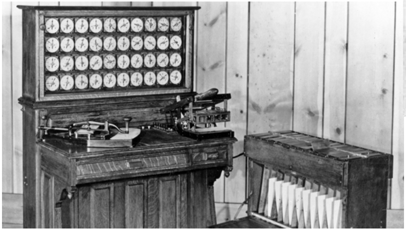
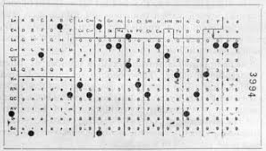
- The algorithm can be implemented in such a way that buckets hold elements with different keys.
- In such a case we have to take care of the additional cost of sorting the elements in each bucket.
- A typical implementation assumes that the input is drawn from a uniform distribution on $[0, 1)$, divides the range of values, from lowest to highest key value, into n equal sized ranks. In the worst case the algorithm has cost $O(n \lg n)$ and average cost $O(n)$.

A bit of history

LSD Radix and counting sort ideas are due to Herman Hollerith.
In 1890 he invented the card sorter that, for ex., allowed to process the US census in 5 weeks, using punching cards.





A bit of history

Counting/Radix sort
H. H Seward
Enhanced Generic Key-Address Mapping Sort Algorithm
MIT 1954.

Bucket sort
E. J. Isaac and R. C. Singleton
Sorting by Address Calculation
JACM 1956

Upper and lower bounds on time complexity of a problem.

- A problem has a time upper bound $T(n)$ if there is an algorithm $\mathcal{A}$ such that, for any input x of size n, $A(x)$ gives the correct answer in $\leq T(n)$ steps.

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- A problem has a time lower bound $L(n)$ if there is NO algorithm which solves the problem in time $< L(n)$, for any input e of size n.
- Lower bounds are hard to prove, as we have to consider every possible algorithm.

Upper and lower bounds on time complexity of a problem.

- Upper bound: $\exists \mathcal{A}, \forall x \; t_{\mathcal{A}}(x) \leq T(|x|)$,
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To prove an upper bound: produce an A so that the bound holds for any input x ($n = |x|$).

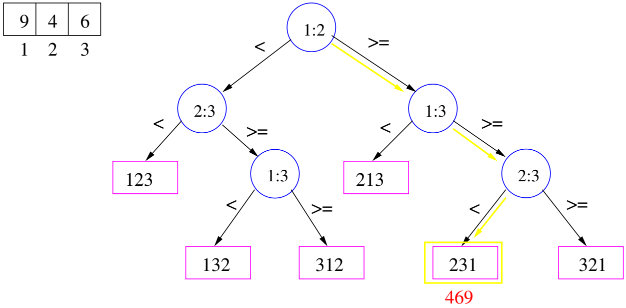
To prove a lower bound , show that for any possible algorithm, the time on one input is greater than or equal to the lower bound.

Lower bound for **comparison based** sorting algorithm.

To prove the lower bound, we consider binary decision trees a way to represent the comparisons made by a sorting algorithm to distinguish the possible inputs of size n.

- each leaf represents one of the $n!$ possible permutations $(a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(n)})$. The tree has exactly $n!$ leaves as the algorithm has to sort correctly all possible permutations.
- In a particular example, each internal node can be labeled by a comparison $a_i : a_j$, the leaves in the left subtree verify $a_i < a_j$ and the ones in the right subtree verify $a_i \geq a_j$.

An example of binary decision tree for $n = 3$



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Proof.

- Equivalent to prove: Any decision tree that sorts n elements must have height $\Omega(n \lg n)$.
- Let h the height of a decision tree with $n!$ leaves,

$$n! \leq 2^h \Rightarrow h \geq \lg(n!) > \lg(\frac{n}{e})^n = \Omega(n \lg n).$$

□