Parameterized Complexity

Maria Serna

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FPT

- $\mathcal{A}$ is an FPT algorithm with respect to κ if there are a computable function f and a polinomial function p such that for each $x \in \Sigma^*$, $\mathcal{A}$ on input x requires time $f(\kappa(x))p(|x|)$

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  - There is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that $\forall x \in \Sigma^* \kappa'(R(x)) \leq g(\kappa(x))$

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- We note $(L, \kappa) \leq^{fpt} (L', \kappa')$ when there is a FPT-reduction from (L, κ) to (L', κ')

FPT-reductions

Lemma

FPT is closed under FPT-reductions

FPT-reductions and complexity classes

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  Exercise

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- If $\mathcal{C}$ is a class of parameterized problems
  - (L,κ) is $\mathcal{C}$-hard if $\mathcal{C} \subseteq [(L,\kappa)]^{fpt}$.
  - (L,κ) is $\mathcal{C}$-complete if $(L,\kappa) \in \mathcal{C}$ and (L,κ) is $\mathcal{C}$-hard.

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- $[(L,\kappa)]^{fpt}$ defines a class of parameterized problems for which (L,κ) is complete
- if (L,κ) is $\mathcal{C}$-complete and $\mathcal{C}$ is closed under FPT reductions, then $\mathcal{C} = [(L,\kappa)]^{fpt}$

FPT-equivalent problems

The class paraNP

- Let (L, κ) be a parameterized problem
- (L, κ) belongs to paraNP if there is a non-deterministic algorithm $\mathcal{A}$ that decides $x \in L$ in time $f(\kappa(x))p(|x|)$,
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- If $L \in$ NP, for each parameterization κ, $(L, \kappa) \in$ paraNP
  p-Clique, p-Vertex Cover, ... belong to paraNP.

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Theorem

If $(L, \kappa) \in$ paraNP is not trivial and has a NP-complete slice, then (L, κ) is paraNP-complete under FPT reductions.

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- paraNP-completeness separates *all slices* in P from *a slice* is NP-hard.

The class XP

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  for some computable function f.
- p-CLIQUE, p-VERTEX COVER, p-HITTING SET, p-HITTING SET, p-DOMINATING SET belong to XP.
- XP is the counterpart of EXP in classic complexity.

XP-complete problems

p-Exp-DTM-Halt

Input: A deterministic TM $\mathbb{M}$, $x \in \Sigma^*$ and an integer k,

Parameter: k

Question: Does $\mathbb{M}$ on input x stop in no more than $|x|^k$ steps?

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Theorem

p-Exp-DTM-Halt *is XP-complete but does not belong to FPT unless* $P = NP$.

Relationships among classes

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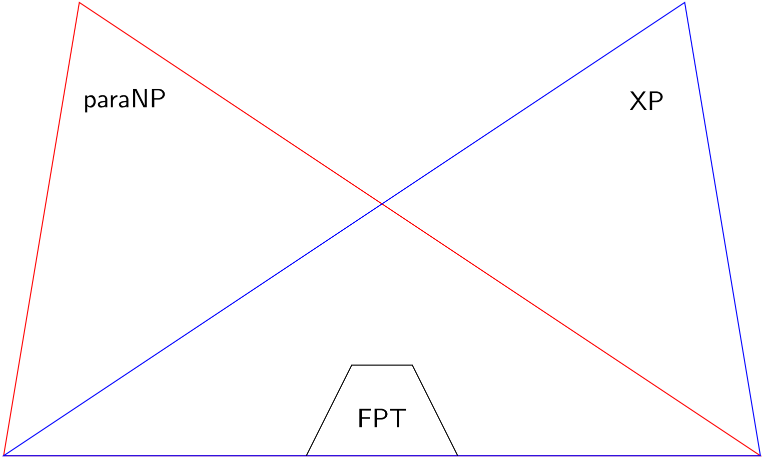
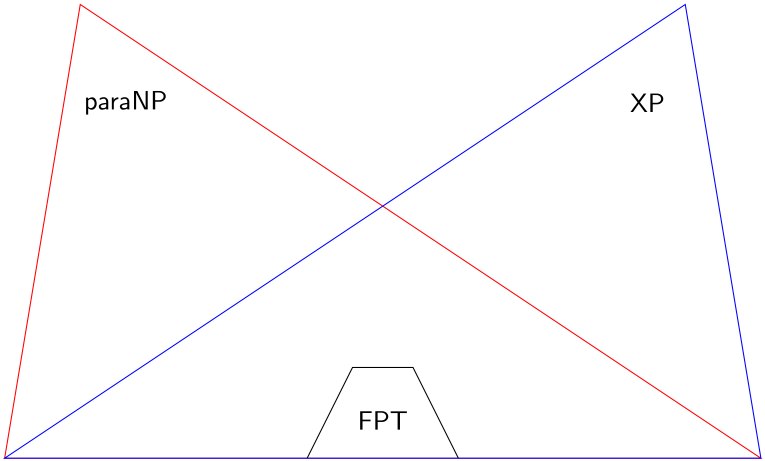


The W-hierarchy



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- Note that $\text{depth}(C) \geq \text{weft}(C)$

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p-WSAT(FAM)
Input: A circuit/formula C/F in family FAM and an integer k,
Parameter: k
Question: Is C/F k-satisfiable?

W-classes

Families of circuits/formulas

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- For $d \geq t \geq 0$, define

$$\mathcal{C}_{t,d} = \{c \mid C \in \textsc{Circ} \text{ and } \mathrm{weft}(C) \leq t \text{ and } \mathrm{depth}(C) \leq d\}$$

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Theorem

- $W[P] \subseteq paraNP \cap XP$
- $W[SAT] \subseteq W[P]$
- *For* $i \geq 1$, $W[i] \subseteq W[SAT]$ *and* $W[i] \subseteq W[i+1]$

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Theorem

If $FPT = W[P]$ *then* CIRCUITSAT *for circuits with* n *inputs and* m *gates can be decided in* $2^{o(n)} m^{O(1)}$ *time.*

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  In fact both problems are $W[2]$-complete

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- We wish to get results like:
  If there is an $f(k)\ n^{o(k)}$ time algorithm for problem XXX, then ETH fails.

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Lemma

If VERTEX COVER can be solved in time $2^{o(k)}\ n^{O(1)}$, then ETH fails.

Proof.

There is a polynomial-time reduction from m-clause 3SAT to m-vertex VERTEX COVER. The assumed algorithm would solve the latter problem in time $2^{o(m)}\ n^{O(1)}$, violating ETH. □

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  Output: $(1 + \epsilon)$-approximate solution
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- Efficient PTAS (EPTAS) running time is $f(1/\epsilon)|x|^{O(1)}$
- For some problems, there is a PTAS, but no EPTAS is known.

  Can we show that no EPTAS is possible?

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Lemma

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Proof.

Suppose an $f(1/\epsilon)\ n^{O(1)}$ time EPTAS exists.
Running this EPTAS with $\epsilon = 1/(k+1)$ decides if the optimum is at most/at least k. □

Parameterized complexity

- Possibility to give evidence that certain problems are not FPT.
- Parameterized reduction.
- The W-hierarchy.
- ETH gives much stronger and tighter lower bounds.
- PTAS vs. EPTAS
- Kernel size lower bounds