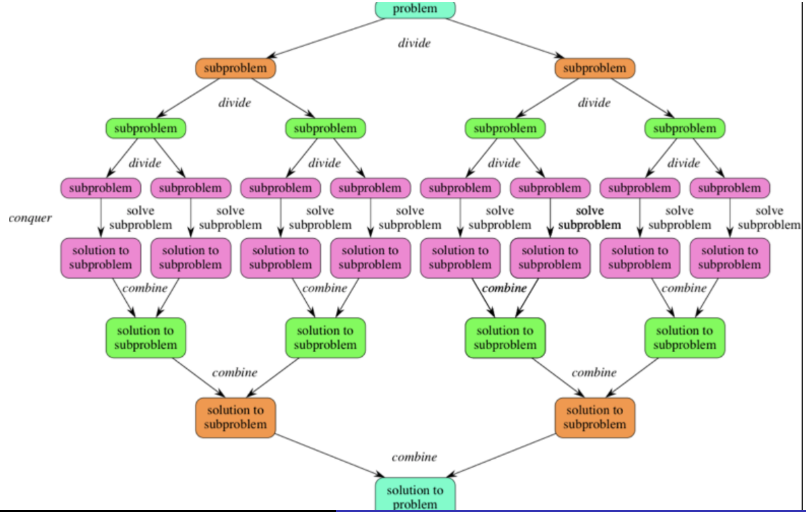
Divide-and-conquer: Selection



Selection

From 9.3 in CLRS

Selection Problem: Given an array A of n unordered distinct keys, and $i \in \{1, \dots, n\}$, select the ith-smallest element in A, that is the key that is larger than exactly $i-1$ other keys in A.

We use the term rank for the position that occupies an element after sorting A.

Notice that i can be any rank value, in particular when:

1. $i = 1$, the MINIMUM element
2. $i = n$, the MAXIMUM element
3. $i = \lfloor \frac{n+1}{2} \rfloor$, the MEDIAN
4. $i = \lfloor 0.25\, n \rfloor \Rightarrow$ *order statistics*

A first algorithm

Sort A in $(O(n \lg n))$ steps, then the i-th smallest key is $A[i]$.

Can we do it faster? in linear time?

Yes, selection is easier than sorting

The algorithm: High level

- Chose a split element x.
- Let k be the rank of x, if $k = i$, we found the i-th element. Otherwise,
- Use x to determine a partition of A, smaller than x to the left and larger to the right.
- Compute recursively the i-th element in the left part, when $i < k$, or the $i - k$-th element in the right part, when $i > k$.

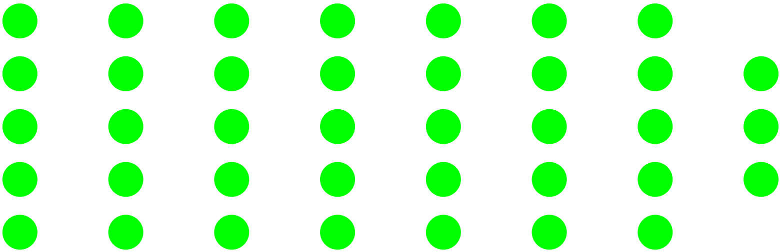
The algorithm is correct, independently of the rule used to determine x, as x's rank is correctly computed.

The time depends on the quality of the splitting element to divide fairly the elements

The problem
Algorithm idea
Computing a good split element
The algorithm
The cost

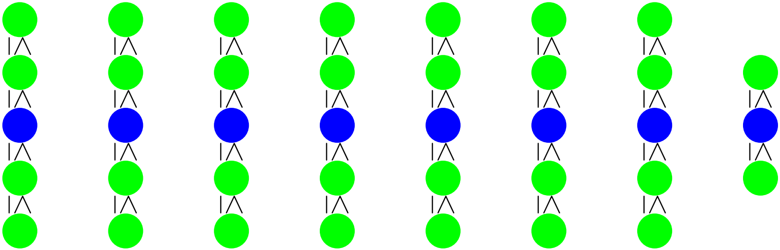
If $n \leq 5$ return their median.

Otherwise, divide the n elements in $\lceil n/5 \rceil$ groups, each with 5 elements except one group that might have < 5 elements).



Selection: Finding a splitting element

Sort the elements in each group and find its median.
(Each sort needs ≤ 25 comparisons, i.e. $\Theta(1)$).
Call x_j the median of the j-th group.



The splitting element x is the median of the set of medians,
$\{x_j \mid 1 \leq j \leq \lceil n/5 \rceil\}$.

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The algorithm

Select(A, i)
Divide A into $m = \lceil n/5 \rceil$ groups, all but at most one with 5 elements
$X[j]$ = median of group j, $j = 1, \ldots, m$
$x =$ **Select**$(X, \lfloor (m+1)/2 \rfloor)$ i.e. the median of X
Let k be the rank of x in A
if $i = k$ **then**
  return x
else
  L = the elements of A smaller than x (left)
  R = the elements of A bigger than x (right)
  if $i < k$ **then**
    return **Select**(L, i)
  else
    return **Select**$(R, i - k)$

Example: Find the median

Let $n = 15$, we want to get the 5-th element on the following input:

$$A = \quad 3\ 13\ 9\ 4\ 5 \quad 1\ 15\ 12\ 10\ 2 \quad 6\ 14\ 8\ 17\ 11$$

An example

Let $n = 15$, we want to get the 5-th element on the following input:

$$A = \boxed{3\ 13\ 9\ 4\ 5}\ \boxed{1\ 15\ 12\ 10\ 2}\ \boxed{6\ 14\ 8\ 17\ 11}$$

3	1	6
4	2	8
5	10	11
9	12	14
13	15	17

The median of $X = (5, 10, 11)$ is 10 which has rank 9
As $5 < 9$, recursively ask for the 5-th element in the left part with respect to $x = 10$, i.e., $(3, 9, 4, 5, 1, 2, 6, 8)$

Example: Find the median

The problem

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Computing a good split element

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The cost

In the next call $n = 8$, we look for the 5-th element in the following input:

$$A = \;\; 3\;9\;4\;5\;1\quad 2\;6\;8$$

An example

In the next call $n = 8$, we look for the 5-th element in the following input:

$$A = \boxed{3\ 9\ 4\ 5\ 1}\boxed{2\ 6\ 8}$$

1	
3	2
4	6
5	8
9	

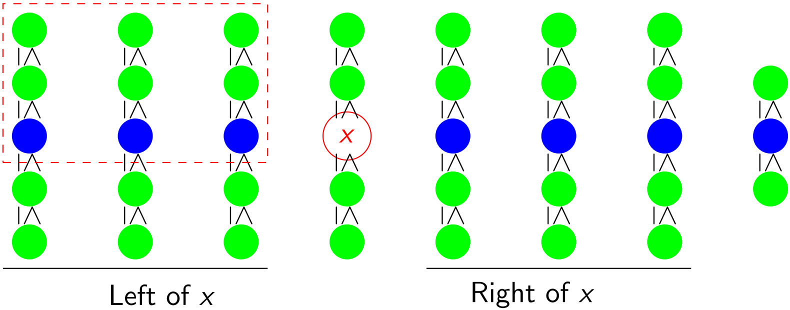
The median of $X = (4, 6)$ is 4 which has rank 4.
As $5 > 4$ the algorithm looks for the 1st element in the right part $(5, 6, 8, 9)$, which is 5.

Selection algorithm: Cost

Select(A, i)
Divide A into $m = \lceil n/5 \rceil$ groups, all but at most one with 5 elements $O(n)$
$X[j]$ = median of group j, $j = 1, \ldots, m$ $O(n)$
$x =$ **Select**$(X, \lfloor (m+1)/2 \rfloor)$ i.e. the median of X $T(n/5)$
Let k be the rank of x in A
if $i = k$ **then**
  return x
else
  L = the elements of A smaller than x $O(n)$
  R = the elements of A bigger than x $O(n)$
  if $i < k$ **then**
    return **Select**(L, i) $T(?)$
  else
    return **Select**$(R, i - k)$ $T(?)$

Selection algorithm: elements smaller than x

At least $3(\frac{1}{2}(\lceil n/5 \rceil - 2)) \geq \frac{3n}{10} - 6$ of the elements are $< x$.



Selection algorithm: elements bigger than x

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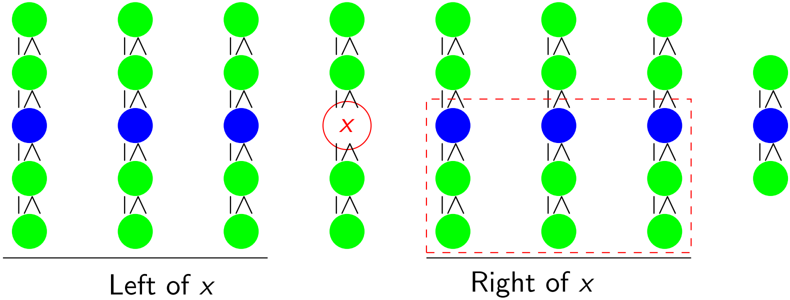
At least $3(\frac{1}{2}(\lceil n/5 \rceil - 2)) \geq \frac{3n}{10} - 6$ of the elements are $> x$.



Selection algorithm: the recurrence

- As at least $\geq \frac{3n}{10} - 6$ of the elements are $> x$ ($< x$), at most $n - (\frac{3n}{10} - 6) = 6 + 7n/10$ elements are $\leq x$ ($\geq x$).
- In the worst case, **Select** recursively calls on a vector with size $\leq 6 + 7n/10$. So, step 5 takes time $\leq T(6 + 7n/10)$.

Therefore, selecting 50 as the size to stop the recursion, we have

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 50, \\ T(\lceil n/5 \rceil) + T(6 + 7n/10) + \Theta(n) & \text{if } n > 50. \end{cases}$$

Solving we get $T(n) = \Theta(n)$ How?

Solving the recurrence

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- Use substitution.
- Assume that $T(n) \leq c\,n$, for some constant c and $n \leq 50$. Note that $6 + 7n/10 < n$, for $n > 12$.
- Prove that $T(n) \leq c\,n$ by induction. As usual we replace a $\Theta(n)$ term by $d\,n$, for an adequate constant d.

$$\begin{aligned} T(n) &\leq T(\lceil n/5 \rceil) + T(6 + 7n/10) + d\,n \\ &\leq c\lceil n/5 \rceil + c(6 + 7n/10) + d\,n \\ &\leq c(n/5 + 1) + c(6 + 7n/10) + d\,n \\ &\leq 9\,c\,n/10 + 7c + d\,n \leq cn \end{aligned}$$

Taking $c = 10d$, for large n, the inequality holds.

Remarks on the cardinality of the groups

Notice:

- If we make groups of 7, the number of elements $\geq x$ is $\frac{2n}{7}$, which yield $T(n) \leq T(n/7) + T(5n/7) + O(n)$ with solution $T(n) = O(n)$.
- However, if we make groups of 3, then $T(n) \leq T(n/3) + T(2n/3) + O(n)$, which has a solution $T(n) = O(n \ln n)$.