Parameterized algorithms: Tree width and dynamic programming

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1. Parameterizing by treewidth
2. Nice tree decomposition
3. Algorithmic meta theorems

Equivalent tw parameterizations for property P

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Input: A graph G, a tree decomposition (T, X) of G
and an integer k,

Parameter: $width(T, X) + k$

Question: Is P(G, k) true?

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Input: A graph G and integers w and k,
Parameter: $w + k$
Question: Is $tw(G) \leq w$ and $\mathrm{P}(G, k)$ true?

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  - repeating the above procedure we get a small tree decomposition.

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- We can make use the FPT algorithm that given a (G, w) decides iwhether $tw(G) > w$ or produces a tree decomposition of width $4w + 4$.
- We can further assume that such a tree decomposition is small.

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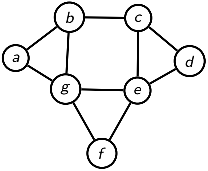
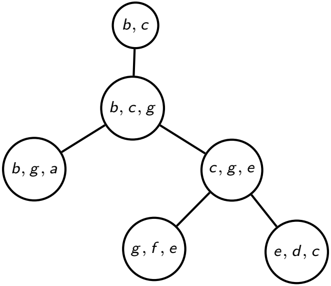
- Notice that X_v is a separator in G.





Exercise For this rooted tree decomposition. Draw the graphs associated to each node in the tree. What are the differences among parent-child graphs?

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- Let (T, X) be a rooted tree decomposition of G with width w.
  - For every $v \in V(T)$ and every proper k-coloring α of $G[Xv]$, define $P_v(\alpha) = 1$ iff $G(v)$ has an α-compatible k-coloring β.

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- Our algorithm computes $P_v(\alpha)$, for each node in T, from leaves to root.

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Lemma

$P_u(\alpha) = 1$ iff for all children v of u, there is an α-compatible coloring β of $G[X_v]$ with $P_v(\beta) = 1$.

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Proof.

- ($\Rightarrow$) Let γ be an α-compatible coloring of $G(u)$. $G(v)$ is a subgraph of $G(u)$, so restricting to X_v gives the desired coloring β.

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    - The same can be done for all children of u simultaneously.

EndProof

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    with $X_v \subseteq X_u$ and $|Xu| = |Xv| + 1$ (introduce)

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    - with $X_u \subseteq X_v$ and $|X_u| = |Xv| - 1$ (forget)
    - with $X_v \subseteq X_u$ and $|Xu| = |Xv| + 1$ (introduce)
  - u has two children v and w with $X_u = X_v = X_w$ (join)

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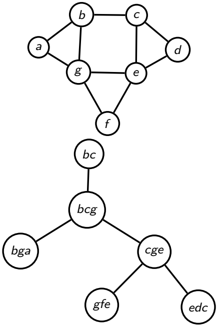

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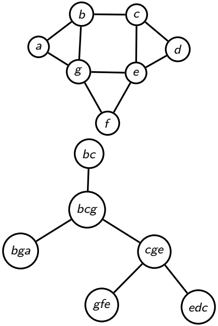

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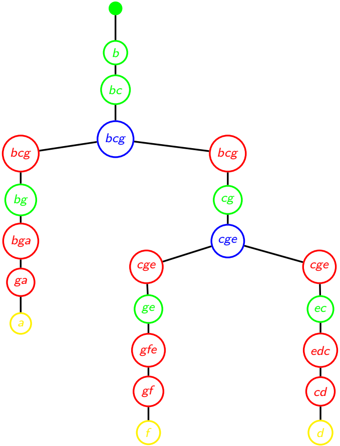

Nice tree decomposition

Lemma

Computing a rooted nice tree decomposition with width at most k, given a small tree decomposition of width at most k takes $O(kn)$ time.

Nice tree decomposition

- Nodes in the tree
  node u holds a subset of vertices X_u, and has a subgraph G_u associated to it.
- the root r has $X_r = \emptyset$ and $G_r = G$.
- nodes can be of four types:
  Start Introduce Forget Join

A parameterization for Min Vertex Cover

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TW-MIN VERTEX COVER
Input: A graph G, a tree decomposition (T, X),
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Question: Compute a minimum size vertex cover of G

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- For each node $v \in V(T)$ we keep a table $s_v(C)$ for each $C \subseteq X_v$ holding the minimum size of a vertex cover C' of $G(v)$ with $C' \cap X_v = C$ if such a C' exists, and $s_v(C) = \infty$ otherwise.

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- The value of $s_r(\emptyset)$ is the size of a minimum vertex cover of G.

- We deal with each type of node separately

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Claim

Let u be a leaf of T with $X_u = \{x\}$.
$s_u(\{x\}) = 1$ and $s_u(\emptyset) = 0$.

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  All neighbors of x in $G(u)$ are in X_v and therefore in C since C is a vertex cover of $G[X_u]$.

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  - If $x \in C'$ then similarly $|C'| \geq s_v(C + x)$.
- $(\leq)$ Let C_1 and C_2 be the $\textsc{vc}$s that determine $s_v(C)$ and $s_v(C + x)$ respectively.

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  - If $x \notin C'$ then $C' \cap X_u = C$ so $|C'| \geq s_v(C)$.
  - If $x \in C'$ then similarly $|C'| \geq s_v(C + x)$.
- $(\leq)$ Let C_1 and C_2 be the $\textsc{vc}$s that determine $s_v(C)$ and $s_v(C + x)$ respectively. C_1, C_2 are $\textsc{vc}$ of $G(u)$ compatible with C, so $s_v(C) \leq \min\{|C_1|, |C_2|\}$.

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Claim

Let u be a join node of T with children v and w.
Then for all $C \subseteq X_u$: $s_u(C) = s_v(C) + s_w(C) - |C|$.

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- ($\geq$) If C' is a vertex cover of $G(u)$ with $C' \cap X_u = C$, then $C' \cap V(v)$ is a vertex cover of $G(v)$ and $C' \cap V(w)$ is a vertex cover of $G(w)$, which share $|C|$ vertices.

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- ($\leq$) Two C-compatible vertex covers of $G(v)$ and $G(w)$ of size $s_v(C)$ and $s_w(C)$ can be combined to a vertex cover of $G(u)$ of size $s_v(C) + s_w(C) - |C|$.

□

Min Vertex cover parameterized by treewidth

Theorem

Let (T, X) be a rooted nice tree decomposition of width w of a graph G on n vertices. In time $2^{w+1}n^{O(1)}$ the size of a minimum vertex cover of G can be computed.

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- We can construct a minimum vertex cover as well, by tracing back through the tree decomposition.
- $+ \; O(f(k)n^{O(1)})$ to get the tree decomposition if needed.

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- To get an algorithm, as we have done, you should working out all the details of the DP!

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- Main uses: quick complexity classification tools, mapping the limits of applicability for specific techniques.
- Usually they are grounded in logics or other properties

First Order Logic on graphs

- We express graph properties using logic

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$$\exists x_1 \exists x_2 \forall y \forall z \; E(y, z) \rightarrow (y = x_1 \vee y = x_2 \vee z = x_1 \vee z = x_2$$

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  $\exists x_1 \exists x_2 \exists x_3 \; E(x_1, x_2) \wedge E(x_1, x_3) \wedge E(x_2, x_3)$

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- Rule of thumb: FO = local properties

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$$\exists V_1 \exists V_2 \forall x \forall y \; E(x, y) \to (x \in V_1 \leftrightarrow y \in V_2)$$

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Recall: No algorithm is constructed!