Max-flow and min-cut problems

Dinic and Edmonds-Karp algorithm

J.Edmonds, R. Karp: *Theoretical improvements in algorithmic efficiency for network flow problems*. Journal ACM 1972.

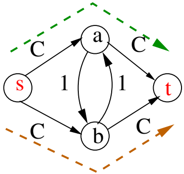
Yefim Dinic: *Algorithm for solution of a problem of maximum flow in a network with power estimation*. Doklady Ak.N. 1970

Choosing a good augmenting path can lead to a faster algorithm.
Use BFS to find an augmenting paths in G_f.

Edmonds-Karp algorithm

FF algorithm but using BFS: choose the augmenting path in G_f with the smallest length (number of edges).

Edmonds-Karp(G, c, s, t)
For all $e = (u, v) \in E$ let $f(u, v) = 0$
$G_f = G$
while there is an $s \rightsquigarrow t$ path in G_f
do
  $P = \text{BFS}(G_f, s, t)$
  $f = \text{Augment}(f, P)$
  Compute G_f
return f



The BFS in EK will choose: → or →

BFS paths on G_f

For $\mathcal{N} = (V, E, c, s, t)$ and a flow f in $\mathcal{N}$, assuming that G_f has an augmenting path, let f' be the next flow after executing one step of the EK algorithm.

- The path from s to t in a BFS traversal starting at s, is a path $s \rightsquigarrow t$ with minimum number of edges, i.e., a shortest length path.
- For $\in V$, let $\delta_f(s, v)$ denote length of a shortest length path from s to v in G_f.

Some properties of G_f and $G_{f'}$

How can we have $(u, v) \in E_{f'}$ but $(u, v) \notin E_f$?

- (u, v) is a forward edge saturated in f and not in f''.
- (u, v) is a backward edge in G_f and $f(v, u) = 0$

In any of the two cases, the augmentation must have modified the flow from v to u, so (u, v) must form part of the augmenting path.

EK and the shortest length distances

Lemma

If the EK-algorithm runs on $\mathcal{N} = (V, E, c, s, t)$, for all vertices $v \neq s$, $\delta_f(s, v)$ increases monotonically with each flow augmentation.

Proof. By contradiction.

Let f be the first flow such that, for some $u \neq s$,

$$\delta_{f'}(s, u) < \delta_f(s, u).$$

EK and the shortest length distances

Proof (cont)

Let v be the vertex with the minimum $\delta_{f'}(s, v)$ whose distance was decreased.

- Let $P : s \rightsquigarrow u \rightarrow v$ be a shortest length path from s to v in $G_{f'}$
- Then, $\delta_{f'}(s, v) = \delta_{f'}(s, u) + 1$ and $\delta_{f'}(s, u) \geq \delta_f(s, u)$.
- If $(u, v) \in E_f$,
  $\delta_f(s, v) \leq \delta_f(s, u) + 1 \leq \delta_{f'}(s, u) + 1 = \delta_{f'}(s, v)$
- So, $(u, v) \notin E_f$

EK and the shortest length distances

Proof (cont)

How can we have ?

- $(u, v) \in E_{f'}$ but $(u, v) \notin E_f$
- If so, (v, u) appears in the augmenting path.
- Then, the shortest length path from s to u in G_f has (v, u) as it last edge.
  $\delta_f(s, v) \leq \delta_f(s, u) - 1 \leq \delta_{f'}(s, u) - 1 = \delta_{f'}(s, v) - 1 - 1$
- which contradicts $\delta_{f'}(s, v) < \delta_f(u, v)$. $\square$

Some properties of G_f and $G_{f'}$

Let P be an augmenting path in G_f.

$(u, v) \in P$ is critical if $b(P) = c_f(u, v)$.

Critical edges do not appear in $G_{f'}$.

- (u, v) forward, $f'(u, v) = c(u, v)$
- (u, v) backward, $f'(v, u) = 0$

EK and critical edges

Lemma

In the EK algorithm, each one of the edges can become critical at most $|V|/2$ times.

Proof:

- Let $(u, v) \in E$, when (u, v) is critical for the first time, $\delta_f(s, v) = \delta_f(s, u) + 1$
- After this step (u, v) disappears from the residual graph until after the flow in (u, v) changes.
- At this point, (v, u) forms part of the augmenting path in $G_{f'}$, and $\delta_{f'}(s, u) = \delta_{f'}(s, v) + 1$,

$$\delta_{f'}(s, u) = \delta_{f'}(s, v) + 1 \geq \delta_f(s, v) + 1 \geq \delta_f(s, u) + 2$$

- So, the distance has increased by at least 2. $\square$

Complexity of Edmonds-Karp algorithm

Theorem

The EK algorithms runs in $O(mn(n+m))$ steps. Therefore it is a polynomial time algorithm.

Proof:

- Need time $O(m+n)$ to find the augmenting path using BFS.
- By the previous Lemma, there are $O(mn)$ augmentations. $\square$

Finding a min-cut

Given (G, s, t, c) to find a min-cut:

1. Compute the max-flow f^* in G.
2. Obtain G_{f^*}.
3. Find the set $S = \{v \in V | s \rightsquigarrow v\}$ in G_{f^*}.
4. Output the cut
$(S, V - \{S\}) = \{(v, u) | v \in S and u \in V - \{S\}\}$ in G.

The running time is the same than the algorithm to find the max-flow.

The max-flow problems: History

- Ford-Fulkerson (1956) $O(mC)$, where C is the max flow val.
- Dinic (1970) (blocking flow) $O(n^2 m)$
- Edmond-Karp (1972) (shortest augmenting path) $O(nm^2)$
- Karzanov (1974), $O(n^2 m)$ Goldberg-Tarjant (1986) (push re-label preflow + dynamic trees) $O(nm \lg(n^2/m))$ (uses parallel implementation)
- King-Rao-Tarjan (1998) $O(nm \log_{m/n \lg n} n)$.
- J. Orlin (2013) $O(nm)$ (clever follow up to KRT-98)
- Chen, Kyng, Liu, Peng, Gutenberg, Sachdeva (2022) $O(m^{1+o(1)})$ (polynomially bounded integral capacities) You can read Quanta Magazine article.

Final's Scheduling

We have as input:

- n courses, each one with a final. Each exam must be given in one room. Each course c_i has $E[i]$ students.
- r rooms. Each r_j has a capacity $S[j]$,
- τ time slots. For each room and time slot, we only can schedule one final.
- p professors to watch exams. Each exam needs one professor in each class and time. Each professor has its own restrictions of availability and no professor can oversee more than 6 finals. For each p_ℓ and τ_k define a Boolean variable $A[k, \ell] = T$ if p_ℓ is available at τ_k.

Design an efficient algorithm that correctly schedules a room, a time slot and a professor to every final, or report that not such schedule is possible.

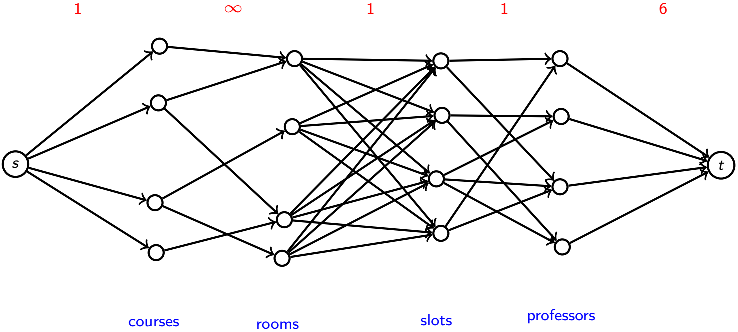
Construction of the network

Construct the network $\mathcal{N}$ with vertices $\{s, t, \{c_i\}, \{r_j\}, \{\tau_k\}, \{p_\ell\}\}$. Edges and capacities:

- (s, c_i) with capacity 1 (each course has one final)
- (c_i, r_j), if $E[i] \leq S[j]$, with capacity ∞ (can go to a room in which it fits)
- $\forall j, k$, (r_j, τ_k), with capacity 1 (one final per room and time slot).
- (τ_k, p_ℓ), if $A[k, \ell] = T$, capacity 1 (p can watch one final, if p is available at τ_k).
- (p_ℓ, t), capacity 6 (each p can watch ≤ 6 finals)

Notice that neither rooms nor time slots have individual restrictions.

Final's Scheduling: Flow Network



Final's Scheduling

- Notice the input size to the problem is $N = n + r + \tau + p + 2$. and size of the network is $O(N)$ vertices and $O(N^2)$ edges, why?
- Every path $s \rightsquigarrow t$ is an assignment of room-time-professor to a final, and any assignment room-time-professor to a final can be represented by a path $s \rightsquigarrow t$.
- Every integral flow identifies a collection of $|f|$ (s, t)-paths leading to a valid assignment for $|f|$ finals and viceversa.

FINAL'S SCHEDULING

- To maximize the number of finals to be given, we compute the max-flow f^* from s to t.
- If $|f^*| = n$, then we can schedule all finals, otherwise we can not.
- To recover the assignment we have to consider the edges with positive flow and extract assignment from the n (s, t)-paths
- Complexity:
  - To construct $\mathcal{N}$, we need $O(N^2)$.
  - As $|f^*| \leq n$ integral, we can use Ford-Fulkerson to compute f^*, with cost $O(nN^2)$.
  - The second part requires $O(N^2)$ time.
  - So, the cost of the algorithm is $O(nN^2) = O(n(n + r + \tau + p)^2)$.

Applications: Generalized assignment problems

- Consider a generalized assignment problem $\mathcal{GP}$ where, we have as input d finite sets $X_1, \ldots, X_d$, each representing a different set of resources.
- Our goal is to chose the "largest" number of d-tuples, each d-tuple containing exactly one element from each X_i, subject to the constrains:
  - For each $i \in [d]$, each $x \in X_i$ can appears in at most $c(x)$ selected tuples.
  - For each $i \in [d]$, any two $x \in X_i$ and $y \in X_{i+1}$ can appear in at most $c(x, y)$ selected tuples.
  - The values for $c(x)$ and $c(x, y)$ are either in $\mathbb{Z}^+$ or ∞.
- Notice that only pairs of objects between adjacent X_i and X_{i+1} are constrained.

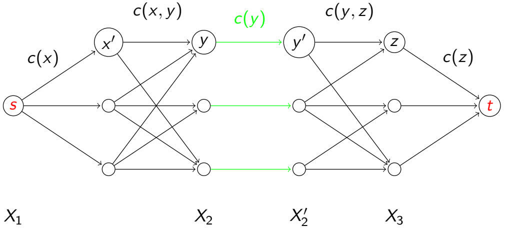
Applications: Generic reduction to Max-Flow

Make the reduction from $\mathcal{GP}$ to the following network $\mathcal{N}$:

- V contains a vertex x, for each element x in each X_i, and a copy x', for each element $x \in X_i$ for $1 \leq i < d$.
- We add vertex s and vertex t.
- Add an edge $s \to x$ for each $x \in X_1$ and add an edge $y \to t$ for every $y \in X_d$. Give capacities $c(s, x) = c(x)$ and $c(y, t) = c(y)$.
- Add an edge $x' \to y$ for every pair $x \in X_i$ and $y \in X_{i+1}$. Give a capacity $c(x, y)$. Omit the edges with capacity 0.
- For every $x \in X_i$ for $1 \leq i < d$, add an edge $x \to x'$ with $c(x, x') = c(x)$.

Every path $s \rightsquigarrow t$ in $\mathcal{N}$ identifies a feasible d-tuple, conversely every d-tuple determines a a path $s \rightsquigarrow t$.

Flow Network: The reduction



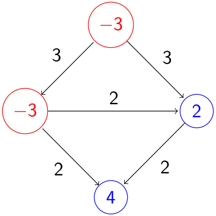
- To solve $\mathcal{GP}$, we construct $\mathcal{N}$, and then we find an integer maximum flow f^*.
- In the subgraph formed by edges with $f^*(e) > 0$, we find a (s,t) path P (a d-tuple), decrease in 1 the flow in each edge of P, remove edges with 0 flow.
- Repeat $|f^*|$ times. In this way we obtain a set of d-tuples with maximum size verifying all the restrictions.

Circulation with demands

- We introduce another flow problem, to deal with supply and demand inside a network.
- Instead of having a pair source/sink the new setting consider a producer/consumer scenario.
- Some nodes are able to produce a certain amount of flow.
- Some nodes are willing to consume flow.
- The question is whether it is possible to route “all” the produced flow to the consumers. When possible the flow assignment is called a circulation

Network with demands

A network with demands $\mathcal{N}$ is a tuple (V, E, c, d) where c assigns a positive capacity to each edge, and d is a function associating a demand $d(v)$, to $v \in V$.



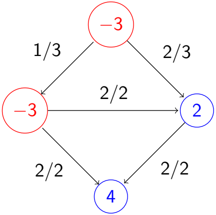
- When $d(v) > 0$, v can receive $d(v)$ units of flow more than it sends, v is a sink,.
- If $d(v) < 0$, v can send $d(v)$ units of flow more than it receives, v is a source.
- If $d(v) = 0$, v is neither a source or a sink.
- Define S to be the set of sources and T the set of sinks.

Network with demands: circulation

Given a network $\mathcal{N} = (V, E, c, d)$, a circulation is a flow assignment $f : E \to \mathbb{R}^+$ s.t.

1. capacity: For each $e \in E$, $0 \leq f(e) \leq c(e)$,
2. conservation: For each $v \in V$,

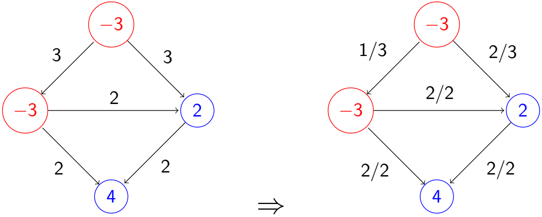
$$\sum_{(u,v)\in E} f(u, v) - \sum_{(v,z)\in E} f(v, z) = d(v).$$



Take into account that a circulation might not exist.

Network with demands: circulation problem

Circulation problem: Given $\mathcal{N} = (V, E, c, d)$ with $c > 0$, obtain a circulation provided it does exists.



A first conditions for a circulation to exists

If f is a circulation for $\mathcal{N} = (V, E, c, d)$,

$$\sum_{v \in V} d(v) = \sum_{v \in V} \left(\underbrace{\sum_{(u,v) \in E} f(u,v)}_{\text{edges to } v} - \underbrace{\sum_{(v,z) \in E} f(v,z)}_{\text{edges out of } v} \right).$$

For $e = (u, v) \in E$, $f(e)$ appears in the sum of edges to v and in the sum of edges out of u. Both terms cancel!

Then, $\sum_{v \in V} d(v) = 0$.

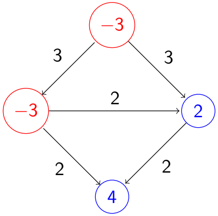
A first conditions for a circulation to exists

If there is a circulation, then $\sum_{v \in V} d(v) = 0$.

Recall that
$S = \{v \in V | d(v) < 0\}$ and
$T = \{v \in V | d(v) > 0\}$.

Define $D = -\sum_{v \in S} d(v) = \sum_{v \in T} d(v)$.

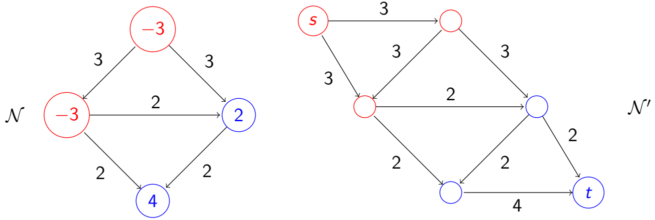
D is the total amount of extra flow that has to be transported from the sources to the sinks.



Circulation problem: reduction to Max-flow

From $\mathcal{N} = (V, E, c, d)$, define a flow network $\mathcal{N}' = (V', E', c', s, t)$:

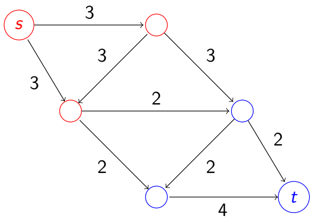
- $V' = V \cup \{s, t\}$, we add a source s and a sink t.
- For $v \in S$ $(d(v) < 0)$, add (s, v) with capacity $-d(v)$.
- For $v \in T$ $(d(v) > 0)$, add (v, t) with capacity $d(v)$.
- Keep E and, for $e \in E$, $c'(e) = c(e)$.



Circulation problem: reduction to Max-flow

1.- Every flow f' in $\mathcal{N}'$ verifies $|f'| \leq D$

The capacity $c'(\{s\}, V) = D$, by the capacity restriction on flows, $|f'| \leq D$.



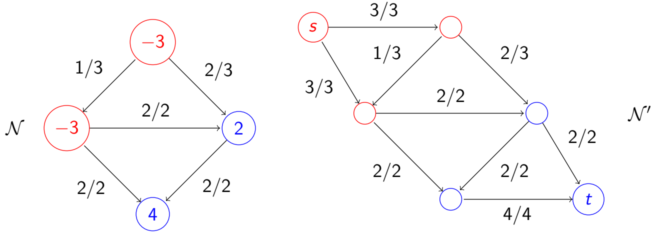
$\mathcal{N}'$

Circulation problem: reduction to Max-flow

2.- If there is a circulation f in $\mathcal{N}$, we have a max-flow f' in $\mathcal{N}'$ with $|f'| = D$.

Extend f to a flow f', assigning $f'(s, v) = -d(v)$, for $v \in S$, and $f'(u, t) = d(u)$, for $u \in T$.

By the circulation condition, f' is a flow in $\mathcal{N}'$. Furthermore, $|f'| = D$.

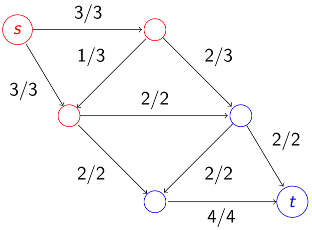


Analysis

3.- If there is a flow f' in $\mathcal{N}'$ with $|f'| = D$, $\mathcal{N}$ has a circulation

For $e \in E$, define $f(e) = f'(e)$.

- As $|f'| = D$, all edges $(s, v) \in E'$ and $(u, t) \in E'$ are saturated by f'.
- By flow conservation, f satisfies $d(v) = \underbrace{\sum_{(u,v)\in E} f(u, v)}_{\text{edges to } v} - \underbrace{\sum_{(v,z)\in E} f(v, z)}_{\text{edge.s out of } v}$.
- So, f is a circulation for $\mathcal{N}$.



Circulation: main results

From the previous discussion, we can conclude:

Theorem (Necessary and sufficient condition)

There is a circulation for $\mathcal{N} = (V, E, c, d)$ iff the maxflow in $\mathcal{N}'$ has value D.

Theorem (Circulation integrality theorem)

If all capacities and demands are integers, and there exists a circulation, then there exists an integer valued circulation.

Sketch Proof Max-flow formulation + integrality theorem for max-flow $\square$

Circulation: main results

Theorem

There is a polynomial time algorithm to solve the circulation problem.

The cost of the algorithm is the same as the cost of the algorithm used for the MaxFlow computation.

Theorem

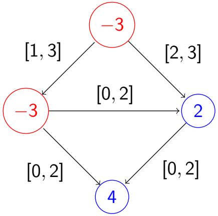
If all capacities and demands are integers, and there exists a circulation, then we can obtain an integer valued circulation in time $O(Dm)$.

Networks with demands and lower bounds

Generalization of the previous problem: besides satisfy demands at nodes, we want to force the flow to use certain edges.

Introduce a new constrain $\ell(e)$ on each $e \in E$, indicating the min-value the flow must be on e.

A network $\mathcal{N}$ with demands and lower bounds is a tuple (V, E, c, ℓ, d) with $c(e) \geq \ell(e) \geq 0$, for each $e \in E$,

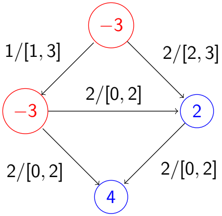


Networks with demands and lower bounds: circulation

Given a network $\mathcal{N} = (V, E, c, \ell, d)$ a circulation as a flow assignment $f : E \to \mathbb{R}^+$ s.t.

1. capacity: For each $e \in E$, $\ell(e) \leq f(e) \leq c(e)$,
2. conservation: For each $v \in V$,

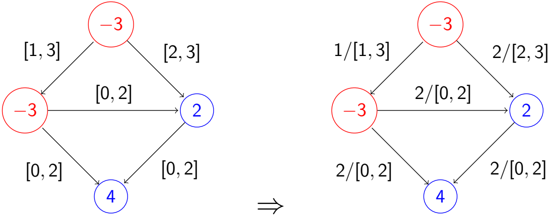
$$\sum_{(u,v)\in E} f(u,v) - \sum_{(v,z)\in E} f(v,z) = d(v).$$



A circulation might not exist.

Circulations with demands and lower bounds problem

Circulation with demands and lower bounds problem: Given $\mathcal{N} = (V, E, c, \ell, d)$, obtain a circulation for $\mathcal{N}$, provided it does exists



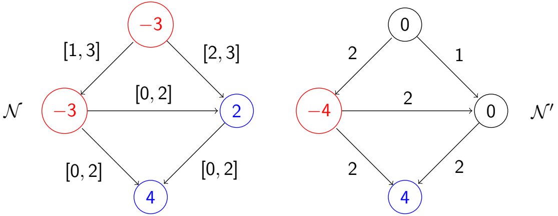
We devise an algorithm to the problem by a reduction to a circulation with demands problem.

Circulations with demands and lower bounds: the reduction

Let $\mathcal{N} = (V, E, c, \ell, d)$, construct a network $\mathcal{N}' = (V, E, c', d')$ with only demands as follows:

Initially set $c' = c$ and $d' = d$.
For each $e = (u, v) \in E$, with $\ell(e) > 0$:

- $c'(e) = c(e) - \ell(e)$.
- Update the demands on both ends of e: $d'(u) = d(u) + \ell(e)$ and $d'(v) = d(v) - \ell(e)$



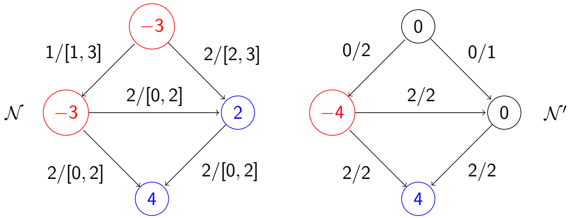
Circulations with demands and lower bounds: the reduction

1.- If f is a circulation in $\mathcal{N}$, $f'(e) = f(e) - \ell(e)$, for $e \in E$, is a circulation in $\mathcal{N}'$.

By construction of $\mathcal{N}'$, f' verifies the capacity constraint.

Besides, for (u, v) with $\ell(u, v) > 0$, the flow out of u and the flow in v is decreased by $\ell(u, v)$.

f is a circulation in $\mathcal{N}$ so, the flow imbalance of f' matches the demand d' at each node.



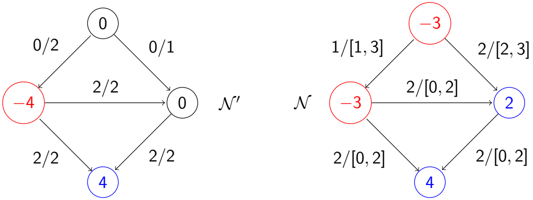
Circulations with demands and lower bounds: the reduction

2.- If f' is a circulation in $\mathcal{N}'$, $f(e) = f'(e) + \ell(e)$, for $e \in E$, is a circulation in $\mathcal{N}$.

f' verifies the capacity constraint $f'(e) \geq 0$, so $f'(e) \geq \ell(e)$.

f' is a circulation, the f' imbalance at u is $d'(u)$.

Therefore, for (u, v) with $\ell(u, v) > 0$, the increase of flow in (u, v) balances $\ell(u, v)$ units of flow out of u with $\ell(u, v)$ units of flow entering v. Thus the f imbalance at u is $d(u)$.



Main result

Theorem

There exists a circulation in $\mathcal{N}$ iff there exists a circulation in $\mathcal{N}'$. Moreover, if all demands, capacities and lower bounds in $\mathcal{N}$ are integers, and $\mathcal{N}$ admits a circulation, there is a circulation in $\mathcal{N}$ that is integer-valued.

The integer-valued circulation part is a consequence of the integer-value circulation Theorem for f' in G'.

Circulation with demands and lower bounds: main results

Theorem

There is a polynomial time algorithm to solve the circulation with demands and lower bounds problem.

The cost of the algorithm is the same as the cost of the algorithm used for the circulation with demands computation.

Theorem

If all capacities, lower bounds, and demands are integers, and there exists a circulation, then we can obtain an integer valued circulation in time $O((D+L)m)$ where L is the sum of al lower bounds.

Survey Design problem

Problem: Design a survey among customers of products (KT-7.8)

- Customer i can only be asked about a bought product and must receive a questionnaire for at least c_i such products, those values are determined as function of the purchased products.
- For each product j, we want to collect data from a minimum of p_j customers.
- The c and p values cannot exceed the number of

Survey Design problem

The input to the problem is:
A set C of n customers and a set P of m products.

- For each customer $i \in C$, a list of purchased products and the two values $c_i \leq c'_i$.
- For each product $j \in P$, two values p_j and p'_j.

Alternatively,

- The information about purchases can be represented as a bipartite graph $G = (C \cup P, E)$, where C is the set of customers and P is the set of products.
- $(i, j) \in E$ means $i \in C$ has purchased product $j \in P$.

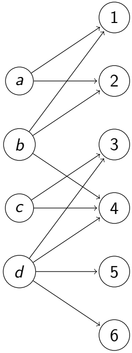
Survey Design: Input

Customers $C = \{a, b, c, d\}$
Products $P = \{1, 2, 3, 4, 5, 6\}$

Customer	Bought	c
a	1,2	1
b	1,2,4	1
c	3,6	1
d	3,4,5,6	2

Prod.	1	2	3	4	5	6
d	1	1	1	1	0	1

G



Survey Design: Circulation with lower bounds formulation

We construct a network $\mathcal{N} = (V', E', c, \ell)$ from G as follows:

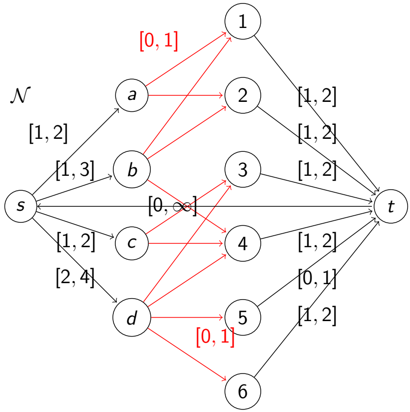
- Nodes: $V' = V \cup \{s, t\}$
- Edges: E' contains E and edges $s \to \{C\}$, $\{P\} \to t$, and (t, s).
- Capacities and lower bounds:
  - $c(t, s) = \infty$ and $\ell(t, s) = 0$
  - For $i \in C$, $\ell(s, i) = c_i$ and $c(s, i) =$ the number of purchased products.
  - For $j \in P$, $\ell(j, t) = p_j$ and $c(j, t) =$ number of customers that purchased j.
  - For $(i, j) \in E$, $c(i, j) = 1$, and $\ell(i, j) = 0$.

Survey Design: Circulation with lower bounds formulation

Customers $C = \{a, b, c, c\}$
Products $P = \{1, 2, 3, 4, 5, 6\}$

Customer	Bought	c
a	1,2	1
b	1,2,4	1
c	3,6	1
d	3,4,5,6	2

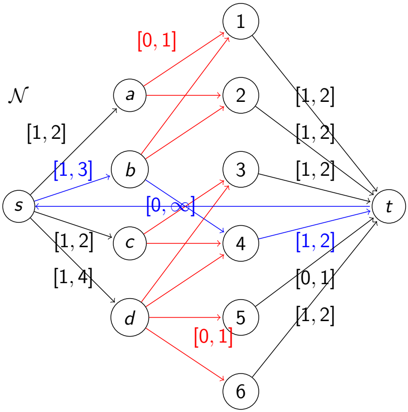
Prod.	1	2	3	4	5	6
d	1	1	1	1	0	1



Survey Design: Circulation interpretation

If f is a circulation in $\mathcal{N}$:

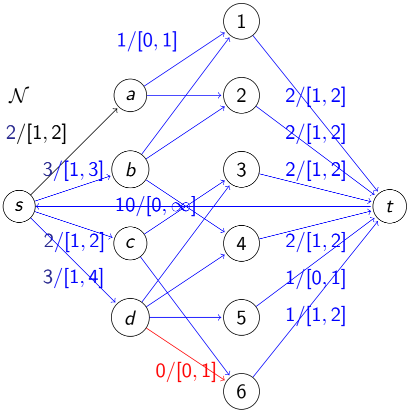
- one unit of flow circulates $s \to i \to j \to t \to s$.
- $f(i,j) = 1$ means ask i about j,
- $f(s,i)$ # products to ask i for opinion,
- $f(j,t) = \#$ customers to be asked to review j,
- $f(t,s)$ is the total number of questionnaires.



Survey Design: Circulation vs solutions

A solution

- Ask a about $1, 2$.
- Ask b about $1, 2, 4$.
- Ask c about $3, 6$.
- Ask d about $3, 4, 5$.



Main result

Theorem $\mathcal{N}$ has a circulation iff there is a feasible way to design the survey.

Proof if there is a feasible way to design the survey:

- if i is asked about j then $f(i,j) = 1$,
- $f(s,i) =$ number questions asked to i ($\geq c_i$).
- $f(j,t) =$ number of customers who were asked about j ($\geq d_j$),
- $f(t,s) =$ total number of questions.
- easy to verify that f is a circulation in $\mathcal{N}$

If there is an integral circulation in $\mathcal{N}$:

- if $f(i,j) = 1$ then i will be asked about j,
- the constrains will be satisfied by the capacity rule. $\square$

Cost of the algorithm

- $\mathcal{N}$ has $N = n + m + 2$ vertices and $E = n + m + nm$ edges
- $L = \sum_e \ell(e) \leq nm$.
- Obtain $\mathcal{N}$ and extract the information from the circulation has cost $O(nm)$.
- FF analysis, the cost of obtaining a circulaton $O(L(N + M)) = O(n^2 m^2)$.
- EK analysis, the cost of obtaining a circulaton $O(NM(N + M)) = O((n + m)n^2 m^2)$.
- The algorithm has cost $O(n^2 m^2)$.

Joint rounding

Consider a matrix $A = (a_{ij})$ with dimension $n \times n$, where each $a_{ij} \in \mathbb{R}^+ \cup \{0\}$ and furthermore the sum of each row/column is an integer value. We want to round, if possible, each value a_{ij} to $\lfloor a_{ij} \rfloor$ or $\lceil a_{ij} \rceil$ without changing the value of the sum per row/column.

$$\begin{pmatrix} 10.9 & 2.5 & 1.3 & 9.3 \\ 3.8 & 9.2 & 2.2 & 11.8 \\ 7.9 & 5.2 & 7.3 & 0.6 \\ 3.4 & 13.1 & 1.2 & 6.3 \end{pmatrix} F \quad \rightarrow \quad \begin{pmatrix} 11 & 3 & 1 & 9 \\ 4 & 9 & 2 & 12 \\ 7 & 5 & 8 & 1 \\ 4 & 13 & 2 & 6 \end{pmatrix}$$

Such a rounding is called a joint rounding.

Joint rounding

Note that:

- The elements in A that are integers cannot be modified.
- Let $r_i = \sum_{j=1}^{n}(a_{ij} - \lfloor a_{ij} \rfloor)$ and $c_j = \sum_{i=1}^{n}(a_{ij} - \lfloor a_{ij} \rfloor)$
- As the rows/columns of A add up to an integer, r_i and c_j are integer values.
- Furthermore, $\sum_i r_i = \sum_j c_j$.

In order to solve the problem we perform a reduction from this problem to a circulation problem.
One unit of flow can be seen as indicating that a decimal part rounded up to 1.
Zero flow indicates that a decimal part is rounded down to 0.

Build the network with demands $\mathcal{N} = (V, E, c, d)$ where:

- **Vertices:** $V = \{x_i, y_i | 1 \leq i \leq n\}$. x's vertices represent rows and y vertices columns.
- **Edges:** $E = \{(x_i, y_j) | 1 \leq i, j \leq n \text{ i } a_{i,j} \notin \mathbb{Z}\}$
- **Capacities:** $c(x_i, y_j) = 1$.
- **Demands:** $d(x_i) = -r_i$, $1 \leq i \leq n$, i $d(y_j) = c_j$, $1 \leq j \leq n$.

$\mathcal{N}$ has $O(n)$ vertices and $O(n^2)$ edges.

If there is a joint rounding of A, $\mathcal{N}$ has a circulation with integer values in $\{0,1\}$.

- Let B a joint rounding of A, we define a new matrix D where
$$d_{ij} = \begin{cases} 1 & \text{if } b_{ij} > a_{ij} \\ 0 & \text{otherwise} \end{cases}$$
- As B is a joint rounding, $\sum_j d_{ij} = r_i$ i $\sum_i d_{ij} = c_i$.
- Therefore, the flow assignment $f(i,j) = d_{ij}$ is a circulation in $\mathcal{N}$.

If $\mathcal{N}$ has a circulation with integer values in $\{0, 1\}$, there is a joint rounding of A.

- Let f be a circulation in $\mathcal{N}$,
- we define matrix B as

$$b_{i,j} = \begin{cases} a_{ij} & \text{if } a_{ij} \in \mathbb{Z} \\ \lceil a_{i,j} \rceil & \text{if } a_{ij} \notin \mathbb{Z} \text{ i } f(i,j) = 1 \\ \lfloor a_{i,j} \rfloor & \text{otherwise} \end{cases}$$

- As f is a circulation, $\sum_j b_{ij} = \sum_j a_{ij}$ and $\sum_i b_{ij} = \sum_i a_{ij}$.
- Therefore, B is a joint rounding of A.

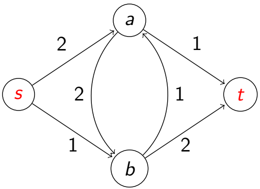
The construction of $\mathcal{N}$ has cost $O(n^2)$.

Ford-Fulkerson algorithm requires $O(D|E|)$, where D is the sum of the positive demands, i.e., $D = \sum r_i = O(n^2)$. As $|E| = O(n^2)$, the total cost is $O(n^4)$.

Flow Network with costs

A network $\mathcal{N} = (V, E, c, \$, s, t)$ is formed by

- a digraph $G = (V, E)$,
- a source vertex $s \in V$
- a sink vertex $t \in V$,
- and edge capacities $c : E \to \mathbb{R}^+$
- and unit flow cost $\$: E \to \mathbb{R}^+$



e	\$	e	\$
(s, a)	0.2	(s, b)	0.1
(a, b)	0.1	(a, t)	0.1
(b, a)	0.5	(b, t)	0.2

A flow in a network

Given a network $\mathcal{N} = (V, E, c, s, t)$
A Flow is an assignment $f : E \to \mathbb{R}^+ \cup \{0\}$ that follows the Kirchoff's laws:

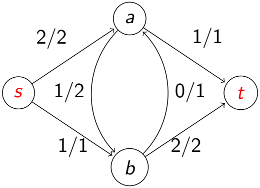
- $\forall (u, v) \in E$, $0 \le f(u, v) \le c(u, v)$,
- (Flow conservation) $\forall v \in V - \{s, t\}$, $\sum_{u \in V} f(u, v) = \sum_{z \in V} f(v, z)$

The value of a flow f is

$$|f| = \sum_{v \in V} f(s, v) = f(s, V) = f(V, t).$$

The cost of a flow f is

$$\$(f) = \sum_{e \in E} \$(e) f(e).$$

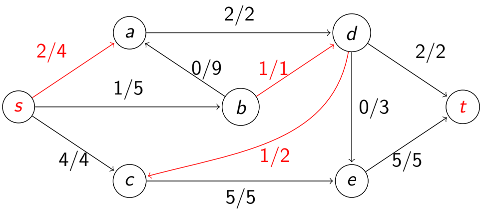


e	\$	e	\$
(s, a)	0.2	(s, b)	0.1
(a, b)	0.1	(a, t)	0.1
(b, a)	0.5	(b, t)	0.2

$\$(f) = 0.4 + 0.1 + 0.1 + 0.1 + 0.4 = 1.1$

The Min cost Maximum flow problem

INPUT: A network flow with costs $\mathcal{N} = (V, E, c, \$, s, t,)$
QUESTION: Find a flow of maximum value on $\mathcal{N}$ having minimum cost.

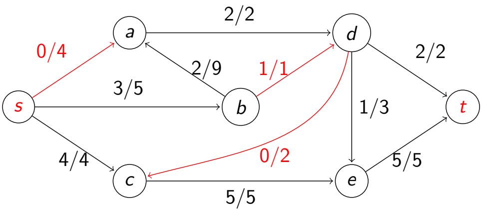


Red edges have unit cost 0.5 and all others unit cost 0.1

$$|f| = 7 \text{ (it is maximum)}$$
$$\$(f) = 1.9 + 2 = 3.9$$

The Min cost Maximum flow problem

INPUT: A network flow with costs $\mathcal{N} = (V, E, c, \$, s, t,)$
QUESTION: Find a flow of maximum value on $\mathcal{N}$ having minimum cost.



Red edges have unit cost 0.5 and all others unit cost 0.1

$|f| = 7$ (it is maximum)
$\$(f) = 2.4 + 0.5 = 3.4$

Flows and cycles in the residual graph

Given a network with costs $\mathcal{N} = (V, E, s, \$, t, c)$ together with a flow f on it, the residual graph, $(G_f = (V, E_f, c_f, \$_f)$ is a weighted digraph on the same vertex set and with edge set:

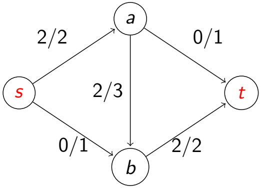
- if $c(u,v) - f(u,v) > 0$, then $(u,v) \in E_f$ and $c_f(u,v) = c(u,v) - f(u,v) > 0$ and $\$(u,v) = \$(u,v)$ (forward edges)
- if $f(u,v) > 0$, then $(v,u) \in E_f$ and $c_f(v,u) = f(u,v)$ and $\$(v,u) = -\$(u,v)$ (backward edges).

Let C be a (simple) cycle in G_f, the bottleneck, $b(C)$, is the minimum (residual) capacity of the edges in P.

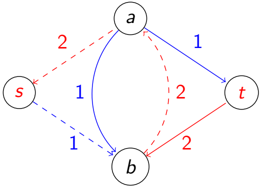
Cycle redistribution

Let $\mathcal{N} = (V, E, c, s, \$, t)$ and let f be a flow in $\mathcal{N}$,

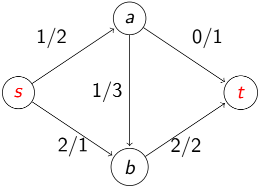
Redistribute(C, f)
b=bottleneck (C)
for each $(u, v) \in C$ **do**
  if (u, v) is a forward edge **then**
    Increase $f(u, v)$ by b
  else
    Decrease $f(v, u)$ by b
return f



$\mathcal{N}, f$



$G_f,\ P = (s, a, t),\ b(P) = 1$



$\mathcal{N}, f'$

Redistribute: flow conservation

Lemma

Let f' =Redistribute(C, f), then f' is a flow in $\mathcal{N}$ and $|f'| = |f|$.

Proof.

We have to prove the two flow properties.

- Capacity law
  - Forward edges $(u, v) \in P$, we increase $f(u, v)$ by b, as $b \leq c(u, v) - f(u, v)$ then $f'(u, v) = f(u, v) + b \leq c(u, v)$.
  - Backward edges $(u, v) \in P$ we decrease $f(v, u)$ by b, as $b \leq f(v, u), f'(v, u) = f(u, v) - b \geq 0$.

Redistribute: flow conservation

- Conservation law, $\forall v \in P \setminus \{s, t\}$ let u be the predecessor of v in P and let w be its successor.
- As the cycle is simple only the alterations due to (u, v) and (v, w) can change the flow that goes trough v. As we did for the algorithm Augment, for augmenting path, a case by case analysis shows that the conservation law is preserved.

Redistribute: flow conservation

Now we have to prove that the value of the flow does not change. We have two cases:

- $s \notin C$. As Redistribute only changes the in/out flow of the vertices in C, the flow out of s is not changed. Therefore, $|f'| = |f|$.
- $s \in C$. As s has not incoming edges, any cycle in G_f containing s must involve a backward edge entering s and a forward edge out of s. Therefore, we reduce by b the flow in one edge out of s and increase by b another such edges. Again, $|f'| = |f|$.

EndProof

Redistribute: cost

Lemma

Let $f' =$Redistribute(C, f), then $\$(f') = \$(f) + b\,\$_f(C)$.

Proof.

The changes done by Redistribute are (1) subtract b units of flow from the backward edges in C and (2) add b units of flow to the forward edges in C.
According to the definition of $\$_f$ the total change in cost is given by $b\,\$_f(C)$. $\square$

Min cost Max Flow

Theorem

f is a minimum cost maximum flow for $\mathcal{N} = (V, E, c, s, \$, t)$ iff f is a maximum flow in $\mathcal{N} = (V, E, c, s, t)$ and the residual graph G_f has no negative cost cycles.

Proof.

- If there is a negative cost cycle C, f has maximum value but not minimum cost as Redistribute(f, C) will provide a flow with maximum value and smaller cost.

- If there is no negative cycle,
  - We can compute the shortest distance $\delta(v)$ from s to every node v in G_f according to edge weight $\$_f$.
  - As we have seen when discussing Johnson's algorithm, under the reduced cost $c(v,w) = \$_f(v,w) + \delta(v) - \delta(w)$, all edges in G_f have non-negative costs. This means that any change in f, cannot decrease the reduced cost of f.
  - By the path/cycle invariant of the reduced cost, any change in f cannot decrease its cost.

EndProof.

Cycle-canceling algorithm

Note that when $f' =$Redistribute(F, C) for some negative cost cycle C in G_f, C does not form in $G_{f'}$.

Morton Klein, *A Primal Method for Minimal Cost Flows with Applications to the Assignment and Transportation Problems*, Management Science, INFORMS, vol. 14(3), pages 205-220, November 1967.

Cycle Canceling$(G, s, t, c, \$)$
$f =$ MaxFlow(G, s, t)
Compute G_f
while there is a negative cost cycle C in G_f **do**
  $f =$ **Redistribute**(f, C, G_f)
  Compute G_f
return f

Networks with integer capacities

Using the same arguments as for the Ford Fulkerson algorithm.

Lemma (Integrality invariant)

Let $\mathcal{N} = (V, E, c, \$, s, t)$ where $c : E \to \mathbb{Z}^+$. At every iteration of the Cycle Canceling algorithm, the flow values $f(e)$ are integers.

Theorem (Integrality theorem)

Let $\mathcal{N} = (V, E, c, \$, s, t)$ where $c : E \to \mathbb{Z}^+$. There exists a min cost max-flow f^* such that $f^*(e)$ is an integer, for any $e \in E$.

Networks with integer capacities and costs

Lemma

Let $\mathcal{N} = (V, E, c, \$, s, t)$ where $c, \$: E \to \mathbb{Z}^+$. Let C be the min cut capacity, the Cycle Canceling algorithm terminates after finding at most C augmenting paths and after performing at most $\$(C)$ redistribution calls.

Proof.

The value of the flow increases by ≥ 1 after each augmentation and the cost of a maximum flow after a redistribute call decreases at least by 1. $\square$

Networks with integer capacities and costs: running time

- Computing the MaxFlow f^* takes $O(|f^*|(n+m))$.
- For the second part of the Cycle Canceling algorithm:
  - Constructing G_f, takes $O(m)$ time.
  - $O(nm)$ time to decide if G_f has a negative cycle and if so computing one (use Bellman-Ford algorithm).
  - A call to Redistribute requires $O(m)$ steps
- Let $C = \max_{e \in E} c(e)$ and $K = \max_{e \in E} \$(e)$, $\$(f^*) \leq CKm$
- Total running time is $O(|f^*|(n+m) + CKnm)$
- Thus, we have a pseudo polynomial algorithm.

Improving the cost

- Like Ford-Fulkerson algorithm, more careful choices of which cycle to cancel lead to more efficient algorithms.
- In 1980, Goldberg and Tarjan developed an algorithm that cancels the minimum-mean cycle, the cycle whose average cost per edge is smallest. A clever implementation of the algorithm achieves running time $O(nm^2 \log V)$
- Combining Edmonds-Karp algorithm with Goldberg and Tarjan's, we get a polynomial time algorithm solving the Min cost Maximum flow problem.
- The , Chen, Kyng, Liu, Peng, Gutenberg, Sachdeva (2022) $O(m^{1+o(1)})$ solves also the Min cost Maximum flow algorithm.

Min cost circulations

- The Cycle Canceling algorithm can be extended to compute min cost circulations in flow networks with demands and lower bounds, provided a circulation exists.
- The algorithms, have the same asymptotic cost as the ones for the minimum cost maximum flow problem