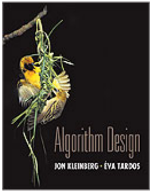
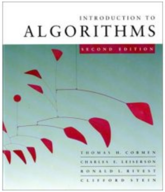
Algorithms and Complexity

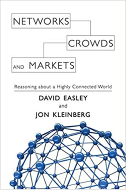
Fall 2023

Algorithmics: Basic references

- Kleinberg, Tardos. Algorithm Design, Pearson Education, 2006.
- Cormen, Leisserson, Rivest and Stein. Introduction to algorithms. Second edition, MIT Press and McGraw Hill 2001.
- Easley, Kleinberg. Networks, Crowds, and Markets: Reasoning About a Highly Connected World, Cambridge University Press, 2010

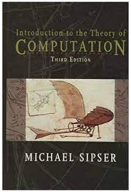


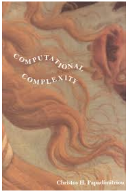


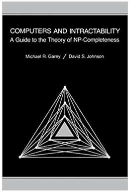


Computational Complexity: Basic references

- Sipser Introduction to the Theory of Computation 2013.
- Papadimitriou Computational Complexity 1994.
- Garey and Johnson Computers and Intractability: A Guide to the Theory of NP-Completeness 1979







Alphabets and languages

- Alphabet: a non-empty finite set.
- Symbol: an element of an alphabet.
- Word: a finite sequence of symbols.
  λ denotes the empty word, a sequence with 0 symbols.
- Language: set of words over an alphabet.

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  - For $x, y \in \Sigma^*$, $x \cdot y$ is the word obtained placing the symbols in x followed by the symbols in y.
  - For example, if $\Sigma = \{0, 1\}$, $x = 001000$ and $y = 11101$, $xy = 00100011101$.
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  - $|x \cdot y| = |x| + |y|$
  - $|\lambda| = 0$
- A language L is a subset of Σ^*.
  We can extend concatenation to languages in the usual form.

$$L_1 \cdot L_2 = \{x \cdot y \mid x \in L_1, y \in L_2\}$$

Alphabets and languages: enumerability

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- We cannot use alphabetical order $a, aa, aaa, aaaa, \ldots$
- We use lexicographic order
  - Order words by length.
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  - Among words with the same length order them according to alphabetical order.
- For $\Sigma = \{0, 1\}$ we can enumerate $\{0.1\}^*$ as

$$\lambda, 0, 1, 00, 01, 10, 11, 000, \ldots, 111, 0000, \ldots$$

Problem types

- Decision

  Input x

  Property $P(x)$

  Example: Given a graph and two vertices, is there a path joining them?

- Function

  Input x

  Compute y *such that* $Q(x, y)$

  Example: Given a graph and two vertices, compute the minimum distance between them.

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  By coding inputs/outputs on alphabet Σ a deterministic algorithm solving a problem determines a function $f : \Sigma^* \to \Sigma^*$ s.t., for any x, $Q(x, f(x))$ is true.

Decision problem classes

- Undecidable
  No algorithm can solve the problem.
- Decidable
  There is an algorithm solving them.

Turing machines

- A Turing machine (TM) M is a tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\mathrm{F}})$, where
  - Q is a finite set of states,
  - Σ is the input alphabet.
  - Γ is the tape alphabet, $\Gamma = \Sigma \cup \{\mathtt{b}, \blacktriangleright\}$, with $\mathtt{b}, \blacktriangleright \notin \Sigma$.
  - $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\mathtt{l}, \mathtt{r}, \mathtt{n}\}$ is the transition function.
  - q_0 is the initial state.
  - q_{F} is the final or accepting state.

Turing machines: Recognizing languages

- Let $M = (Q, \Sigma, \Gamma, \delta, q_0, q_\mathrm{F})$ be a TM and $x \in \Sigma^*$
- The computation of M with input x goes as follows:
  - Initially: the state is q_0; the tape has $\blacktriangleright x$ and all remaining cells in the tape hold a $\mathtt{b}$; the head has access to the first symbol of x.
  - While there is a transition in δ for the combination state, symbol accessed by the head, the transition is applied.

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- Assuming that Q and Γ are disjoint, the word $\blacktriangleright \alpha q \beta$ is a configuration in which $\blacktriangleright \alpha\beta$ are the tape contents ($\mathtt{b}$ outside), q is a state and the head is accessing the tape cell holding the first symbol in β.

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- The computation of M on x is a sequence of configurations, starting with $\blacktriangleright q_0 x$, so that we can pass from one configuration to the next applying δ.

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- The computation of a TM on input x is a sequence of configurations.
- If the sequence of configurations is finite, we say the M halts on input x, we note this as $M(x) \downarrow$,
  otherwise, the computation diverges or does not halt, $M(x) \uparrow$.

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- When $M(x)\downarrow$,
  - the number of configurations in the computation is the computation time.
  - Let $\blacktriangleright \alpha q \beta$ be the last configuration in the computation.
    - $M(x) = \alpha\beta$ is the output,
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    - $M(x) = \alpha\beta$ is the output,
    - M halts on input x in state q.
- $L(M) \subseteq \Sigma^*$ is the set of words that M accepts, i.e, M on input x halts in state q_{F}.
- A language $L \subseteq \Sigma^*$ is recognizable iff there is a TM M with $L = L(M)$.

The Church-Turing thesis: Problems and programs

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- The set of TM is enumerable (easy to see), so the set of all programs is enumerable.
- So, we can identify TMs with the natural numbers. Let M_x be the x-th TM.

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- The universal TM designed by Turing, has input x, y and outputs the result of executing TM x on input y. This has an equivalent in the language interpreter.
- Technically x, y is not a well defined input. Nevertheless, $\mathbb{N} \times \mathbb{N}$ is enumerable, so we have a bijection to $\mathbb{N}$ (or Σ^*) that allow us to recover the original words. We denote this effective codification as $\langle x, y\rangle \in \Sigma^*$.

The Entscheidungsproblem.

- David Hilbert, ICM Bolonia-1929, The Entscheidungsproblem: Find a procedure to decide the validity of a given a first-order logic expression, in a finite number of operations.

  Ex: Is it true that
  $\neg\exists x, y, z, n \in \mathbb{N} : (n \geq 3) \wedge (x^n + y^n = z^n)$?

- Reflecting Hilbert's dream of a mechanical procedure that can prove or refute any mathematical claim.

  If *the Entscheidungsproblem* is solved $\Rightarrow$ all mathematics could be mechanized.

Hilbert program fails: Gödel's theorem

- The framework
  - Formal system: a language, a finite set of axioms and a set of inference rules used to derive an expression from a set of axioms.
  - A statement S in a formal system is decidable: if there is a finite proof, according to the inference rules, which will correctly prove that S is true or not. Otherwise the statement is undecidable
  - A formal system is complete: every statement can be proved or disproved. Otherwise the system is incomplete.

- K. Gödel's incompleteness theorem (1931):
  In any formal system sufficiently powerful to include ordinary arithmetic, there will be undecidable statements.

  There are statements in the arithmetic ($\mathbb{N}$ with $+, x$), which can not be proved or disproved.

The Entscheidungsproblem unsolvable: Turing's approach

- Consider the set of TMs, i.e., tuples $(Q, \Sigma, \Gamma, \delta, q_0, q_F)$, this set is enumerable.
- The set of languages is $\mathcal{P}(\Sigma^*)$, which is not enumerable.
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- A language $L \subseteq \Sigma^*$ is decidable iff there is a decider M such that $L = L(M)$.

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Theorem

The language $A_{TM} = \{\langle x, w\rangle \mid w \in L(M_x)\}$ *is undecidable.*

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  $M = M_z$ accepts z iff M_y rejects $\langle z, z\rangle$ iff M_z rejects z.

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$\overline{A_{TM}}$ is not recognizable.

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If $A \leq_m B$

- B is decidable $\Rightarrow$ A is decidable.
- A is undecidable $\Rightarrow$ B is undecidable.

Some undecidable problems

- HALT: Given a program P and an input x, does P halt on input x?
- Given a program P, does P halt on input 0?
- Given a program P, is the set of inputs x on which P halts finite?
- Given a program P, is there an input x s.t. $P(x) = x$?

Semi-Thue systems

Introduced by Axel Thue (1863-1922)

- A rewriting system or semi-Thue system is a tuple (Σ, R) where
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  $s \underset{R}{\longrightarrow} t$ iff there exist $x, y, u, v \in \Sigma^*$ s.t. $s = xuy$, $t = xvy$, and $u \overset{R}{\to} v$.
- A zero-or-more-steps rewriting is captured by the reflexive transitive closure of $\underset{R}{\longrightarrow}$, $\overset{*}{\underset{R}{\longrightarrow}}$.

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words	start aba endl ba
solution	$aba \to aaba \to aaaba \to aaaaba \to aaaaaba \to aaaaaaba$ $\to bbbbba \to bbbba \to bbba \to bba \to ba$

WP is undecidable

Theorem

$A_{TM} \leq_m \mathrm{WP}$

- Let $\langle M, x\rangle$ be an input to A_{TM}, $x \in \Sigma^*$ and $M = (Q, \Sigma, \Gamma, \delta, q_0, q_F)$.

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- From M and x, we construct the following input to WP with alphabet the union of Γ, Q and an additional symbol $\#$
  - The rewriting rules are

$$\begin{array}{ll} \multicolumn{2}{l}{\text{for } q, p \in Q \text{, with } q \neq q_F \text{ and for } a, b, c \in \Gamma} \\ (qa, bp) & \text{si } \delta(q, a) = (p, b, \mathtt{r}) \\ (cqa, pcb) & \text{si } \delta(q, a) = (p, b, \mathtt{l}) \\ (q\#, bp\#) & \text{si } \delta(q, \mathtt{b}) = (p, b, \mathtt{r}) \\ (cq\#, pcb\#) & \text{si } \delta(q, \mathtt{b}) = (p, b, \mathtt{l}) \\ \multicolumn{2}{l}{\text{for } a \in \Gamma,\ (aq_F, q_F) \text{ and } (q_F a, q_F)} \end{array}$$

  - The start word is $u = \blacktriangleright q_0 x\#$, and the final word is $v = q_F\#$

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  - The start word is $u = \blacktriangleright q_0 x\#$, and the final word is $v = q_F\#$
- $x \in L(M)$ iff there is a finite computation $\blacktriangleright q_0 x, \ldots, \alpha q_F \beta$ iff $\blacktriangleright q_0 x\# \xrightarrow[R]{*} q_F\#$

Post Correspondence Problem

Introduced by by Emil Post in 1946.

PCP: Given two lists of words with the same length $A = (x_1, \ldots, x_n)$ and $B = (y_1, \ldots, y_n)$. Is there a finite sequence $(i_1, \ldots, i_r)$, $r \geq 1$ such that $x_{i_1} \cdots x_{i_r} = y_{i_1} \cdots y_{i_r}$?

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	A	B
1	01	011
2	10	010
3	001	01
4	1011	10

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Is a "yes" instance as $(1, 2, 2, 1, 3, 4, 3)$ gives 01101001001101 1001 in both systems.

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Is a "no" instance, the number of ones in words 1,3,4 is higher in A than in B and equal for word 2.

Modified Post Correspondence Problem

MPCP: Given two lists of words with the same length $A = (x_1, \dots, x_n)$ and $B = (y_1, \dots, y_n)$. Is there a finite sequence $(1, i_2 \dots, i_r)$, $r \geq 1$ such that $x_1 x_{i_2} \cdots x_{i_r} = y_1 y_{i_2} \cdots y_{i_r}$?

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- The main difference is that we are forcing to start with the first word in A and B,
- MPCP is a subproblem of PCP,
- If PCP is decidable, MPCP is decidable.
- We wan to prove that MPCP is undecidable
- Then, extend this result to show that PCP is undecidable.

MPCP is undecidable

Theorem

$WP \leq_m MPCP$

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$\mathit{WP} \leq_m \mathit{MPCP}$

Proof

- Let (R, u, v) be an input to WP with $R = \{(u_1, v_1), \ldots, (u_n, v_n)\}$ over an alphabet Σ.

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- Let (R, u, v) be an input to WP with $R = \{(u_1, v_1), \ldots, (u_n, v_n)\}$ over an alphabet Σ.
- Let $\#$ be a symbol not in Σ.

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- Let $\#$ be a symbol not in Σ.
- Define (A, B) as

A	B	
$\#$	$\#u\#$	
u_i	v_i	$(u_i, v_i) \in R$
a	a	$a \in \Sigma$
$\#$	$\#$	
$v\#\#$	$\#$	

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$v\#\#$	$\#$	

- When $(R, u, v) \in \mathtt{WP}$, there is a word $\#u\#x_1\# \dots \#x_m\#v\#$, where each intermediate word is obtained by applying a rule to the previous word.

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- We can reach this word starting from the first rule, and applying the corresponding rewriting rule. In B we go one word/letter in advance with respect to A. Using the last rule, we balance at the end.

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- We can reach this word starting from the first rule, and applying the corresponding rewriting rule. In B we go one word/letter in advance with respect to A. Using the last rule, we balance at the end.
- So, $(A, B) \in \texttt{MCPC}$.

MPCP is undecidable

- Assume $(A, B) \in \texttt{MCPC}$.

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u_i	v_i	$(u_i, v_i) \in R$
a	a	$a \in \Sigma$
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$v\#\#$	$\#$	

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- Assume $(A, B) \in \texttt{MCPC}$.
- The solution must start by the first rule and finalize with the last, so there is a common word $\#u\#x_1\#\dots\#x_m\#v\#$.

MPCP is undecidable

A	B
$\#$	$\#u\#$
u_i	v_i
a	a
$\#$	$\#$
$v\#\#$	$\#$

$(u_i, v_i) \in R$

$a \in \Sigma$

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- As the correspondence copies or rewrites according to the Thue system, each intermediate word is obtained by applying a rule to the previous word.

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End Proof