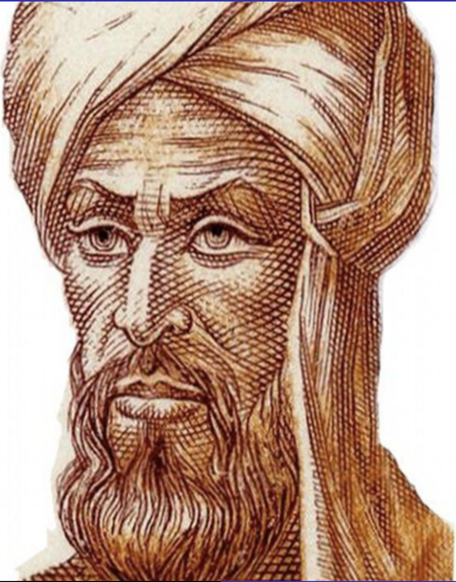
Algorithmics: Basic definitions and concepts



The Algorithmics course

Already known (EDA level)

- Algorithms cost and Asymptotic notation
- Sorting algorithms: Mergesort, Quicksort, ...
- Divide and conquer, recurrences, master theorem
- Complexity, P and NP, reductions
- Foundations on probability
- Basic data structures: Arrays, lists, stacks, queues, heaps, hashing ...
- Basics on graph theory, graph data structures
- Graph and digraph traversals (BFS, DFS) and applications.
- Backtracking algorithms

The Algorithmics course

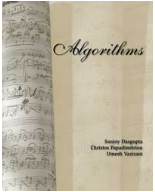
Topics to cover:

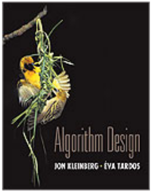
- Divide and conquer: Linear Selection
- Sorting in linear time (when? how?)
- Greedy algorithms
- Dynamic programming
- Distances in graphs
- Flow networks: problems, algorithms and applications
- Linear Programming
- Approximation algorithms
- Streaming algorithms

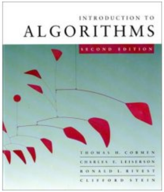
Provide models to solve real problems

References

Main references:







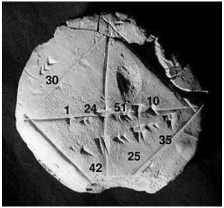
"The algorithmic lenses: C. Papadimitriou"

- In 1936 Alan Turing demonstrated the universality of computational principles with his mathematical model of the Turing machine.
- Theoretical Computer Science views computation as a ubiquitous phenomenon, not one that it is limited to computers.
- Algorithms themselves have evolved into a complex set of techniques, for instances self-learning, Web services, concurrent, distributed or parallel, etc... Each of them with ad-hoc relevant computational limitations and social implications.
- However, this course will be a course on classical algorithms, which are the core needed to understand more advanced computational material.

Algorithms

Algorithm: Precise recipe for a precise computational task. Each step of the process must be clear and unambiguous, and it should always yield a clear answer.

Sqrt (n)
$x_0 = 1$
for $i = 1$ to 6 **do**
  $x_i = (x_{i-1} + n/x_{i-1})/2$
end for



Babilònia (XVI BC)

For $n = 20$, x's are 1 10.5 6.2023 4.7134 4.4783

Once we designed an algorithm: What do we want to know?

- Correctness, it always does what it should?
- Performance,
  - computing time,
  - memory use
  - communication cost, ...

For an algorithm $\mathcal{A}$, $t_{\mathcal{A}}(x)$ is the computing time on input x.

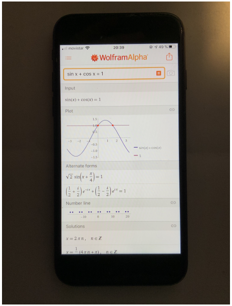
In this course, we use a worst case analysis: Given a problem, for which you designed an algorithm, you assume that your meanest adversary gives you the worst possible input.
We use as measure of time complexity or cost the function

$$T(n) = \max_{|x|=n} t_{\mathcal{A}}(x)$$

.

Time complexity

The time complexity must be independent of the "used" machine



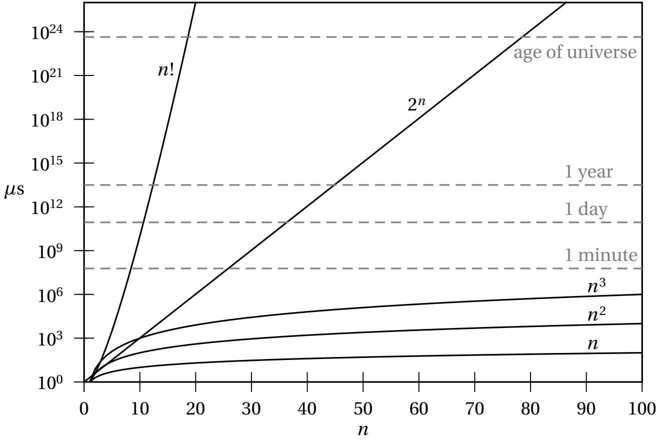
We must consider carefully how operations scale with respect to size.

Typical computation times

We study the behavior of $T(n)$ when n can take very large values (i.e., $n \to \infty$)

- if $n = 10$, $n^2 = 100$ and $2^n = 1024$;
- if $n = 100$, $n^2 = 10000$ and
  $$2^n = 1267650600228224401469670320537б;$$
- if $n = 10^3$, $n^2 = 10^6$ and 2^n is a number with 302 digits.
- As a comparison, 10^{64} is estimated to be the number of atoms in hearth ($< 2^{213}$).

Computation time assuming that an input with size $n = 1$ can be solved in 1 μsecond:



From: Moore-Mertens, The Nature of Computation

An example

- Assume that the input to an algorithm is an integer x that uses 64 bits.
- The cost of the algorithm is $O(x)$ and the time units are nanoseconds.
- Thus, processing this input takes more than 500 years

Note that the cost of this algorithm is a polynomial function on the input value not on the input size.
Such algorithms are classified as pseudopolynomial.

Efficient algorithms and practical algorithms

- We say that an algorithm is feasible if its cost is polynomial.
- However $n^{10^{10}}$ is a polynomial but this computing time could be prohibitive!
- In the same way, if we have cn^2 for constant $c = 10^{64}$, then c dominates inputs up to a size of $n > 10^{64}$.
- In this course, we will not enter in the analysis up to constants, but keep in mind that constants matter!!!!
- In practice, even a feasible algorithms with time complexity of for example n^4 could be too slow for $n \geq 1000$.

Asymptotic notation

Symbol	$L = \lim_{n\to\infty} \frac{f(n)}{g(n)}$	intuition
$f(n) = O(g(n))$	$L < \infty$	$f \leq g$
$f(n) = \Omega(g(n))$	$L > 0$	$f \geq g$
$f(n) = \Theta(g(n))$	$0 < L < \infty$	$f = g$
$f(n) = o(g(n))$	$L = 0$	$f < g$
$f(n) = \omega(g(n))$	$L = \infty$	$f > g$

Names used for specific function classes

Name	Definition
polylogarithmic	$f = O(\log^c n)$ (c constant)
polynomial	$f = O(n^c)$ (c constant) or $n^{O(1)}$
subexponential	$f = o(2^{n^\epsilon})$ $(0 < \epsilon < 1)$
exponential	$f = 2^{\mathrm{poly}(n)}$
double exponential	$f = 2^{\exp(n)}$

Notation:

$\lg \equiv \log_2$; $\ln \equiv \log_e$; $\log \equiv \log_{10}$.

Some math you should remember

Given an integer $n > 0$ and a real $a > 1$ and $a \neq 0$:

- Arithmetic summation: $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.
- Geometric summation: $\sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a}$.

Logarithms and Exponents: For $a, b, c \in \mathbb{R}^+$,

- $\log_b a = c \Leftrightarrow a = b^c \Rightarrow \log_b 1 = 0$
- $\log_b ac = \log_b a + \log_b c$, $\log_b a/c = \log_b a - \log_b c$.
- $\log_b a^c = c \log_b a \Rightarrow c^{\log_b a} = a^{\log_b c} \Rightarrow 2^{\log_2 n} = n$.
- $\log_b a = \log_c a / \log_c b \Rightarrow \log_b a = \Theta(\log_c a)$

Stirling: $n! = \sqrt{2\pi n}(n/e)^n + 0(1/n) + \gamma \Rightarrow n! + \omega((n/2)^n)$.

n-Harmonic: $H_n = \sum_{i=1}^{n} 1/i \sim \ln n$.

Graphs

See for ex. Chapter 3 of Dasgupta, Papadimitriou, Vazirani (DPV).

Graph: $G = (V, E)$, where V is the set of vertices, $n = |V|$, and $E \subset V \times V$ is the set of edges, $m = |E|$,

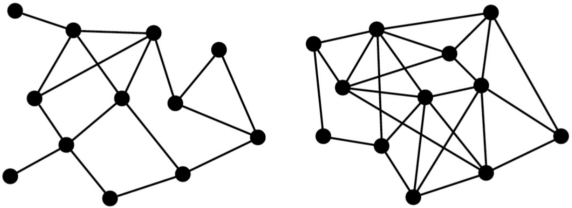
- Graphs: *undirected graphs (graphs)* and *directed graphs (digraphs)*
- The degree of v ($d(v)$) is the number of edges which are incident to v.
- A clique on n vertices (K_n) is a complete graph (with $m = n(n-1)/2$).
- A undirected G is said to be connected if there is a path between any two distinct vertices.
- If G is connected, then $m \geq n - 1$.

Directed graphs

- Edges are directed.
- The connectivity concept in digraphs is the so called strong connectivity: There is is a directed path between any two vertices.
- In a digraph $m \leq n(n-1)$.

Density of a graph

A G with n vertices is said to be dense when $m = \Theta(n^2)$.
When $m = o(n^2)$, G is said to be sparse.



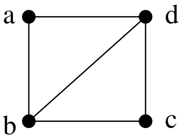
Common graph's data structures

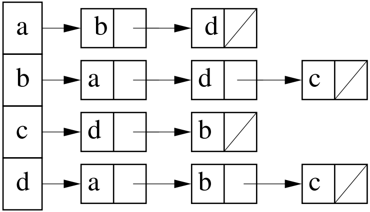
Let G be a graph with $V = \{1, 2, \ldots, n\}$.

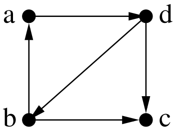
Adjacency list

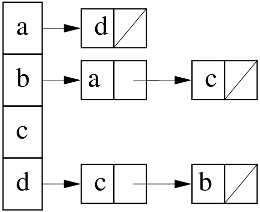
Adjacency matrix

Adjacency list





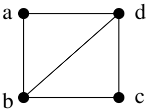




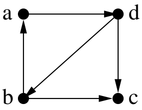
Adjacency matrix

Given G with $|V| = n$ define its adjacency matrix as the $n \times n$ matrix:

$$A[i,j] = \begin{cases} 1 & \text{if } (i,j) \in E, \\ 0 & \text{if } (i,j) \notin E. \end{cases}$$



$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$



$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Adjacency matrix

- If G is undirected, its adjacency matrix $A(G)$ is symmetric.
- If A is the adjacency matrix of G, then A^2 gives, for $i, j \in V$, whether there is a path between i and j in G, with length 2.
  For $k > 0$, A^k indicates if there is a path with length k from i to j.
- If G has weights on edges, i.e. $w_{i,j}$ for each $(i, j) \in E$, $A(G)$ keeps w_{ij} in position (i, j).
- Adjacency matrices allow the use of tools from linear algebra.

Comparison between the matrix and the list DS

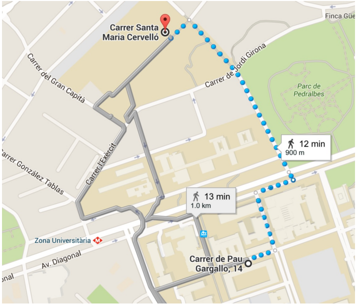
- The adjacency list uses a register per vertex and two per edge. As each register needs 64 bits, then the space to represent a graph is $\Theta(n+m)$.
- The use of the adjacency matrix needs n^2 bits ($\{0,1\}$), so for an unweighted graph G, we need $\Theta(n^2)$ bits.
- For weighted G, we need $64n^2$ bits (assuming weights are reasonably "small").
- In general, for unweighted dense graphs, the adjacency matrix is better, otherwise the adjacency list is a shorter representation.

Complexity issues between matrix and list DS

- Adding a new edge to G: In both data structures, we need $\Theta(1)$.
- Edge query, for u and v in $V(G)$, $(u, v) \in E(V)$?:
  For matrix representation: $\Theta(1)$.
  For list representation: $O(n)$.
- Explore all neighbours of vertex v:
  For matrix representation: $\Theta(n)$
  For list representation: $\Theta(|d(v)|)$
- Erase an edge in G: The same as Edge query.
- Erase a vertex in G:
  For matrix representation: $\Theta(n)$.
  For list representation: $O(m)$.

Searching a graph: Breadth First Search

1. Start with vertex v, visit v and all its neighbors.
2. Then, the non-visited neighbors of visited ones.
3. Repeat until all vertices are visited.



BFS use a QUEUE, (FIFO) to keep the neighbors of visited vertices.

Recall that vertices are labeled to avoid visiting them more than once.

Searching a graph: Depth First Search

explore

1. From current vertex, move to a neighbor.
2. Until you get stuck.
3. Then backtrack till new place to explore.

DFS use a STACK, (LIFO)

Time Complexity of DFS and BFS

For graphs given by adjacency lists: $O(|V| + |E|)$

For graphs given by adjacency matrix: $O(|V|^2)$

Therefore, both procedures can be implemented in linear time with respect to the size of the input graph.

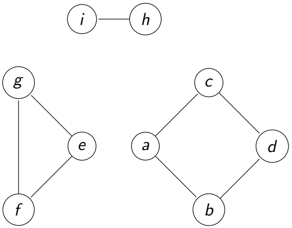
Connected components in undirected graphs

A connected component is a maximal connected subgraph of G.

A connected graph has a unique connected component.

Connected Components Problem
INPUT: undirected graph G
GOAL: Find all the connected components of G.

To find connected components in G use DFS/BFS and keep track of the set of vertices visited in each **explore** call.



The problem can be solved in $O(|V| + |E|)$.

Strongly connected components in a digraph

A digraph $G = (V, E)$, is *strongly connected*, if for all $u, v \in V$, there are paths $u \to v$ and $v \to u$.

A strongly connected component is a maximal strongly connected graph.

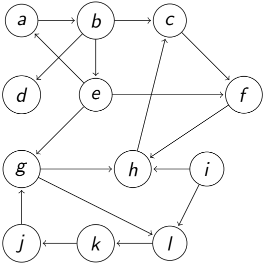
Strongly Connected Components Problem
INPUT: digraph G
GOAL: Find the strongly connected components of G.

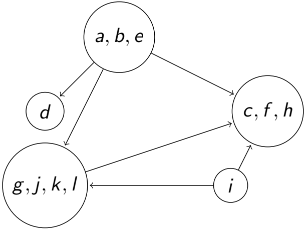
Kosharaju-Sharir's algorithm: Uses DFS (twice). Complexity $T(n) = O(|V| + |E|)$

Tarjan's algorithm: Using DFS (once). Complexity $T(n) = O(|V| + |E|)$

Strongly connected components in a digraph

A nice property: For every digraph, the graph on its strongly connected components is *acyclic*.





The classes P and NP

- Recall that a problem belong to the class P if there exists an algorithm that is polynomial in the worst-case analysis, (for the worst input given by a malicious adversary)
- A problem given in decisional form belong to the class NP non-deterministic polynomial time if, given a certificate of a solution, we can verify in polynomial time that indeed the certificate is a valid solution to the problem and those certificates have polynomial size
- It is easy to se that P$\subseteq$ NP, but it is an open problem to prove that P$=$NP or that P$\neq$NP.
- The class NP-complete is the class of most difficult problems in decisional form that are in NP. Most difficult in the sense that if one of them is proved to be in P then P$=$NP.

Beyond worst-case analysis

- Under the hypothesis that P$\neq$NP, if the decision version of a problem is NP-complete, then the optimization problem will require at least exponential time, for some inputs.
- The classification of a problem as NP-complete is a case of worst-case analysis, and for many problems the "expensive inputs" are few, and far from practical typical inputs. We will see some examples through the course.
- Therefore, there are alternative ways to get in practice, solutions for NP-complete problems, with the use of alternative algorithmic techniques, as approximation (we will see some examples), heuristics and self-learning algorithms, that are deferred to other courses.

A powerful tool to solve problems: Reductions

You have been introduced in previous courses to the concept of reduction between decision problems, to define the class NP-complete.

We have to extend the concept to function problems.

Reductions

Given problems A and B, assume we have an algorithm $\mathcal{A}_B$ to solve the problem B on any input y.

A polynomial time reduction $A \leq B$ is a pair of polynomial time functions (f, g) such that

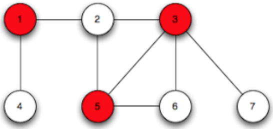
- f maps any input x to A, in polynomial time, to an input $f(x)$ to problem B in such a way that x has a valid solution for A iff $f(x)$ has a valid solution for B.
- g maps solutions to $f(x)$ into solutions to x.

Therefore if we have that $A \leq B$, as there is an algorithm $\mathcal{A}_B$ to solve problem B in polynomial time, then we have an algorithm $\mathcal{A}_A$, for any input x of A: Compute $g(\mathcal{A}_B(f(x)))$, that runs in polynomial time.

The VERTEX COVER problem

VERTEX COVER: Given a graph $G = (V, E)$ with $|V| = n, |E| = m$, find the minimum set of vertices $S \subseteq V$ such that it covers every edge of G.

Example:



The VERTEX COVER problem is known to be in NP-hard.

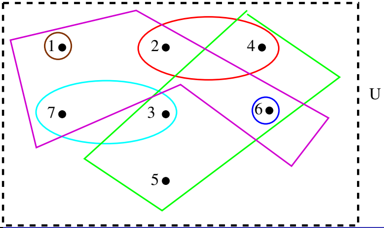
The SET COVER problem

SET COVER: Given a set U of m elements, a collection $S = \{S_1, \ldots, S_n\}$ where each $S_i \subseteq U$, select de minimum number of subsets in such a way that their union is equal to U.

There is a weighted version of the problem, but this simpler version already is NP-hard.

Example: Given $U = \{1, 2, 3, 4, 5, 6, 7\}$ $(m = 7)$, with $S_1 = \{3, 7\}$, $S_2 = \{2, 4\}$, $S_3 = \{3, 4, 5, 6\}$, $S_4 = \{6\}$, $S_5 = \{1\}$, $S_6 = \{1, 2, 6, 7\}$ $(n = 6)$.

Solution: $\{S_3, S_6\}$



Set Cover

The Vertex Cover problem is a special case of the Set Cover problem. As a model, the Set Cover has important practical applications.
To understand the computational complexity of Set Cover it is important to understand first the complexity of special cases as Vertex Cover.

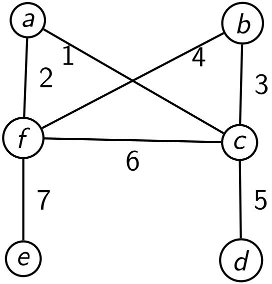
VERTEX COVER $\leq$ SET COVER

Given a input to VERTEX COVER, $G = (V, E)$ of size $|V| + |E| = n + m$, we want to construct in polynomial time on $n + m$ a specific input $f(G) = (U, S)$ to SET COVER such that if there exist a polynomial algorithm $\mathcal{A}$ to find a min set cover in G, then $\mathcal{A}(f(G))$ is an efficient algorithm to find an optimal solution to vertex cover.

REDUCTION f:

- Consider U as the set E of edges.
- For each vertex $i \in V$, S_i is the set of edges incident to i. Therefore $|S| = n$ and for each S_i, $|S_i| \leq m$.
- The cost of the reduction from G to (U, S) is $O(n + m)$

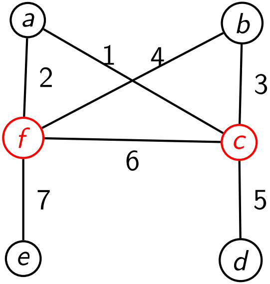
Example for the reduction



$\overbrace{\Rightarrow}^{f}$

$$\begin{aligned}
&U = \{1, 2, 3, 4, 5, 6, 7\} \\
&S = \{S_a, S_b, S_c, S_d, S_e, S_f\} \\
&S_a = \{1, 2\}, S_b = \{3, 4\}, \\
&S_c = \{1, 3, 5, 6\},\ S_d = \{5\}, \\
&S_e = \{7\},\ S_f = \{2, 4, 6, 7\}.
\end{aligned}$$

Example for the reduction



$\overset{f}{\overbrace{\Rightarrow}}$

$U = \{1, 2, 3, 4, 5, 6, 7\}$
$S = \{S_a, S_b, S_c, S_d, S_e, S_f\}$
$S_a = \{1, 2\}, S_b = \{3, 4\},$
$S_c = \{1, 3, 5, 6\}, S_d = \{5\},$
$S_e = \{7\}, S_f = \{2, 4, 6, 7\}.$

If there is an algorithm to solve the SET COVER, the same algorithm apply to $f(G) = (U, S)$ will yield a solution for VERTEX COVER on input G.

As VERTEX COVER is known to be NP-hard, this shows that SET COVER is also NP-hard.

The divide-and-conquer strategy.

1. Break the problem into smaller subproblems,
2. recursively solve each problem,
3. appropriately combine their answers.

Julius Caesar (I-BC)
"Divide et impera"

Known Examples:

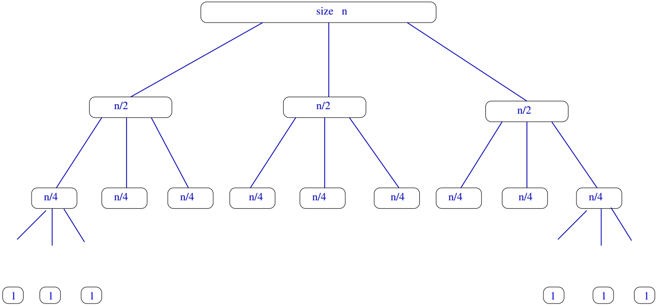
- Binary search
- Merge-sort
- Quicksort
- Strassen matrix multiplication

J. von Neumann (1903-57) Merge sort

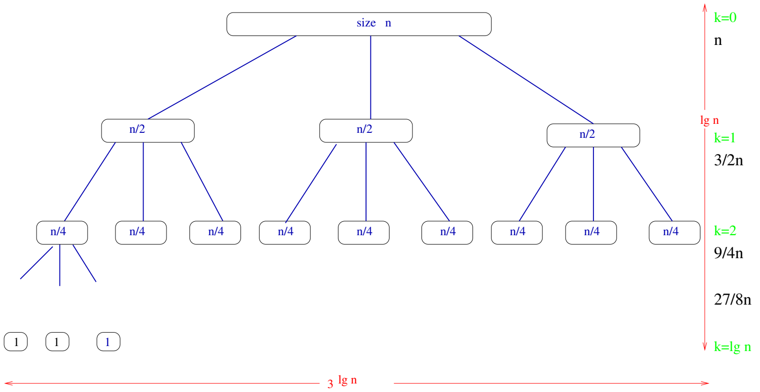
Recurrences Divide and Conquer

$T(n) = 3T(n/2) + O(n)$

The algorithm under analysis divides input of size n into 3 subproblems, each of size $n/2$, at a cost (of dividing and joining the solutions) of $O(n)$



$T(n) = 3T(n/2) + O(n)$.



T(n)=3T(n/2)+O(n)

At depth k of the tree there are 3^k subproblems, each of size $n/2^k$.

For each of those problems we need $O(n/2^k)$ (splitting time + combination time).
Therefore, for some constant c, the cost at depth k is:

$$3^k \times \left(\frac{n}{2^k}\right) = \left(\frac{3}{2}\right)^k \times c\, n.$$

with max. depth $k = \lg n$, so $T(n)$ is

$$\left(1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 + \cdots + \left(\frac{3}{2}\right)^{\lg n}\right) c\, n$$

From $T(n) = c\,n\left(\sum_{k=0}^{\lg n}\left(\frac{3}{2}\right)^k\right)$,

We have a geometric series of ratio $3/2$, starting at 1 and ending at $\left(\left(\frac{3}{2}\right)^{\lg n}\right) = \frac{n^{\lg 3}}{n^{\lg 2}} = \frac{n^{1.58}}{n} = n^{0.58}$.

As the series is increasing, $T(n)$ is dominated by the last term:

$$T(n) = c\,n\left(\frac{n^{\lg 3}}{n}\right) = O(n^{1.58}).$$

A basic Master Theorem

There are several versions of the Master Theorem to solve D&C recurrences. The one presented below is taken from DPV's book.

Theorem (DPV-2.2)

If $T(n) = aT(\lceil n/b \rceil) + O(n^d)$ for constants $a \geq 1, b > 1, d \geq 0$, then has asymptotic solution:

$$T(n) = \begin{cases} O(n^d), & \text{if } d > \log_b a, \\ O(n^d \lg n), & \text{if } d = \log_b a, \\ O(n^{\log_b a}), & \text{if } d < \log_b a. \end{cases}$$

Master Theorems

- This basic Master Theorem does not provide always exact bounds.
- A different one can be found in CLRS's book providing exact bounds but leaving cases outside.
- For stronger versions look at Akra-Bazi Theorem or Salvador Roura Theorems