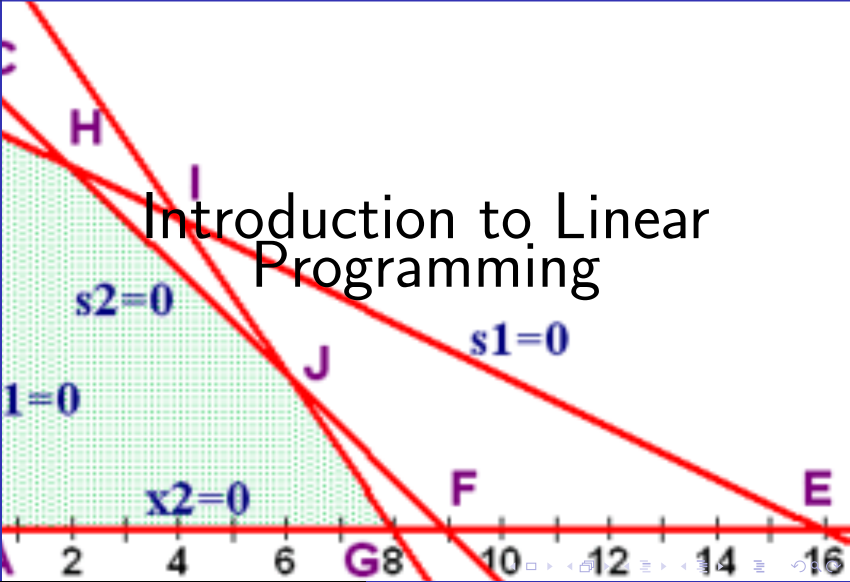
Introduction to Linear Programming



Linear Programming.

In a linear programming problem, we are given a set of variables, an objective function a set of linear constrains and want to assign real values to the variables as to:

- satisfy the set of linear equations,
- maximize or minimize the objective function.

LP is of special interest because many combinatorial optimization problems can be reduced to LP: Max-Flow; Assignment problems; Matchings; Shortest paths; MinST; ...

Example.

A company produces 2 products P1, and P2, and wishes to maximize the profits.

Each day, the company *can produce* x_1 units of P1 and x_2 units of P2.
The company *makes a profit* of 1 for each unit of P1; and a profit of 6 for each unit of P2.

Due to supply limitations and labor constrains we have the following additional constrains: $x_1 \leq 200, x_2 \leq 300$ and $x_1 + x_2 \leq 400$.

What are the best levels of production?

We express this problem as a linear program:

$$\begin{aligned} \text{Objective function:} \quad & \max(x_1 + 6x_2) \\ \text{subject to the constraints:} \quad & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0. \end{aligned}$$

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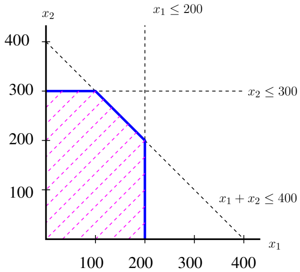
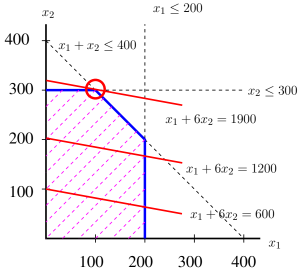
Recall a linear equation in x_1 and x_2 defines a line in $\mathbb{R}^2$. A linear inequality defines a half-space.
The feasible region of this LP are the (x_1, x_2) in the convex polygon defined by the linear constrains.


x_2
$x_1 \le 200$
400
300
200
100
$x_2 \le 300$
$x_1 + x_2 \le 400$
x_1
100
200
300
400


x_2
$x_1 \le 200$
400
$x_1 + x_2 \le 400$
300
$x_2 \le 300$
$x_1 + 6x_2 = 1900$
200
$x_1 + 6x_2 = 1200$
100
$x_1 + 6x_2 = 600$
x_1
100
200
300
400

In a linear program *the optimum is achieved at a vertex of the feasible region*.

A LP is infeasible if

- The constrains are so tight that there are impossible to satisfy all of them. For ex. $x \geq 2$ and $x \leq 1$,
- The constrains are so loose that the feasible region is unbounded. For ex. $\max(x_1 + x_2)$ with $x_1, x_2 \geq 0$

Higher dimensions.

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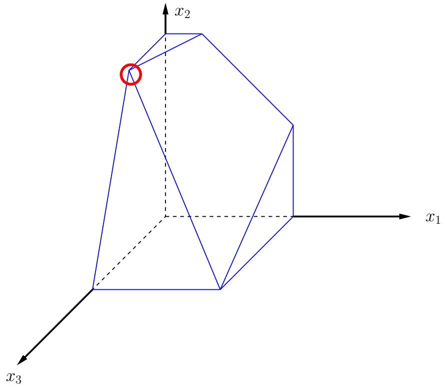
$$\begin{aligned} \max(x_1 + 6x_2 + 13x_3) \\ x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 + x_3 &\leq 400 \\ x_2 + 3x_3 &\leq 600 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$


x_2
x_1
x_3

Standard form of a Linear Program.

INPUT: Given real numbers $(c_i)_{i=1}^n, (a_{ji})_{1\leq j\leq m \& 1\leq i\leq n} (b_j)_{j=1}^m$
OUTPUT: real values for variables $(x_i)_{i=1}^n$ such that

- the objective function $\sum_{i=1}^n c_i x_j$ is minimized under the values verifying,
- for $1 \leq j \leq m$, $\sum_i a_{ji} x_i \geq b_j$

A linear programming problem is the problem or maximizing (minimizing) a linear function the objective function subject to a finite set of linear constraints

A LP is in standard form if the following are true:

- We want to minimize the objective function.
- Non-negative constraints for all variables.
- All remaining constraints are expressed as $\geq$ constraints.

Equivalent formulations of LP.

In principle LP has many degrees of freedom:

1. It can be a maximization or a minimization problem.
2. Its constrains could be equalities or inequalities.
3. The variables are often restricted to be non-negative, but they also could be unrestricted.

Most of the ”*real life*” constrains are given as inequalities.
The main reason to convert a LP into standard form is because the solvers for LP starts with a LP in standard form.

Transformation rules

- *To convert inequality* $\sum_{i=1}^{n} a_i x_i \leq b$ *into equality:*

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- *To convert inequality $\sum_{i=1}^{n} a_i x_i \leq b$ into equality:* introduce a slack variable $s \geq 0$ and replace inequality by $\sum_{i=1}^{n} a_i x_i + s = b$.
  The slack variable s_i measures the amount of "non-used resource."

  Ex: $x_1 + x_2 + x_3 \leq 40$ is replaced by $s \geq 0$ and
  $x_1 + x_2 + x_3 + s = 40$
  So that, $s = 40 - (x_1 + x_2 + x_3)$

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- *To convert inequality $\sum_{i=1}^{n} a_i x_i \geq -b$ into equality:* introduce a surplus variable $s \geq 0$ and $\sum_{i=1}^{n} a_i x_i - s = b$.
  The surplus variable $s \geq 0$ measures the extra amount of used resource.
  Ex: $-x_1 + x_2 - x_3 \geq 4 \Rightarrow -x_1 + x_2 - x_3 - s_1 = 4$

Transformations among LP forms (cont.)

- *To to deal with an unrestricted variable x (i.e. x can be positive or negative):*

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- *To turn max. problem into min. problem:* multiply the coefficients of the objective function by -1.
  Ex: $\max(10x_1 + 60x_2 + 140x_3)$ is equivalent to $\min(-10x_1 - 60x_2 - 140x_3)$.

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Applying these transformations, we can rewrite any LP into standard form, in which variables are all non-negative, the constrains are equalities, and the objective function is to be minimized.

Algebraic representation of LP

Let $c = (c_1, \ldots, c_n)$ $x = (x_1, \ldots, x_n)$, $b = (b_1, \ldots, b_m)$ and let $A = (a_{ji})$ be the $m \times n$ matrix of the coefficients involved in the constrains.

A LP can be represented using matrix and vectors:

$$
\begin{array}{rcl}
\min \sum_{i=1}^{n} c_i x_j & & \min \sum_{i=1}^{n} c^T x \\
\text{subject to} & \Rightarrow & \text{subject to} \\
\sum_{i=1}^{n} a_{ji} x_i \geq b_j, \ 1 \leq j \leq m & & Ax \geq b \\
x_i \geq 0, \ 1 \leq i \leq n & & x \geq 0
\end{array}
$$

Given a LP

min $c^T x$
subject to
$Ax \geq b$
$x \geq 0$

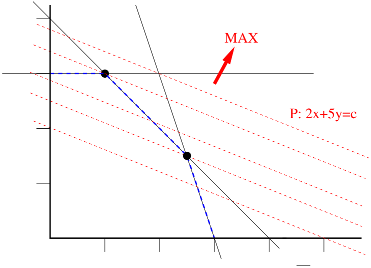
Any x that satisfies the constraints is a *feasible solution*.
A LP is *feasible* if there exists a feasible solution. Otherwise is said to be *infeasible*.
A feasible solution x^* is an optimal solution if

$$c^T x^* = \min\{c^T x \,|\, Ax \geq b, x \geq 0\}$$

The Geometry of LP

Consider P:

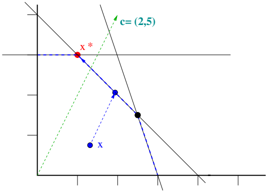
$$\begin{aligned} \max 2x{+}5y \\ 3x + y \leq 9 \\ y \leq 3 \\ x + y \leq 4 \\ x, y \geq 0 \end{aligned}$$



Theorem

If there exists an optimal solution to P, x, then there exists one that is a vertex of the polytope.

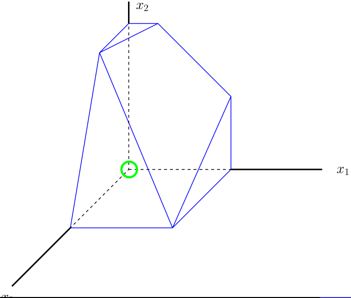
Intuition of proof If x is not a vertex, move in a non-decreasing direction until reach a boundary. Repeat, following the boundary.

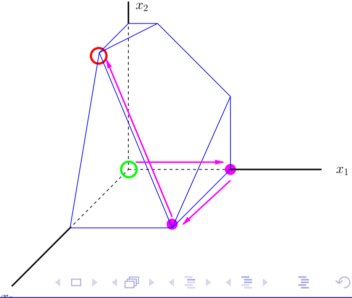


The Simplex algorithm

LP can be solved efficiently: George Dantzing (1947)

It uses hill-climbing: Start in a vertex of the feasible polytope and look for an adjacent vertex of better objective value. Until reaching a vertex that has no neighbor with better objective function.





Complexity of LP:

Input to LP: The number n of variables in the LP.

Simplex could be exponential on n: there exists specific input (the Klee-Minty cube [1970]) where the usual versions of the simplex algorithm may actually "cycle" in the path to the optimal. (see Ch.6 in Papadimitriou-Steiglitz, *Comb. Optimization: Algorithms and Complexity*)

In practice, the simplex algorithm is quite efficient and can find the global optimum (if certain precautions against cycling are taken).

It is known that simplex solves "typical" (random) problems in $O(n^3)$ steps.
Simplex is the main choice to solve LP, among engineers.

But some software packages use interior-points algorithms, which guarantee poly-time termination.

Primal, Dual and Weak Duality

Consider a LP in n variables $x = (x_1, \dots, x_n)$ with m constraints represented by matrix A, independent terms b, and objective function b.

Primal

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

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The dual is an effort to construct the best lower bound for the primal objective function.

Searching for a lower bound: The best one?

LP (PRIMAL)

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if x^* opt, $y^T Ax$ is a general linear combination of equations, if we can select y so that $y^T Ax^* = c^T x^*$, $c^T x^* \geq y^T b$

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The best lower bound, for any x?

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y = c \\ & y \geq 0 \end{aligned}$$

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LP (PRIMAL)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

if x^* opt, $y^T A x$ is a general linear combination of equations, if we can select y so that $y^T A x^* = c^T x^*$, $c^T x^* \geq y^T b$

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$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y \leq c \\ & y \geq 0 \end{array} \qquad \text{DUAL}$$

Primal - Dual: an example

- Working from the dual trying to get the best lower bound we come back to the primal.

Primal - Dual: an example

Let $G = (V, E)$ be a graph.

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LP primal

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad e = (i, j) \in E \\ & x_i \geq 0 \quad i \in V \end{aligned}$$

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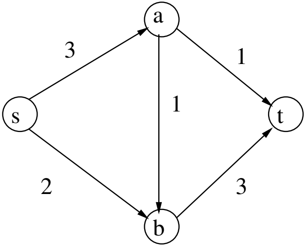
LP dual

$$\begin{array}{ll} \max & \displaystyle\sum_{e \in E} z_e \\ \text{s.t.} & \displaystyle\sum_{i \in e} z_e \leq 1 \quad \text{for all } i \in V \\ & z_e \geq 0 \quad \text{for all } e \in E \end{array}$$

Example: The Max-Flow problem



$$
\begin{aligned}
\max \ f_{sa} + f_{sb} & \\
f_{sa} &\le 3 \\
f_{sb} &\le 2 \\
f_{ab} &\le 1 \\
f_{at} &\le 1 \\
f_{bt} &\le 3 \\
f_{sa} - f_{ab} - f_{at} &= 0 \\
f_{sb} + f_{ab} - f_{bt} &= 0 \\
f_{sa}, f_{sb}, f_{ab}, f_{at}, f_{bt} &\ge 0.
\end{aligned}
$$

The Min Cut problem

$$\begin{aligned}
\min\ \ 3y_{sa} + 2y_{sb} + y_{ab} + y_{at} + y_{bt} & \\
y_{sa} + u_a &\geq 1 \\
y_{sb} + u_b &\geq 1 \\
y_{ab} - u_a + u_b &\geq 0 \\
y_{at} - ua &\leq 1 \\
y_{bt} - u_b &\leq 3 \\
y_{sa}, y_{sb}, y_{ab}, y_{at}, y_{bt}, u_a, u_b &\geq 0.
\end{aligned}$$

This D - LP defines the min-cut problem where for $x \in \{a, b\}$, $u_x = 1$ iff vertex $x \in S$, and $y_{xz} = 1$ iff $(x, z) \in$ cut (S, T).

Strong and Weak duality theorem

There are additional conditions for a pair (x, y) of primal-dual optimal/feasible solutions.

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Theorem (Strong duality)

If the primal has an optimal solution x^* then the dual has an optimal solution y^* such that $c^T x^* = b^T y^*$

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Theorem (Weak Duality)

For every feasible solution x to the primal and every solution z to the dual,

$$\sum_{i=1}^{n} c_i x_i \geq \sum_{j=1}^{m} b_j z_j$$

Conditions for optimality: Complementary slackness

Let x be a feasible solution to the primal and let z be a feasible solution to the dual.

Primal complementary slackness

If $x_i > 0$, then $\sum_{j=1}^{m} a_{ij} z_j = c_i$.

Dual complementary slackness

If $z_j > 0$, then $\sum_{i=1}^{n} a_{ij} x_i = b_j$.

Conditions for optimality: Complementary slackness

Theorem

If (x, y) satisfies complementary slackness, then x and y are optimal solutions for primal and dual problems, respectively.

Proof.

$$\sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n}(\sum_{j=1}^{m} a_{ij} z_j) x_i = \sum_{j=1}^{m}(\sum_{i=1}^{n} a_{ij} x_i) z_j = \sum_{j=1}^{m} b_j z_j$$

$\square$

Max Flow and LP

Min Cost Max Flow: Given a flow network and a valuation of the cost of transporting a unit of flow along each edge. Find a maximum flow with minimum cost.

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- Max-Flow Min Cut theorem follows from strong duality
- It is easy to adapt the LP for MaxFlow to ensure that the flow value is F and incorporate the cost in the objective function.
  Add the equation $f(s, V) = F$
  Objective function: minimize $\sum_{e \in E} c_e f_e$
- This approach provides a polynomial time algorithm for the Min Cost Max Flow problem.

Integer Linear Programming (ILP)

Consider the Min Vertex Cover problem: Given an undirected $G = (V, E)$ with $|V| = n$ and $|E| = m$, want to find $S \subseteq V$ with minimal cardinality s.t.. it covers all edges $e \in E$.

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- This problem can be expressed as a linear program on $\{0, 1\}$ variables, interpreting a solution as
  Let $x \in \{0, 1\}^n$ be seen as a set S, in the usual way, for $i \in V$:

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

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- Under this interpretation we the constraints $\forall (i, j) \in E$ $x_i + x_j \geq 1$ are equivalent to say that S is a vertex cover. The constraints give $Ax \geq 1$.

Integer Linear Programming

We can express the min VC problem as:

$\min \sum_{i \in V} x_i$
subject to
$x_i + x_j \geq, (i,j) \in E$
$x_i \in \{0,1\}, i \in V$

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$$\min \sum_{i \in V} x_i$$
subject to
$$x_i + x_j \geq , \ (i,j) \in E$$
$$x_i \in \{0,1\}, \ i \in V$$

where we have a new constrain, we require the solution to be 0,1. This can be replaced by requiring the variables to be positive integers (as we are minimizing).

Asking for the best possible integral solution for a LP is known as the Integer Linear Programming:

Integer Linear Programming

The ILP problem is defined:

Given $A \in \mathbb{Z}^{n \times m}$ together with $b \in \mathbb{Z}^n$ and $c \in \mathbb{Z}^m$, find a x that max (min) c^T subject to:

$$\min\ c^T x$$

subject to

$$Ax \geq 1$$

$$x \in \mathbb{Z}^m,$$

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$$\begin{aligned} &\min\ c^T x \\ &\text{subject to} \\ &\quad Ax \geq 1 \\ &\quad x \in \mathbb{Z}^m, \end{aligned}$$

Big difference between LP and ILP:
Ellipsoidal/Interior point methods solve LP in polynomial time but ILP is NP-hard.

Solvers for LP

Due to the importance of LP and ILP as models to solve optimization problem, there is a very active research going on to design new algorithms and heuristics to improve the running time for solving LP (algorithms) IPL (heuristics).

There are a myriad of solvers packages:

- CPLEX: http://ampl.com/products/solvers/solvers-we-sell/cplex/
- GUROBI Optimizer: http://www.gurobi.com/products/gurobi-optimizer