1. Give a TM deciding the language $\{|x| \text{ is even} \mid x \in \{0,1\}^*\}$.
2. Give a TM that having as input a natural number n in binary outputs $n+1$ (also in binary).
3. Show that $\leq_m$ is a reflexive and transitive relation on $\mathcal{P}(\Sigma^*)$. Is it symmetric?
4. Show that the following problem is undecidable.

   EQUIV: Given two TMs M and N, is $L(M) = L(N)$?
5. Given a language $L \subset \Sigma^*$, consider the following set

   $$M(L) = \{M \in TM \mid L(M) = L\}.$$

   Prove that $M(L)$ is either empty or infinite.
6. The Post Correspondence Problem (PCP) [Emil Post,1946] is defined as follows:

   PCP: Given two lists of words with the same length $A = (x_1, \ldots, x_n)$ and $B = (y_1, \ldots, y_n)$. Is there a finite sequence $(i_1, \ldots, i_r)$, $r \geq 1$ such that $x_{i_1} \cdots x_{i_r} = y_{i_1} \cdots y_{i_r}$?

   - Find a solution in the instance of PCP having $A = (ab, b, aba, aa)$ and $B = (abab, a, b, a)$.
   - Show that when $\Sigma = \{0\}$, the PCP problem is decidable.
7. The Modified Post Correspondence Problem (MPCP) is defined a s follows:

   MPCP: Given two lists of words with the same length $A = (x_1, \ldots, x_n)$ and $B = (y_1, \ldots, y_n)$. Is there a finite sequence $(1, i_2 \ldots, i_r)$, $r \geq 1$ such that $x_1 x_{i_2} \cdots x_{i_r} = y_1 y_{i_2} \cdots y_{i_r}$?

   Let (A, B) an instance of `MPCP` with $A = (x_1, \ldots, x_n)$ and $B = (y_1, \ldots, y_n)$ and consider the following construction:

   - Assume $x_i = a_{i1} \ldots a_{ik_i}$ and $y_i = b_{i1} \ldots b_{ir_i}$.
   - Let, # and \$ be two new symbols, define
     - $v_0 = \#a_{11}\# \ldots \#a_{1k_1}\#$, $w_0 = \#b_{11}\# \ldots \#b_{1r_1}$, $1 \leq i \leq n$,
     - $v_i = a_{i1}\# \ldots \#a_{ik_i}\#$, $w_i = \#b_{i1}\# \ldots \#b_{ir_i}$, $1 \leq i \leq n$,
     - $v_{n+1} = \$$ and $w_{n+1} = \#\$$.
   - Let $A' = (v_0, \ldots, v_{n+1})$ and $B' = (w_0, \ldots, w_{n+1})$.

   Show that $(A, B) \in$ `PCP` iff $(A', B') \in$ `MPCP`.

   Does this construction allow us to show that MPCP$\leq_m$PCP?