- You can assume that we have fixed an effective enumeration of the set of TM, i.e., TM= $\{M_1, M_2, \dots\}$.
- We use *a recursively enumerable language* as synonym of *a recognizable language*.

1. Show that $\leq_m$ is a reflexive and transitive relation on $\mathcal{P}(\Sigma^*)$. Is it symmetric?
2. Given a language $L \subset \Sigma^*$, consider the following set
$$M(L) = \{x \in \mathbb{N} \mid L(M_x) = L\}.$$
Prove that $M(L)$ is either empty or infinite. Is $M(L)$ decidable?
3. Show that the language $K = \{x \in \mathbb{N} \mid M_x(x) \downarrow\}$ is recognizable but undecidable.
4. Consider the following problem:
EQUIV: Given two TMs M and N, is $L(M) = L(N)$?
   (a) Provide a definition of the language L_{EQUIV} formed by the inputs in which the answer is yes.
   (b) Show that L_{EQUIV} is undecidable.
5. Consider the following languages:
$A = \{x \mid$ the computation of M_x on input x repeat some configuration$\}$,
$B = \{x \mid$ the computation of M_x on input x do no repeat any configuration, and it is infinite$\}$,
$C = \{x \mid$ the computation of M_x on input x do no repeat any configuration, and it is finite.$\}$
   (a) Prove that if i A, B and C are recursive enumerable then $K = \{x \mid M_x(x) \downarrow\}$ will be decidable.
   (b) For A, B i C prove o disprove their membership in the class of recursively enumerable language.
6. The Post Correspondence Problem (PCP) [Emil Post,1946] is defined as follows:
PCP: Given two lists of words with the same length $A = (x_1, \dots, x_n)$ and $B = (y_1, \dots, y_n)$. Is there a finite sequence $(i_1, \dots, i_r)$, $r \geq 1$ such that $x_{i_1} \cdots x_{i_r} = y_{i_1} \cdots y_{i_r}$?
Show that when $\Sigma = \{0\}$, the PCP problem is decidable. For this it will be enough to provide a high level algorithmc solution.
7. The Modified Post Correspondence Problem (MPCP) is defined a s follows:
MPCP: Given two lists of words with the same length $A = (x_1, \dots, x_n)$ and $B = (y_1, \dots, y_n)$. Is there a finite sequence $(1, i_2 \dots, i_r)$, $r \geq 1$ such that $x_1 x_{i_2} \cdots x_{i_r} = y_1 y_{i_2} \cdots y_{i_r}$?
Let (A, B) an instance of `MPCP` with $A = (x_1, \dots, x_n)$ and $B = (y_1, \dots, y_n)$ and consider the following construction:
   - Assume $x_i = a_{i1} \dots a_{ik_i}$ and $y_i = b_{i1} \dots b_{ir_i}$.
   - Let, # and \$ be two new symbols, define
     - $v_0 = \#a_{11}\# \dots \#a_{1k_1}\#$, $w_0 = \#b_{11}\# \dots \#b_{1r_1}$, $1 \leq i \leq n$,
     - $v_i = a_{i1}\# \dots \#a_{ik_i}\#$, $w_i = \#b_{i1}\# \dots \#b_{ir_i}$, $1 \leq i \leq n$,
     - $v_{n+1} = \$$ and $w_{n+1} = \#\$$.
   - Let $A' = (v_0, \dots, v_{n+1})$ and $B' = (w_0, \dots, w_{n+1})$.

   Show that $(A, B) \in$ `PCP` iff $(A', B') \in$ `MPCP`.
   Does this construction allow us to show that MPCP$\leq_m$PCP?