

# Max-flow and min-cut problems

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Karp alg

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Demands

Lower bounds

Survey design

Joint rounding

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# Dinic and Edmonds-Karp algorithm

J.Edmonds, R. Karp: *Theoretical improvements in algorithmic efficiency for network flow problems.* Journal ACM 1972.

Yefim Dinic: *Algorithm for solution of a problem of maximum flow in a network with power estimation.* Doklady Ak.N. 1970



Choosing a **good** augmenting path can lead to a faster algorithm.

Use **BFS** to find an augmenting paths in  $G_f$ .

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# Edmonds-Karp algorithm

FF algorithm but **using BFS**: choose the augmenting path in  $G_f$  with the smallest length ( number of edges).

**Edmonds-Karp**( $G, c, s, t$ )

For all  $e = (u, v) \in E$  let  $f(u, v) = 0$

$G_f = G$

**while** there is an  $s \rightsquigarrow t$  path in  $G_f$

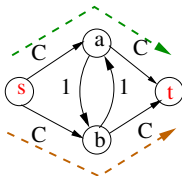
**do**



$P = \text{BFS}(G_f, s, t)$

$f = \text{Augment}(f, P)$

Compute  $G_f$

**return**  $f$



The BFS in EK will choose:  or 

# BFS paths on $G_f$

For  $\mathcal{N} = (V, E, c, s, t)$  and a flow  $f$  in  $\mathcal{N}$ , assuming that  $G_f$  has an augmenting path, let  $f'$  be the next flow after executing one step of the EK algorithm.

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- The path from  $s$  to  $t$  in a BFS traversal starting at  $s$ , is a path  $s \rightsquigarrow t$  with minimum number of edges, i.e., a **shortest length path**.

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# BFS paths on $G_f$

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- The path from  $s$  to  $t$  in a BFS traversal starting at  $s$ , is a path  $s \rightsquigarrow t$  with minimum number of edges, i.e., a **shortest length path**.
- For  $v \in V$ , let  $\delta_f(s, v)$  denote length of a shortest length path from  $s$  to  $v$  in  $G_f$ .

# Some properties of $G_f$ and $G_{f'}$

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How can we have  $(u, v) \in E_{f'}$  but  $(u, v) \notin E_f$ ?

- $(u, v)$  is a forward edge saturated in  $f$  and not in  $f'$ .
- $(u, v)$  is a backward edge in  $G_f$  and  $f(v, u) = 0$

# Some properties of $G_f$ and $G_{f'}$

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- $(u, v)$  is a backward edge in  $G_f$  and  $f(v, u) = 0$

In any of the two cases, the augmentation must have modified the flow from  $v$  to  $u$ , so  $(u, v)$  must form part of the augmenting path.

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## Lemma

*If the EK-algorithm runs on  $\mathcal{N} = (V, E, c, s, t)$ , for all vertices  $v \neq s$ ,  $\delta_f(s, v)$  increases monotonically with each flow augmentation.*

**Proof.** By contradiction.

Let  $f$  be the first flow such that, for some  $u \neq s$ ,

$$\delta_{f'}(s, u) < \delta_f(s, u).$$

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## Proof (cont)

Let  $v$  be the vertex with the minimum  $\delta_{f'}(s, v)$  whose distance was decreased.

- Let  $P : s \rightsquigarrow u \rightarrow v$  be a shortest length path from  $s$  to  $v$  in  $G_{f'}$
- Then,  $\delta_{f'}(s, v) = \delta_{f'}(s, u) + 1$  and  $\delta_{f'}(s, u) \geq \delta_f(s, u)$ .
- If  $(u, v) \in E_f$ ,  
 $\delta_f(s, v) \leq \delta_f(s, u) + 1 \leq \delta_{f'}(s, u) + 1 = \delta_{f'}(s, v)$
- So,  $(u, v) \notin E_f$

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## Proof (cont)

How can we have ?

- $(u, v) \in E_{f'}$  but  $(u, v) \notin E_f$
- If so,  $(v, u)$  appears in the augmenting path.

# EK and the shortest length distances

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## Proof (cont)

How can we have ?

- $(u, v) \in E_{f'}$  but  $(u, v) \notin E_f$
- If so,  $(v, u)$  appears in the augmenting path.
- Then, the shortest length path from  $s$  to  $u$  in  $G_f$  has  $(v, u)$  as its last edge.  
$$\delta_f(s, v) \leq \delta_f(s, u) - 1 \leq \delta_{f'}(s, u) - 1 = \delta_{f'}(s, v) - 1 - 1$$

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- If so,  $(v, u)$  appears in the augmenting path.
- Then, the shortest length path from  $s$  to  $u$  in  $G_f$  has  $(v, u)$  as its last edge.  
$$\delta_f(s, v) \leq \delta_f(s, u) - 1 \leq \delta_{f'}(s, u) - 1 = \delta_{f'}(s, v) - 1 - 1$$
- which contradicts  $\delta_{f'}(s, v) < \delta_f(u, v)$ . □

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Let  $P$  be an augmenting path in  $G_f$ .

$(u, v) \in P$  is **critical** if  $b(P) = c_f(u, v)$ .

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Let  $P$  be an augmenting path in  $G_f$ .

$(u, v) \in P$  is **critical** if  $b(P) = c_f(u, v)$ .

Critical edges do not appear in  $G_{f'}$ .

- $(u, v)$  forward,  $f'(u, v) = c(u, v)$
- $(u, v)$  backward,  $f'(v, u) = 0$

# EK and critical edges

## Lemma

*In the EK algorithm, each one of the edges can become critical at most  $|V|/2$  times.*

Proof:

- Let  $(u, v) \in E$ , when  $(u, v)$  is critical for the first time,  
$$\delta_f(s, v) = \delta_f(s, u) + 1$$

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- After this step  $(u, v)$  disappears from the residual graph until after the flow in  $(u, v)$  changes.

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- After this step  $(u, v)$  disappears from the residual graph until after the flow in  $(u, v)$  changes.
- At this point,  $(v, u)$  forms part of the augmenting path in  $G_{f'}$ , and  $\delta_{f'}(s, u) = \delta_{f'}(s, v) + 1$ ,

$$\delta_{f'}(s, u) = \delta_{f'}(s, v) + 1 \geq \delta_f(s, v) + 1 \geq \delta_f(s, u) + 2$$

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- So, the distance has increased by at least 2.

# Complexity of Edmonds-Karp algorithm

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## Theorem

*The EK algorithm runs in  $O(mn(n + m))$  steps. Therefore it is a polynomial time algorithm.*

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## Theorem

*The EK algorithm runs in  $O(mn(n + m))$  steps. Therefore it is a polynomial time algorithm.*

Proof:

- Need time  $O(m + n)$  to find the augmenting path using BFS.
- By the previous Lemma, there are  $O(mn)$  augmentations.  
□



# Finding a min-cut

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Given  $(G, s, t, c)$  to find a min-cut:

- 1 Compute the max-flow  $f^*$  in  $G$ .
- 2 Obtain  $G_{f^*}$ .
- 3 Find the set  $S = \{v \in V \mid s \rightsquigarrow v\}$  in  $G_{f^*}$ .
- 4 Output the cut  
 $(S, V - \{S\}) = \{(v, u) \mid v \in S \text{ and } u \in V - \{S\}\}$  in  $G$ .

The running time is the same than the algorithm to find the max-flow.

# The max-flow problems: History

- **Ford-Fulkerson (1956)**  $O(mC)$ , where  $C$  is the max flow val.
- **Dinic (1970)** (blocking flow)  $O(n^2m)$
- **Edmond-Karp (1972)** (shortest augmenting path)  $O(nm^2)$
- **Karzanov (1974)**,  $O(n^2m)$  **Goldberg-Tarjant (1986)** (push re-label preflow + dynamic trees)  $O(nm \lg(n^2/m))$  (uses parallel implementation)
- **King-Rao-Tarjan (1998)**  $O(nm \log_{m/n} n)$ .
- **J. Orlin (2013)**  $O(nm)$  (clever follow up to KRT-98)
- **Chen, Kyng, Liu, Peng, Gutenberg, Sachdeva (2022)**  $O(m^{1+o(1)})$  (polynomially bounded integral capacities)  
You can read [Quanta Magazine](#) article.

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# FINAL'S SCHEDULING

We have as input:

- $n$  courses, each one with a final. Each exam must be given in one room. Each course  $c_i$  has  $E[i]$  students.
- $r$  rooms. Each  $r_j$  has a capacity  $S[j]$ ,
- $\tau$  time slots. For each room and time slot, we only can schedule one final.
- $p$  professors to watch exams. Each exam needs one professor in each class and time. Each professor has its own restrictions of availability and no professor can oversee more than 6 finals. For each  $p_\ell$  and  $\tau_k$  define a Boolean variable  $A[k, \ell] = T$  if  $p_\ell$  is available at  $\tau_k$ .

Design an efficient algorithm that correctly schedules a room, a time slot and a professor to every final, or report that not such schedule is possible.

# Construction of the network

Construct the network  $\mathcal{N}$  with vertices  $\{s, t, \{c_i\}, \{r_j\}, \{\tau_k\}, \{p_\ell\}\}$ . Edges and capacities:

- $(s, c_i)$  with capacity 1 (each course has one final)
- $(c_i, r_j)$ , if  $E[i] \leq S[j]$ , with capacity  $\infty$  (can go to a room in which it fits)
- $\forall j, k, (r_j, \tau_k)$ , with capacity 1 (one final per room and time slot).
- $(\tau_k, p_\ell)$ , if  $A[k, \ell] = T$ , capacity 1 ( $p$  can watch one final, if  $p$  is available at  $\tau_k$ ).
- $(p_\ell, t)$ , capacity 6 (each  $p$  can watch  $\leq 6$  finals)

Notice that neither rooms nor time slots have individual restrictions.

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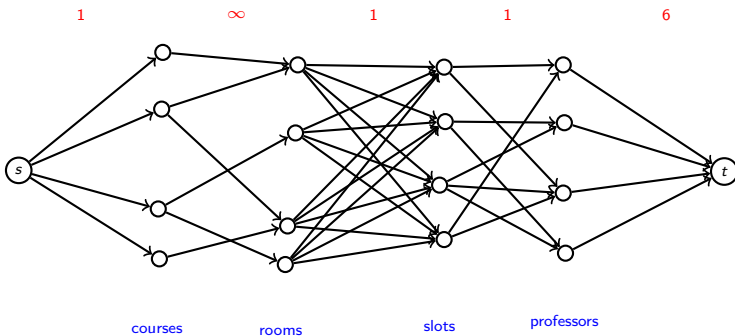
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# FINAL'S SCHEDULING: Flow Network



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- Notice the input size to the problem is  $N = n + r + \tau + p + 2$ . and size of the network is  $O(N)$  vertices and  $O(N^2)$  edges, why?
- Every path  $s \rightsquigarrow t$  is an assignment of room-time-professor to a final, and any assignment room-time-professor to a final can be represented by a path  $s \rightsquigarrow t$ .
- Every integral flow identifies a collection of  $|f|$   $(s, t)$ -paths leading to a valid assignment for  $|f|$  finals and viceversa.

# FINAL'S SCHEDULING

- To maximize the number of finals to be given, we compute the max-flow  $f^*$  from  $s$  to  $t$ .
- If  $|f^*| = n$ , then we can schedule all finals, otherwise we can not.
- To recover the assignment we have to consider the edges with positive flow and extract assignment from the  $n$   $(s, t)$ -paths
- **Complexity:**
  - To construct  $\mathcal{N}$ , we need  $O(N^2)$ .
  - As  $|f^*| \leq n$  integral, we can use Ford-Fulkerson to compute  $f^*$ , with cost  $O(nN^2)$ .
  - The second part requires  $O(N^2)$  time.
  - So, the cost of the algorithm is  $O(nN^2) = O(n(n + r + \tau + p)^2)$ .

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# Applications: Generalized assignment problems

- Consider a **generalized assignment problem**  $\mathcal{GP}$  where, we have as input  $d$  finite sets  $X_1, \dots, X_d$ , each representing a different set of resources.

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- Our goal is to choose the "largest" number of  $d$ -tuples, each  $d$ -tuple containing exactly one element from each  $X_i$ , subject to the constraints:

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- Our goal is to choose the "largest" number of  $d$ -tuples, each  $d$ -tuple containing exactly one element from each  $X_i$ , subject to the constraints:
  - For each  $i \in [d]$ , each  $x \in X_i$  can appear in at most  $c(x)$  selected tuples.
  - For each  $i \in [d]$ , any two  $x \in X_i$  and  $y \in X_{i+1}$  can appear in at most  $c(x, y)$  selected tuples.
  - The values for  $c(x)$  and  $c(x, y)$  are either in  $\mathbb{Z}^+$  or  $\infty$ .

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  - The values for  $c(x)$  and  $c(x, y)$  are either in  $\mathbb{Z}^+$  or  $\infty$ .
- Notice that only pairs of objects between adjacent  $X_i$  and  $X_{i+1}$  are constrained.

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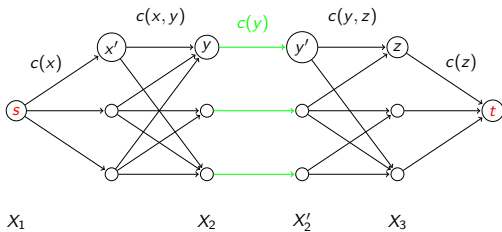
# Applications: Generic reduction to Max-Flow

Make the reduction from  $\mathcal{GP}$  to the following network  $\mathcal{N}$ :

- $V$  contains a vertex  $x$ , for each element  $x$  in each  $X_i$ , and a copy  $x'$ , for each element  $x \in X_i$  for  $1 \leq i < d$ .
- We add vertex  $s$  and vertex  $t$ .
- Add an edge  $s \rightarrow x$  for each  $x \in X_1$  and add an edge  $y \rightarrow t$  for every  $y \in X_d$ . Give capacities  $c(s, x) = c(x)$  and  $c(y, t) = c(y)$ .
- Add an edge  $x' \rightarrow y$  for every pair  $x \in X_i$  and  $y \in X_{i+1}$ . Give a capacity  $c(x, y)$ . Omit the edges with capacity 0.
- For every  $x \in X_i$  for  $1 \leq i < d$ , add an edge  $x \rightarrow x'$  with  $c(x, x') = c(x)$ .

Every path  $s \rightsquigarrow t$  in  $\mathcal{N}$  identifies a feasible  $d$ -tuple, conversely every  $d$ -tuple determines a path  $s \rightsquigarrow t$ .

# Flow Network: The reduction



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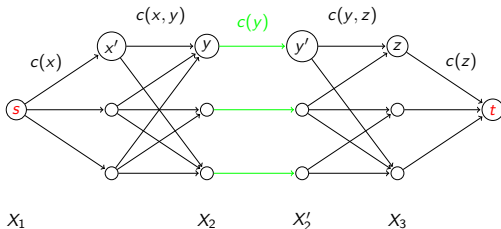
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# Flow Network: The reduction



- To solve  $\mathcal{GP}$ , we construct  $\mathcal{N}$ , and then we find an integer maximum flow  $f^*$ .
- In the subgraph formed by edges with  $f^*(e) > 0$ , we find a  $(s, t)$  path  $P$  (a  $d$ -tuple), decrease in 1 the flow in each edge of  $P$ , remove edges with 0 flow.

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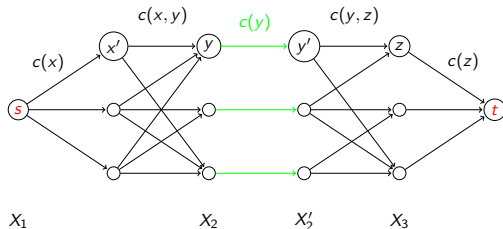
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# Flow Network: The reduction



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- In the subgraph formed by edges with  $f^*(e) > 0$ , we find a  $(s, t)$  path  $P$  (a  $d$ -tuple), decrease in 1 the flow in each edge of  $P$ , remove edges with 0 flow.
- Repeat  $|f^*|$  times. In this way we obtain a set of  $d$ -tuples with maximum size verifying all the restrictions.

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**3 Circulations**

4 Min cost Max Flow

# Circulation with demands

- We introduce another flow problem, to deal with supply and demand inside a network.
- Instead of having a pair source/sink the new setting consider a producer/consumer scenario.

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# Circulation with demands

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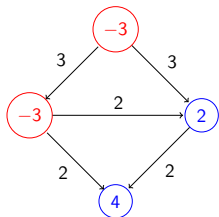
Joint rounding

Min cost Max  
Flow

- We introduce another flow problem, to deal with supply and demand inside a network.
- Instead of having a pair source/sink the new setting consider a producer/consumer scenario.
- Some nodes are able to produce a certain amount of flow.
- Some nodes are willing to consume flow.
- The question is whether it is possible to route “all” the produced flow to the consumers. When possible the flow assignment is called a **circulation**

# Network with demands

A **network with demands**  $\mathcal{N}$  is a tuple  $(V, E, c, d)$  where  $c$  assigns a positive capacity to each edge, and  $d$  is a function associating a demand  $d(v)$ , to  $v \in V$ .



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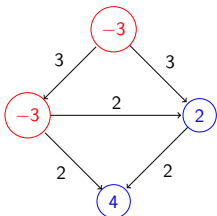
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- When  $d(v) > 0$ ,  $v$  can receive  $d(v)$  units of flow more than it sends,  $v$  is a **sink**.
- If  $d(v) < 0$ ,  $v$  can send  $d(v)$  units of flow more than it receives,  $v$  is a **source**.
- If  $d(v) = 0$ ,  $v$  is neither a source or a sink.



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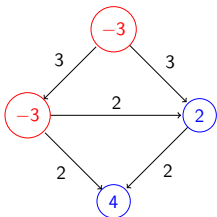
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- If  $d(v) < 0$ ,  $v$  can send  $d(v)$  units of flow more than it receives,  $v$  is a **source**.
- If  $d(v) = 0$ ,  $v$  is neither a source or a sink.
- Define  $S$  to be the set of sources and  $T$  the set of sinks.

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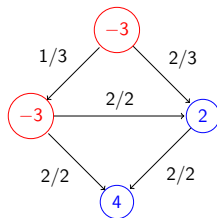
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# Network with demands: circulation

Given a network  $\mathcal{N} = (V, E, c, d)$ , a **circulation** is a flow assignment  $f : E \rightarrow \mathbb{R}^+$  s.t.

- 1 capacity:** For each  $e \in E$ ,  $0 \leq f(e) \leq c(e)$ ,
- 2 conservation:** For each  $v \in V$ ,

$$\sum_{(u,v) \in E} f(u,v) - \sum_{(v,z) \in E} f(v,z) = d(v).$$

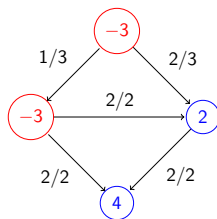


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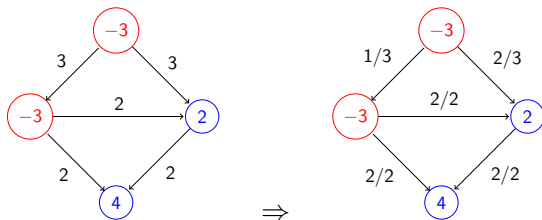


Take into account that a circulation might not exist.



# Network with demands: circulation problem

**Circulation problem:** Given  $\mathcal{N} = (V, E, c, d)$  with  $c > 0$ , obtain a circulation provided it does exist.



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# A first conditions for a circulation to exist

If  $f$  is a circulation for  $\mathcal{N} = (V, E, c, d)$ ,

$$\sum_{v \in V} d(v) = \sum_{v \in V} \left( \underbrace{\sum_{(u,v) \in E} f(u,v)}_{\text{edges to } v} - \underbrace{\sum_{(v,z) \in E} f(v,z)}_{\text{edges out of } v} \right).$$

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For  $e = (u, v) \in E$ ,  $f(e)$  appears in the sum of edges to  $v$  and in the sum of edges out of  $u$ . Both terms cancel!

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For  $e = (u, v) \in E$ ,  $f(e)$  appears in the sum of edges to  $v$  and in the sum of edges out of  $u$ . Both terms cancel!

Then,  $\sum_{v \in V} d(v) = 0$ .

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# A first conditions for a circulation to exists

If there is a circulation, then  $\sum_{v \in V} d(v) = 0$ .

Recall that

$$S = \{v \in V \mid d(v) < 0\} \text{ and}$$

$$T = \{v \in V \mid d(v) > 0\}.$$

Define  $D = -\sum_{v \in S} d(v) = \sum_{v \in T} d(v)$ .

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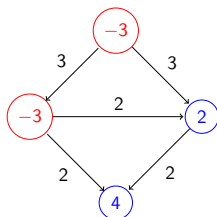
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$S = \{v \in V \mid d(v) < 0\}$  and

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Define  $D = -\sum_{v \in S} d(v) = \sum_{v \in T} d(v)$ .

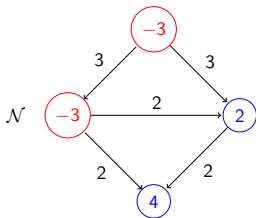
$D$  is the total amount of extra flow that has to be transported from the sources to the sinks.



# Circulation problem: reduction to Max-flow

From  $\mathcal{N} = (V, E, c, d)$ , define a flow network  $\mathcal{N}' = (V', E', c', s, t)$ :

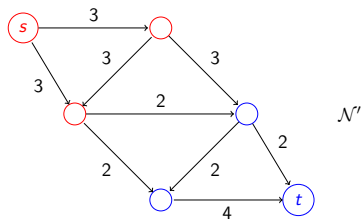
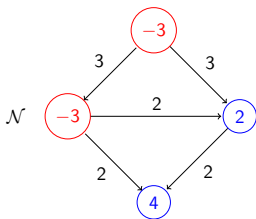
- $V' = V \cup \{s, t\}$ , we add a source  $s$  and a sink  $t$ .
- For  $v \in S$  ( $d(v) < 0$ ), add  $(s, v)$  with capacity  $-d(v)$ .
- For  $v \in T$  ( $d(v) > 0$ ), add  $(v, t)$  with capacity  $d(v)$ .
- Keep  $E$  and, for  $e \in E$ ,  $c'(e) = c(e)$ .



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# Circulation problem: reduction to Max-flow

1.- Every flow  $f'$  in  $\mathcal{N}'$  verifies  $|f'| \leq D$

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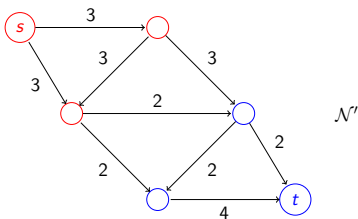
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# Circulation problem: reduction to Max-flow

1.- Every flow  $f'$  in  $\mathcal{N}'$  verifies  $|f'| \leq D$

The capacity  $c'(\{s\}, V) = D$ , by the capacity restriction on flows,  $|f'| \leq D$ .



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# Circulation problem: reduction to Max-flow

2.- If there is a circulation  $f$  in  $\mathcal{N}$ , we have a max-flow  $f'$  in  $\mathcal{N}'$  with  $|f'| = D$ .

Extend  $f$  to a flow  $f'$ , assigning  $f'(s, v) = -d(v)$ , for  $v \in S$ , and  $f'(u, t) = d(u)$ , for  $u \in T$ .

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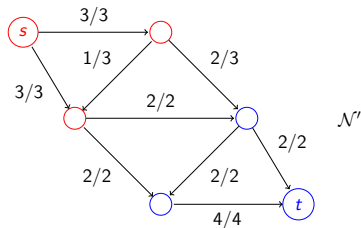
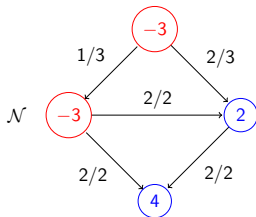
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# Circulation problem: reduction to Max-flow

2.- If there is a circulation  $f$  in  $\mathcal{N}$ , we have a max-flow  $f'$  in  $\mathcal{N}'$  with  $|f'| = D$ .

Extend  $f$  to a flow  $f'$ , assigning  $f'(s, v) = -d(v)$ , for  $v \in S$ , and  $f'(u, t) = d(u)$ , for  $u \in T$ .

By the circulation condition,  $f'$  is a flow in  $\mathcal{N}'$ . Furthermore,  $|f'| = D$ .



# Analysis

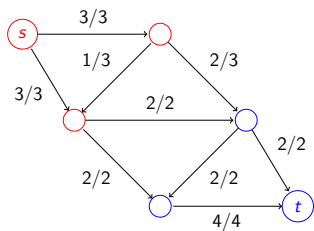
3.- If there is a flow  $f'$  in  $\mathcal{N}'$  with  $|f'| = D$ ,  $\mathcal{N}$  has a circulation

For  $e \in E$ , define  $f(e) = f'(e)$ .

- As  $|f'| = D$ , all edges  $(s, v) \in E'$  and  $(u, t) \in E'$  are saturated by  $f'$ .

- By flow conservation,  $f$  satisfies  $d(v) = \underbrace{\sum_{(u,v) \in E} f(u, v)}_{\text{edges to } v} - \underbrace{\sum_{(v,z) \in E} f(v, z)}_{\text{edges out of } v}$ .

- So,  $f$  is a circulation for  $\mathcal{N}$ .



# Circulation: main results

From the previous discussion, we can conclude:

Theorem (**Necessary and sufficient condition**)

*There is a circulation for  $\mathcal{N} = (V, E, c, d)$  iff the maxflow in  $\mathcal{N}'$  has value  $D$ .*

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Theorem (**Circulation integrality theorem**)

*If all capacities and demands are integers, and there exists a circulation, then there exists an integer valued circulation.*

Sketch Proof Max-flow formulation + integrality theorem for max-flow □

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# Circulation: main results

## Theorem

*There is a polynomial time algorithm to solve the circulation problem.*

The cost of the algorithm is the same as the cost of the algorithm used for the MaxFlow computation.

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# Circulation: main results

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## Theorem

*There is a polynomial time algorithm to solve the circulation problem.*

The cost of the algorithm is the same as the cost of the algorithm used for the MaxFlow computation.

## Theorem

*If all capacities and demands are integers, and there exists a circulation, then we can obtain an integer valued circulation in time  $O(Dm)$ .*

# Networks with demands and lower bounds

Generalization of the previous problem: besides satisfy demands at nodes, we want to force the flow to use certain edges.

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# Networks with demands and lower bounds

Generalization of the previous problem: besides satisfy demands at nodes, we want to force the flow to use certain edges.

Introduce a new constrain  $\ell(e)$  on each  $e \in E$ , indicating the min-value the flow must be on  $e$ .

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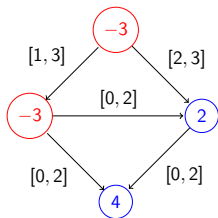
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Generalization of the previous problem: besides satisfy demands at nodes, we want to force the flow to use certain edges.

Introduce a new constrain  $\ell(e)$  on each  $e \in E$ , indicating the min-value the flow must be on  $e$ .

A network  $\mathcal{N}$  with demands and lower bounds is a tuple  $(V, E, c, \ell, d)$  with  $c(e) \geq \ell(e) \geq 0$ , for each  $e \in E$ ,



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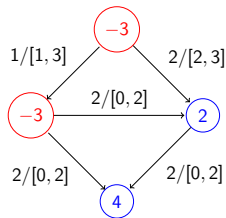
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# Networks with demands and lower bounds: circulation

Given a network  $\mathcal{N} = (V, E, c, \ell, d)$  a **circulation** as a flow assignment  $f : E \rightarrow \mathbb{R}^+$  s.t.

- 1** capacity: For each  $e \in E$ ,  
 $\ell(e) \leq f(e) \leq c(e)$ ,
- 2** conservation: For each  $v \in V$ ,

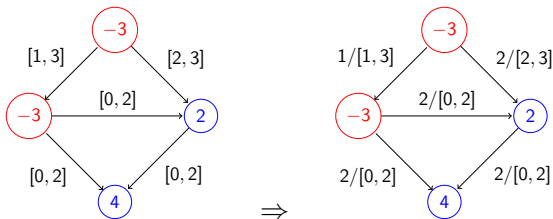
$$\sum_{(u,v) \in E} f(u,v) - \sum_{(v,z) \in E} f(v,z) = d(v).$$



A circulation might not exist.

# Circulations with demands and lower bounds problem

Circulation with demands and lower bounds problem: Given  $\mathcal{N} = (V, E, c, \ell, d)$ , obtain a circulation for  $\mathcal{N}$ , provided it does exist



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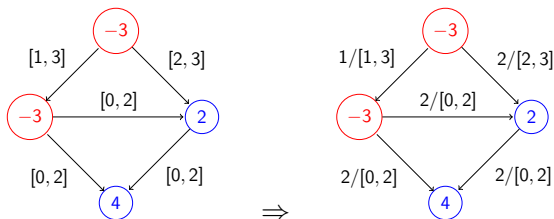
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# Circulations with demands and lower bounds problem

Circulation with demands and lower bounds problem: Given  $\mathcal{N} = (V, E, c, \ell, d)$ , obtain a circulation for  $\mathcal{N}$ , provided it does exist



We devise an algorithm to the problem by a reduction to a circulation with demands problem.

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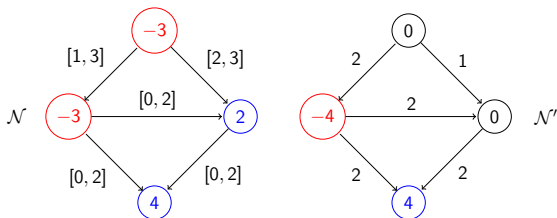
# Circulations with demands and lower bounds: the reduction

Let  $\mathcal{N} = (V, E, c, \ell, d)$ , construct a network  $\mathcal{N}' = (V, E, c', d')$  with only demands as follows:

Initially set  $c' = c$  and  $d' = d$ .

For each  $e = (u, v) \in E$ , with  $\ell(e) > 0$ :

- $c'(e) = c(e) - \ell(e)$ .
- Update the demands on both ends of  $e$ :  
 $d'(u) = d(u) + \ell(e)$  and  $d'(v) = d(v) - \ell(e)$





# Circulations with demands and lower bounds: the reduction

1.- If  $f$  is a circulation in  $\mathcal{N}$ ,  $f'(e) = f(e) - \ell(e)$ , for  $e \in E$ , is a circulation in  $\mathcal{N}'$ .

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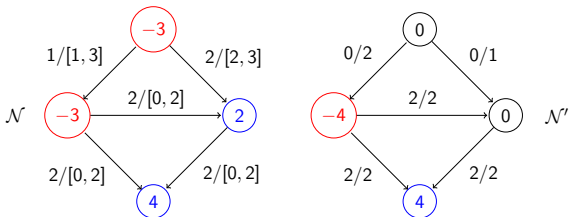
# Circulations with demands and lower bounds: the reduction

1.- If  $f$  is a circulation in  $\mathcal{N}$ ,  $f'(e) = f(e) - \ell(e)$ , for  $e \in E$ , is a circulation in  $\mathcal{N}'$ .

By construction of  $\mathcal{N}'$ ,  $f'$  verifies the capacity constraint.

Besides, for  $(u, v)$  with  $\ell(u, v) > 0$ , the flow out of  $u$  and the flow in  $v$  is decreased by  $\ell(u, v)$ .

$f$  is a circulation in  $\mathcal{N}$  so, the flow imbalance of  $f'$  matches the demand  $d'$  at each node.



# Circulations with demands and lower bounds: the reduction

2.- If  $f'$  is a circulation in  $\mathcal{N}'$ ,  $f(e) = f'(e) + \ell(e)$ , for  $e \in E$ , is a circulation in  $\mathcal{N}$ .

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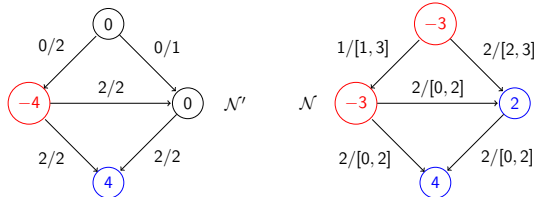
# Circulations with demands and lower bounds: the reduction

2.- If  $f'$  is a circulation in  $\mathcal{N}'$ ,  $f(e) = f'(e) + \ell(e)$ , for  $e \in E$ , is a circulation in  $\mathcal{N}$ .

$f'$  verifies the capacity constraint  $f'(e) \geq 0$ , so  $f(e) \geq \ell(e)$ .

$f'$  is a circulation, the  $f'$  imbalance at  $u$  is  $d'(u)$ .

Therefore, for  $(u, v)$  with  $\ell(u, v) > 0$ , the increase of flow in  $(u, v)$  balances  $\ell(u, v)$  units of flow out of  $u$  with  $\ell(u, v)$  units of flow entering  $v$ . Thus the  $f$  imbalance at  $u$  is  $d(u)$ .



# Main result

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## Theorem

*There exists a circulation in  $\mathcal{N}$  iff there exists a circulation in  $\mathcal{N}'$ . Moreover, if all demands, capacities and lower bounds in  $\mathcal{N}$  are integers, and  $\mathcal{N}$  admits a circulation, there is a circulation in  $\mathcal{N}$  that is integer-valued.*

The integer-valued circulation part is a consequence of the integer-value circulation Theorem for  $f'$  in  $G'$ .

# Circulation with demands and lower bounds: main results

## Theorem

*There is a polynomial time algorithm to solve the circulation with demands and lower bounds problem.*

The cost of the algorithm is the same as the cost of the algorithm used for the circulation with demands computation.

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# Circulation with demands and lower bounds: main results

## Theorem

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The cost of the algorithm is the same as the cost of the algorithm used for the circulation with demands computation.

## Theorem

*If all capacities, lower bounds, and demands are integers, and there exists a circulation, then we can obtain an integer valued circulation in time  $O((D + L)m)$  where  $L$  is the sum of all lower bounds.*

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# SURVEY DESIGN problem

Problem: Design a survey among customers of products (KT-7.8)

- Customer  $i$  can only be asked about a bought product and must receive a questionnaire for at least  $c_i$  such products, those values are determined as function of the purchased products.
- For each product  $j$ , we want to collect data from a minimum of  $p_j$  customers.
- The  $c$  and  $p$  values cannot exceed the number of





# SURVEY DESIGN problem

The input to the problem is:

A set  $C$  of  $n$  customers and a set  $P$  of  $m$  products.

- For each customer  $i \in C$ , a list of purchased products and the two values  $c_i \leq c'_i$ .
- For each product  $j \in P$ , two values  $p_j$  and  $p'_j$ .

Alternatively,

- The information about purchases can be represented as a bipartite graph  $G = (C \cup P, E)$ , where  $C$  is the set of customers and  $P$  is the set of products.
- $(i, j) \in E$  means  $i \in C$  has purchased product  $j \in P$ .

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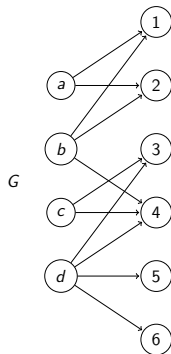
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Flow

# SURVEY DESIGN: Input

Customers  $C = \{a, b, c, d\}$   
Products  $P = \{1, 2, 3, 4, 5, 6\}$

Customer	Bought	$c$
a	1,2	1
b	1,2,4	1
c	3,6	1
d	3,4,5,6	2

Prod.	1	2	3	4	5	6
$d$	1	1	1	1	0	1



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# SURVEY DESIGN: Circulation with lower bounds formulation

We construct a network  $\mathcal{N} = (V', E', c, \ell)$  from  $G$  as follows:

- Nodes:  $V' = V \cup \{s, t\}$
- Edges:  $E'$  contains  $E$  and edges  $s \rightarrow \{C\}$ ,  $\{P\} \rightarrow t$ , and  $(t, s)$ .
- Capacities and lower bounds:
  - $c(t, s) = \infty$  and  $\ell(t, s) = 0$
  - For  $i \in C$ ,  $\ell(s, i) = c_i$  and  $c(s, i) =$  the number of purchased products.
  - For  $j \in P$ ,  $\ell(j, t) = p_j$  and  $c(j, t) =$  number of customers that purchased  $j$ .
  - For  $(i, j) \in E$ ,  $c(i, j) = 1$ , and  $\ell(i, j) = 0$ .

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# SURVEY DESIGN: Circulation with lower bounds formulation

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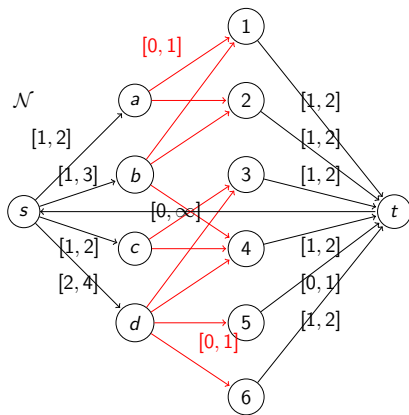
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Customers  $C = \{a, b, c, d\}$   
Products  $P = \{1, 2, 3, 4, 5, 6\}$

Customer	Bought	$c$
a	1,2	1
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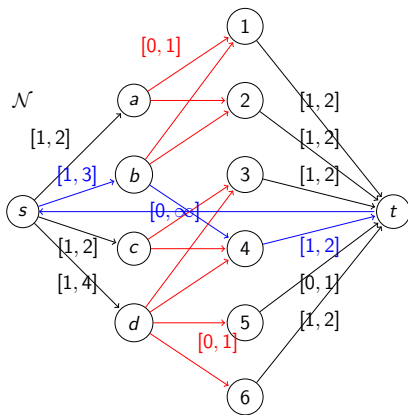
Prod.	1	2	3	4	5	6
$d$	1	1	1	1	0	1



# SURVEY DESIGN: Circulation interpretation

If  $f$  is a circulation in  $\mathcal{N}$ :

- one unit of flow circulates  $s \rightarrow i \rightarrow j \rightarrow t \rightarrow s$ .
- $f(i, j) = 1$  means ask  $i$  about  $j$ ,
- $f(s, i) = \#$  products to ask  $i$  for opinion,
- $f(j, t) = \#$  customers to be asked to review  $j$ ,
- $f(t, s)$  is the total number of questionnaires.



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# SURVEY DESIGN: Circulation vs solutions

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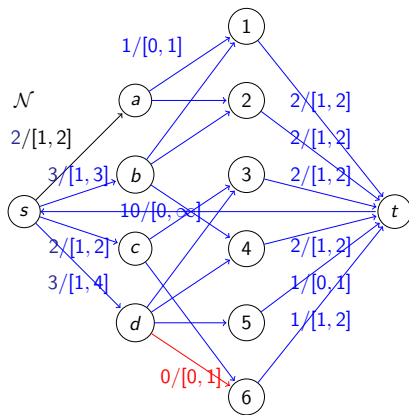
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## A solution

- Ask  $a$  about 1, 2.
- Ask  $b$  about 1, 2, 4.
- Ask  $c$  about 3, 6.
- Ask  $d$  about 3, 4, 5.



# Main result

**Theorem**  $\mathcal{N}$  has a circulation iff there is a feasible way to design the survey.

**Proof** if there is a feasible way to design the survey:

- if  $i$  is asked about  $j$  then  $f(i, j) = 1$ ,
- $f(s, i) =$  number questions asked to  $i$  ( $\geq c_i$ ).
- $f(j, t) =$  number of customers who were asked about  $j$  ( $\geq d_j$ ),
- $f(t, s) =$  total number of questions.
- easy to verify that  $f$  is a circulation in  $\mathcal{N}$

If there is an integral circulation in  $\mathcal{N}$ :

- if  $f(i, j) = 1$  then  $i$  will be asked about  $j$ ,
- the constraints will be satisfied by the capacity rule. □

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# Cost of the algorithm

- $\mathcal{N}$  has  $N = n + m + 2$  vertices and  $E = n + m + nm$  edges
- $L = \sum_e \ell(e) \leq nm$ .
- Obtain  $\mathcal{N}$  and extract the information from the circulation has cost  $O(nm)$ .
- FF analysis, the cost of obtaining a circulator  $O(L(N + M)) = O(n^2m^2)$ .
- EK analysis, the cost of obtaining a circulator  $O(NM(N + M)) = O((n + m)n^2m^2)$ .
- The algorithm has cost  $O(n^2m^2)$ .

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# Joint rounding

Consider a matrix  $A = (a_{ij})$  with dimension  $n \times n$ , where each  $a_{ij} \in \mathbb{R}^+ \cup \{0\}$  and furthermore the sum of each row/column is an integer value. We want to round, if possible, each value  $a_{ij}$  to  $\lfloor a_{ij} \rfloor$  or  $\lceil a_{ij} \rceil$  without changing the value of the sum per row/column.

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$$\begin{pmatrix} 10.9 & 2.5 & 1.3 & 9.3 \\ 3.8 & 9.2 & 2.2 & 11.8 \\ 7.9 & 5.2 & 7.3 & 0.6 \\ 3.4 & 13.1 & 1.2 & 6.3 \end{pmatrix} F \rightarrow \begin{pmatrix} 11 & 3 & 1 & 9 \\ 4 & 9 & 2 & 12 \\ 7 & 5 & 8 & 1 \\ 4 & 13 & 2 & 6 \end{pmatrix}$$

Such a rounding is called a **joint rounding**.

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# Joint rounding

Note that:

- The elements in  $A$  that are integers cannot be modified.
- Let  $r_i = \sum_{j=1}^n (a_{ij} - \lfloor a_{ij} \rfloor)$  and  $c_j = \sum_{i=1}^n (a_{ij} - \lfloor a_{ij} \rfloor)$

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- As the rows/columns of  $A$  add up to an integer,  $r_i$  and  $c_j$  are integer values.
- Furthermore,  $\sum_i r_i = \sum_j c_j$ .

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In order to solve the problem we perform a reduction from this problem to a circulation problem.

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In order to solve the problem we perform a reduction from this problem to a circulation problem.

One unit of flow can be seen as indicating that a decimal part rounded up to 1.

Zero flow indicates that a decimal part is rounded down to 0.

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Build the network with demands  $\mathcal{N} = (V, E, c, d)$  where:

- **Vertices:**  $V = \{x_i, y_i | 1 \leq i \leq n\}$ .  $x$ 's vertices represent rows and  $y$  vertices columns.
- **Edges:**  $E = \{(x_i, y_j) | 1 \leq i, j \leq n \text{ i } a_{i,j} \notin \mathbb{Z}\}$
- **Capacities:**  $c(x_i, y_j) = 1$ .
- **Demands:**  $d(x_i) = -r_i$ ,  $1 \leq i \leq n$ , i  $d(y_j) = c_j$ ,  $1 \leq j \leq n$ .

$\mathcal{N}$  has  $O(n)$  vertices and  $O(n^2)$  edges.

If there is a joint rounding of  $A$ ,  $\mathcal{N}$  has a circulation with integer values in  $\{0, 1\}$ .

- Let  $B$  a joint rounding of  $A$ , we define a new matrix  $D$  where

$$d_{ij} = \begin{cases} 1 & \text{if } b_{ij} > a_{ij} \\ 0 & \text{otherwise} \end{cases}$$

- As  $B$  is a joint rounding,  $\sum_j d_{ij} = r_i$  i  $\sum_i d_{ij} = c_j$ .
- Therefore, the flow assignment  $f(i, j) = d_{ij}$  is a circulation in  $\mathcal{N}$ .



If  $\mathcal{N}$  has a circulation with integer values in  $\{0, 1\}$ , there is a joint rounding of  $A$ .

- Let  $f$  be a circulation in  $\mathcal{N}$ ,
- we define matrix  $B$  as

$$b_{i,j} = \begin{cases} a_{ij} & \text{if } a_{ij} \in \mathbb{Z} \\ \lceil a_{i,j} \rceil & \text{if } a_{ij} \notin \mathbb{Z} \text{ i } f(i,j) = 1 \\ \lfloor a_{i,j} \rfloor & \text{otherwise} \end{cases}$$

- As  $f$  is a circulation,  $\sum_j b_{ij} = \sum_j a_{ij}$  and  $\sum_i b_{ij} = \sum_i a_{ij}$ .
- Therefore,  $B$  is a joint rounding of  $A$ .

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The construction of  $\mathcal{N}$  has cost  $O(n^2)$ .

Ford-Fulkerson algorithm requires  $O(D|E|)$ , where  $D$  is the sum of the positive demands, i.e.,  $D = \sum r_i = O(n^2)$ . As  $|E| = O(n^2)$ , the total cost is  $O(n^4)$ .

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# Flow Network with costs

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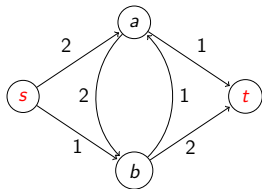
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A network  $\mathcal{N} = (V, E, c, \$, s, t)$  is formed by

- a digraph  $G = (V, E)$ ,
- a source vertex  $s \in V$
- a sink vertex  $t \in V$ ,
- and edge capacities  $c : E \rightarrow \mathbb{R}^+$
- and unit flow cost  $\$ : E \rightarrow \mathbb{R}^+$



$e$	$\$$	$e$	$\$$
$(s, a)$	0.2	$(s, b)$	0.1
$(a, b)$	0.1	$(a, t)$	0.1
$(b, a)$	0.5	$(b, t)$	0.2

# A flow in a network

Given a network  $\mathcal{N} = (V, E, c, s, t)$

A **Flow** is an assignment  $f : E \rightarrow \mathbb{R}^+ \cup \{0\}$  that follows the **Kirchoff's laws**:

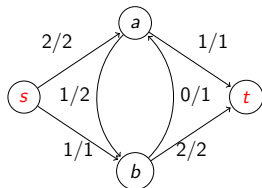
- $\forall (u, v) \in E, 0 \leq f(u, v) \leq c(u, v),$
- (Flow conservation)  $\forall v \in V - \{s, t\},$   
 $\sum_{u \in V} f(u, v) = \sum_{z \in V} f(v, z)$

The **value of a flow**  $f$  is

$$|f| = \sum_{v \in V} f(s, v) = f(s, V) = f(V, t).$$

The **cost of a flow**  $f$  is

$$\$(f) = \sum_{e \in E} \$(e) f(e).$$



$e$	$\$$	$e$	$\$$
$(s, a)$	0.2	$(s, b)$	0.1
$(a, b)$	0.1	$(a, t)$	0.1
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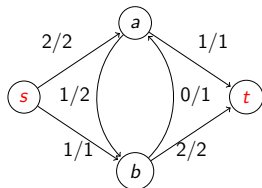
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e	\$	e	\$
(s, a)	0.2	(s, b)	0.1
(a, b)	0.1	(a, t)	0.1
(b, a)	0.5	(b, t)	0.2

$$\$(f) = 0.4 + 0.1 + 0.1 + 0.1 + 0.4 = 1.1$$

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# The Min cost Maximum flow problem

INPUT: A network flow with costs  $\mathcal{N} = (V, E, c, \$, s, t,)$   
QUESTION: Find a flow of maximum value on  $\mathcal{N}$  having minimum cost.

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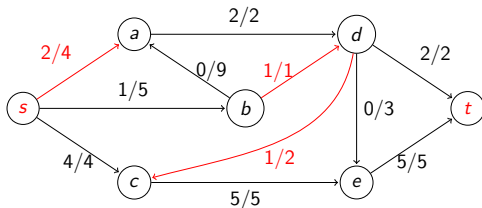
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Red edges have unit cost 0.5 and all others unit cost 0.1

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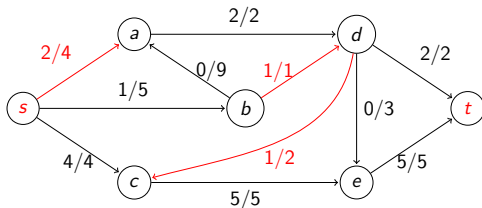
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$$|f| = 7 \text{ (it is maximum)}$$

$$\$(f) = 1.9 + 2 = 3.9$$

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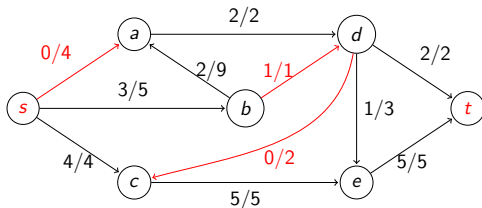
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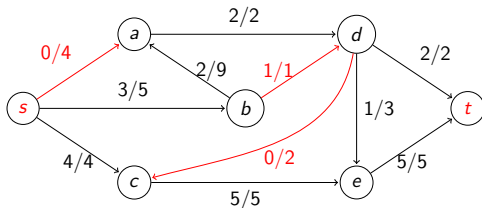
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Red edges have unit cost 0.5 and all others unit cost 0.1

$$|f| = 7 \text{ (it is maximum)}$$

$$\$(f) = 2.4 + 0.5 = 3.4$$

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# Flows and cycles in the residual graph

Given a network with costs  $\mathcal{N} = (V, E, s, \$, t, c)$  together with a flow  $f$  on it, the residual graph,  $(G_f = (V, E_f, c_f, \$_f))$  is a weighted digraph on the same vertex set and with edge set:

- if  $c(u, v) - f(u, v) > 0$ , then  $(u, v) \in E_f$  and  $c_f(u, v) = c(u, v) - f(u, v) > 0$  and  $$(u, v) = $(u, v)$  (forward edges)
- if  $f(u, v) > 0$ , then  $(v, u) \in E_f$  and  $c_f(v, u) = f(u, v)$  and  $$(v, u) = -$(u, v)$  (backward edges).

Let  $C$  be a (simple) cycle in  $G_f$ , the bottleneck,  $b(C)$ , is the minimum (residual) capacity of the edges in  $P$ .

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# Cycle redistribution

Let  $\mathcal{N} = (V, E, c, s, t)$  and let  $f$  be a flow in  $\mathcal{N}$ ,

**Redistribute**( $C, f$ )

$b = \text{bottleneck}(C)$

**for** each  $(u, v) \in C$  **do**

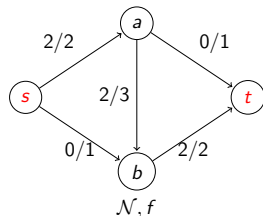
**if**  $(u, v)$  is a forward edge **then**

        Increase  $f(u, v)$  by  $b$

**else**

        Decrease  $f(v, u)$  by  $b$

**return**  $f$



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$b$  = bottleneck ( $C$ )

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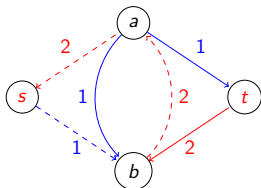
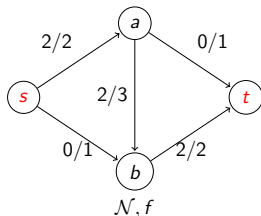
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**return**  $f$



$G_f, P = (s, a, t), b(P) = 1$

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$b = \text{bottleneck}(C)$

**for each**  $(u, v) \in C$  **do**

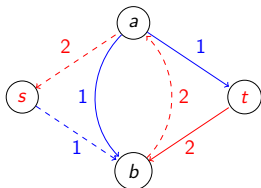
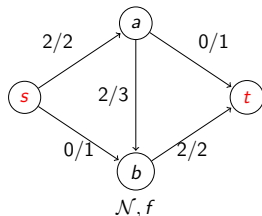
**if**  $(u, v)$  is a forward edge **then**

        Increase  $f(u, v)$  by  $b$

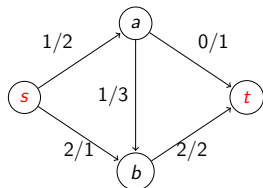
**else**

        Decrease  $f(v, u)$  by  $b$

**return**  $f$



$G_f, P = (s, a, t), b(P) = 1$



# Redistribute: flow conservation

## Lemma

Let  $f' = \text{Redistribute}(C, f)$ , then  $f'$  is a flow in  $\mathcal{N}$  and  $|f'| = |f|$ .

## Proof.

We have to prove the two flow properties.

- **Capacity law**

- Forward edges  $(u, v) \in P$ , we increase  $f(u, v)$  by  $b$ , as  $b \leq c(u, v) - f(u, v)$  then  $f'(u, v) = f(u, v) + b \leq c(u, v)$ .
- Backward edges  $(u, v) \in P$  we decrease  $f(v, u)$  by  $b$ , as  $b \leq f(v, u)$ ,  $f'(v, u) = f(v, u) - b \geq 0$ .



# Redistribute: flow conservation

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- **Conservation law**,  $\forall v \in P \setminus \{s, t\}$  let  $u$  be the predecessor of  $v$  in  $P$  and let  $w$  be its successor.
- As the cycle is simple only the alterations due to  $(u, v)$  and  $(v, w)$  can change the flow that goes through  $v$ . As we did for the algorithm Augment, for augmenting path, a case by case analysis shows that the conservation law is preserved.

# Redistribute: flow conservation

Now we have to prove that the value of the flow does not change. We have two cases:

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# Redistribute: flow conservation

Now we have to prove that the value of the flow does not change. We have two cases:

- $s \notin C$ . As Redistribute only changes the in/out flow of the vertices in  $C$ , the flow out of  $s$  is not changed. Therefore,  $|f'| = |f|$ .

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# Redistribute: flow conservation

Now we have to prove that the value of the flow does not change. We have two cases:

- $s \notin C$ . As Redistribute only changes the in/out flow of the vertices in  $C$ , the flow out of  $s$  is not changed. Therefore,  $|f'| = |f|$ .
- $s \in C$ . As  $s$  has not incoming edges, any cycle in  $G_f$  containing  $s$  must involve a backward edge entering  $s$  and a forward edge out of  $s$ . Therefore, we reduce by  $b$  the flow in one edge out of  $s$  and increase by  $b$  another such edges. Again,  $|f'| = |f|$ .

EndProof

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# Redistribute: cost

## Lemma

*Let  $f' = \text{Redistribute}(C, f)$ , then  $\$(f') = \$(f) + b\$(C)$ .*

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# Redistribute: cost

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## Lemma

Let  $f' = \text{Redistribute}(C, f)$ , then  $\$(f') = \$(f) + b\$(C)$ .

## Proof.

The changes done by Redistribute are (1) subtract  $b$  units of flow from the backward edges in  $C$  and (2) add  $b$  units of flow to the forward edges in  $C$ .

According to the definition of  $\$(C)$  the total change in cost is given by  $b\$(C)$ . □

# Min cost Max Flow

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## Theorem

*$f$  is a minimum cost maximum flow for  $\mathcal{N} = (V, E, c, s, t)$  iff  $f$  is a maximum flow in  $\mathcal{N} = (V, E, c, s, t)$  and the residual graph  $G_f$  has no negative cost cycles.*

## Proof.

- If there is a negative cost cycle  $C$ ,  $f$  has maximum value but not minimum cost as  $\text{Redistribute}(f, C)$  will provide a flow with maximum value and smaller cost.

# Min cost Max Flow

- If there is no negative cycle,
  - We can compute the shortest distance  $\delta(v)$  from  $s$  to every node  $v$  in  $G_f$  according to edge weight  $\$f$ .
  - As we have seen when discussing Johnson's algorithm, under the reduced cost  $c(v, w) = \$f(v, w) + \delta(v) - \delta(w)$ , all edges in  $G_f$  have non-negative costs. This means that any change in  $f$ , cannot decrease the reduced cost of  $f$ .
  - By the path/cycle invariant of the reduced cost, any change in  $f$  cannot decrease its cost.

EndProof.

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# Cycle-canceling algorithm

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# Cycle-canceling algorithm

Note that when  $f' = \text{Redistribute}(F, C)$  for some negative cost cycle  $C$  in  $G_f$ ,  $C$  does not form in  $G_{f'}$ .

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# Cycle-canceling algorithm

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Morton Klein, *A Primal Method for Minimal Cost Flows with Applications to the Assignment and Transportation Problems*, Management Science, INFORMS, vol. 14(3), pages 205-220, November 1967.

```
Cycle Canceling( $G, s, t, c, \$$ )  
 $f = \text{MaxFlow}(G, s, t)$   
Compute  $G_f$   
while there is a negative cost  
cycle  $C$  in  $G_f$  do  
     $f = \text{Redistribute}(f, C, G_f)$   
    Compute  $G_f$   
return  $f$ 
```

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# Networks with integer capacities

Using the same arguments as for the Ford Fulkerson algorithm.

## Lemma (**Integrity invariant**)

Let  $\mathcal{N} = (V, E, c, \$, s, t)$  where  $c : E \rightarrow \mathbb{Z}^+$ . At every iteration of the Cycle Canceling algorithm, the flow values  $f(e)$  are integers.

## Theorem (**Integrity theorem**)

Let  $\mathcal{N} = (V, E, c, \$, s, t)$  where  $c : E \rightarrow \mathbb{Z}^+$ . There exists a min cost max-flow  $f^*$  such that  $f^*(e)$  is an integer, for any  $e \in E$ .

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# Networks with integer capacities and costs

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## Lemma

Let  $\mathcal{N} = (V, E, c, \$, s, t)$  where  $c, \$ : E \rightarrow \mathbb{Z}^+$ . Let  $C$  be the min cut capacity, the Cycle Canceling algorithm terminates after finding at most  $C$  augmenting paths and after performing at most  $\$(C)$  redistribution calls.

## Proof.

The value of the flow increases by  $\geq 1$  after each augmentation and the cost of a maximum flow after a redistribute call decreases at least by 1. □

# Networks with integer capacities and costs: running time

- Computing the MaxFlow  $f^*$  takes  $O(|f^*|(n + m))$ .

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# Networks with integer capacities and costs: running time

- Computing the MaxFlow  $f^*$  takes  $O(|f^*|(n + m))$ .
- For the second part of the Cycle Canceling algorithm:

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  - Constructing  $G_f$ , takes  $O(m)$  time.
  - $O(nm)$  time to decide if  $G_f$  has a negative cycle and if so computing one (use Bellman-Ford algorithm).
  - A call to Redistribute requires  $O(m)$  steps

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- Let  $C = \max_{e \in E} c(e)$  and  $K = \max_{e \in E} \$(e)$ ,  
 $\$(f^*) \leq CKm$

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- Let  $C = \max_{e \in E} c(e)$  and  $K = \max_{e \in E} \$(e)$ ,  
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- Total running time is  $O(|f^*|(n + m) + CKnm)$

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- Total running time is  $O(|f^*|(n + m) + CKnm)$
- Thus, we have a pseudo polynomial algorithm.

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# Improving the cost

- Like Ford-Fulkerson algorithm, more careful choices of which cycle to cancel lead to more efficient algorithms.

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# Improving the cost

- Like Ford-Fulkerson algorithm, more careful choices of which cycle to cancel lead to more efficient algorithms.
- In 1980, Goldberg and Tarjan developed an algorithm that cancels the **minimum-mean cycle**, the cycle whose average cost per edge is smallest. A clever implementation of the algorithm achieves running time  $O(nm^2 \log V)$

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- The , **Chen, Kyng, Liu, Peng, Gutenber, Sachdeva (2022)**  $O(m^{1+o(1)})$  solves also the Min cost Maximum flow algorithm.

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# Min cost circulations

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- The Cycle Canceling algorithm can be extended to compute min cost circulations in flow networks with demands and lower bounds, provided a circulation exists.
- The algorithms, have the same asymptotic cost as the ones for the minimum cost maximum flow problem