A graph parameter: Treewidth

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1. Tree width

Graph parameters

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- A similar parameter measures closeness of a graph to a path: pathwidth.

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- For a vertex $v \in V(G)$, $N(v)$ denotes the set of neighbors of v. $N[v] = N(v) + v$. $d(v) = |N(v)|$.
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- A unicyclic graph has only one cycle.
- A graph is outerplanar if it can be drawn as cycle with non-crossing chords.

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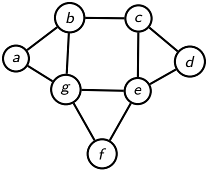

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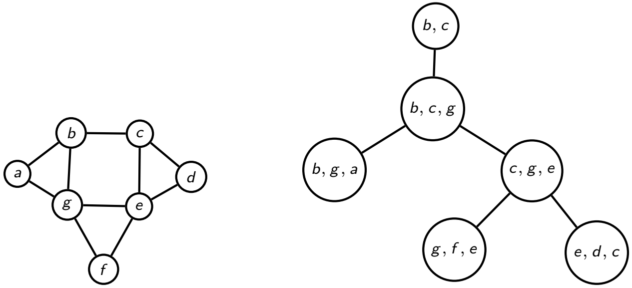

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- Equivalently the second condition can be expressed as:
  - For every $u, v \in V(T)$ and every node $w \in V(T)$ on the path between u and v, $X_u \cap X_v \subseteq X_w$, and
  - every vertex of G appears in at least one X_v .

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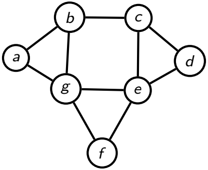

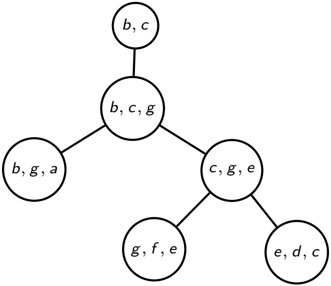

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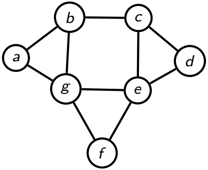

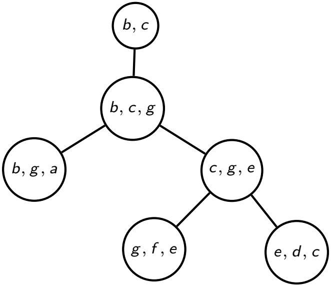

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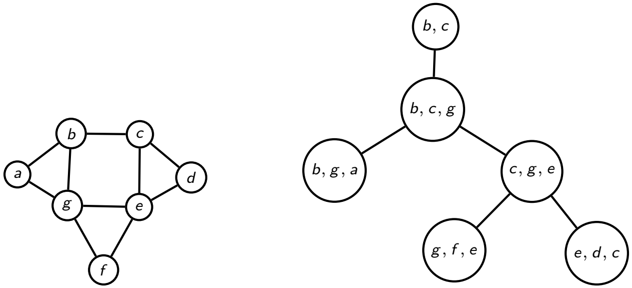

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- The sets X_v are the bags of the tree decomposition.

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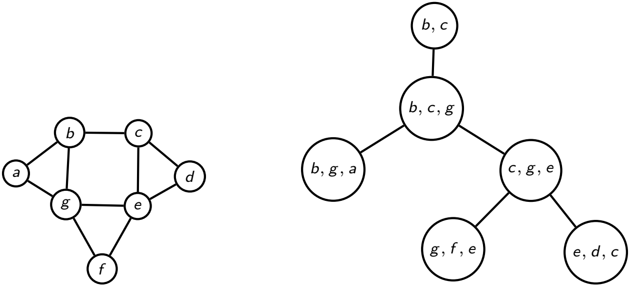
- The width of a tree decomposition (T, X) for G is $\max_{v \in V(T)} |X_v| - 1$.
- The tree width ($tw(G)$) of a graph G is the minimum width over all tree decompositions of G.

A graph with tree width 2



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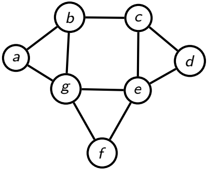

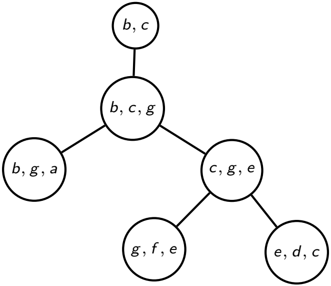

This graph is an outerplanar graph.

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- For K_n, the complete graph on n vertices, $tw(K_n) = n - 1$.

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- Computing a tree decomposition with width at most k (if it exists) takes $O(f(k)n)$ time.
- We present an FPT algorithm that either concludes that a $tw(G) > k$ or provides a tree decomposition with width $\leq 4k + 4$.
  (See section 7.6.2 in M. Cygan et al. Parameterized Algorithms, Springer 2015)

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Claim

Given G, X, Y, the value $\mu(X, Y)$ can be computed in polynomial time, as well as a separator of order $\mu(X, Y)$

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Theorem

Let $G = (V, E), w$ be a vertex weighted connected graph with $tw(G) \leq k$. Then G has a $1/2$-balanced separator X with $|X| \leq k + 1$.

Balanced separators

Proof.

- Let $T = (N, \{X_t\}_{t \in N}, r)$ be a tree decomposition of G with width $\leq k$.
- Let V_t be the vertices in the bags in the subtree rooted at $t \in N$.

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- Furthermore, $w(V \setminus V_t) \leq w(V)/2$.

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- Furthermore, $w(V \setminus V_t) \leq w(V)/2$.
- So, X_t is a balanced separator of order $\leq k+1$.

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Let G, w be a vertex weighted connected with $tw(G) \leq k$. Then G has a $2/3$-balanced separation of order $\leq k + 1$.

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- Assume that $a_1 \geq a_2 \geq \cdots \geq a_p$, and let q be the smallest index such that $S_q = \sum_{i=1}^{q} a_i \geq w(V)/3$ or $q = p$ if this never happens.

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- If $1 < q < p$, $S_{q-1} < w(V)/3$ and $a_q \leq a_{q-1} \leq S_{q-1}$. So, $S_q \leq 2w(V)/3$.

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- Note that if $q = p$, $S_q < w(V)/3 \leq 2w(V)/3$ and that if $q = 1$, $S_q < w(V)/2 \leq 2w(V)/3$.
- If $1 < q < p$, $S_{q-1} < w(V)/3$ and $a_q \leq a_{q-1} \leq S_{q-1}$. So, $S_q \leq 2w(V)/3$.
- (A, B) is $2/3$-balanced.

Balanced separations

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Balanced separators

Corollary 1

Let $G = (V, E)$ be a connected graph with $tw(G) \leq k$. Let $S \subseteq V$ with $|S| = 3k + 4$. Then, there is a partition (S_A, S_B) of S such that $k + 2 \leq |S_A||S_B| \leq 2k + 2$ and $\mu(S_A, S_B) \leq k + 1$.

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- Furthermore, $|(A \setminus B) \cap S||, |(B \setminus A) \cap S| \leq 2k + 2$.

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EndProof.

A component of the algorithm

Assume G is connected.

procedure $\textsc{FindSplit}(W, S)$
- $G = G[W]$
- **if** $|S| < 3k + 4$ **then** ▷ $W \setminus S \neq \emptyset$
  - Choose $u \in W \setminus S$
  - **return** $S \cup \{u\}$
- **else** ▷ $|S| = 3k + 4$
  - **for** $S_A, S_B \subseteq S$ with $k + 2 \leq |S_A||S_B| \leq 2k + 2$ **do**
    - **if** $\mu(S_A, S_B) \leq k + 1$ **then**
      - Find a separation (A, B) separating S_A, S_B with $|A \cap B| \leq k + 1$
      - **return** $S \cup A \cap B$
  - **stop** $tw(G) > k$ ▷ By Coro 1

FindSplit

Claim

Let G be a graph, let $S, W \subseteq V$, and let $k \in \mathbb{N}$. Assume that

1. $S \subsetneq W \subseteq V$, $|S| \leq 3k+4$, $W \setminus S = \emptyset$,
2. $G[W]$ and $G[W \setminus S]$ are connected, and
3. $S = N_G[W \setminus S]$

Then, $\textsc{FindSplit}(S, W)$ either discovers that $tw(G) > k$ or returns a set $\hat{S}$ verifying

1. $S \subsetneq \hat{S}$, $|\hat{S}| \leq 4k+5$,
2. every c.c. of $G[W \setminus \hat{S}]$ is adjacent to at most $3k+4$ vertices of $\hat{S}$.

Furthermore, the running time $\textsc{FindSplit}$ is at most $2^{2(3k+4)}p(n)$.

FindSplit

Proof.

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- When $|S| < 3k + 4$:
  - Clearly $S \subsetneq \hat{S}$, and $|\hat{S}| \leq 3k + 4 \leq 4k + 5$
  - For every c.c. of $G[W \setminus \hat{S}]$ is adjacent only to vertices in $\hat{S}$, so (2) holds.

FindSplit

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FindSplit

- When $|S| = 3k + 4$:
  - FindSplit correctly reports that $tw(G) > k$.
  - As $|A \cap B| \leq k + 1$, $|\hat{S}| \leq 3k + 4 + k + 1 = 4k + 5$.
  - (A, B) separates some $S_A, S_B \subseteq S$ with $k + 2 \leq |S_A||S_B| \leq 2k + 2$

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  - (A, B) separates some $S_A, S_B \subseteq S$ with $k + 2 \leq |S_A||S_B| \leq 2k + 2$
  - As $k + 2 \leq |S_A|, |S_B|$ and $|A \cap B| \leq k + 1$, $S_A \setminus (A \cap B), S_B \setminus (A \cap B) \neq \emptyset$.
  - We can pick $u_A \in S_A \setminus (A \cap B)$ and $u_B \in S_B \setminus (A \cap B)$.
  - As $G[W]$ and $G[W \setminus S]$ are connected and $S = N_G[W \setminus S]$, there is a path from s_A to s_B in $G[W]$ that contains a vertex $u_{AB} \in (W \setminus S) \cap (A \cap B)$.

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  - Therefore, $S \subsetneq \hat{S}$ and condition (1) holds.

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  - Assume that $D \subseteq A \setminus B$.
  - The vertices in $\hat{S}$ that are adjacent to D belong either to $(A \setminus B) \cap S$ or to $A \cap B$.
  - Therefore, D is adjacent to at most $|(A \setminus B) \cap S| + |A \cap B|$ vertices.

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  - As $|(A \setminus B) \cap S| \leq |S| \leq 3k + 4$ and $|A \cap B| \leq k + 1$ condition (2) holds.

EndProof

An approximate algorithm for small treewidth decomposition

Assume $G = (V, E)$ is connected.

procedure $\textsc{Decompose}(W, S)$
  $\hat{S} = \textsc{FindSplit}(G, S)$
  Let $D_1, \dots, D_p$ be the c.c. of $G[W \setminus \hat{S}]$.
  for $i \in [p]$ **do**
    $T_i = \textsc{Decompose}(N[D_i], N(D_i))$
    Construct T from $T_1, \dots, T_p$, adding a rot r having as children of r the roots of the T_i's and setting $X_r = \hat{S}$
  stop $tw(G) > k$ ▷ By Coro 1
  return T

DECOMPOSE

Claim

In every call $\text{DECOMPOSE}(W, S)$ from the initial call $\text{DECOMPOSE}(V, \emptyset)$ the sets W, S verify:

1. $S \subsetneq W \subseteq V$, $|S| \leq 3k + 4$, $W \setminus S = \emptyset$,
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Proof.

Follows directly from the previous claim and the selection of parameters in the recursive call.