Some definitions:
A valuation function $v : \mathcal{C}_n \to \mathbb{R}$ is said to be

- *monotone* when $v(C) \le v(D)$, for $C \subseteq D \subseteq N$.
- *superadditive* when $v(C \cup D) \ge v(C) + v(D)$, for every pair of disjoint coalitions $C, D \subseteq N$.
- *supermodular* when $v(C \cup D) + v(C \cap D) \ge v(C) + v(D)$, for $C, D \subseteq N$.

A coalitional game (N, v) is *convex* iff v is supermodular.

5 Cooperative games

5.1. Consider a cooperative game which is defined on an undirected graph $G = (V, E)$. The players are the vertices in the graph and for $S \subseteq V$, $v(S) = |\{u \in V \mid N(u) \cap S \neq \emptyset\}|$. As usual $N(u) = \{v \mid (u, v) \in E\}$.

(a) Is the valuation function monotone? supperadditive? supermodular?

(b) Is the core empty? Can this property be decided in polynomial time?

5.2. Consider an undirected graph $G = (V, E)$ and two vertices s and t. Assume that $n = |V|$ and $m = |E|$. Consider the cooperative game $\mathcal{C}(s, t) = (N, v)$ where $N = E$ and, for $S \subseteq N$, $v(S)$ is m minus the length of the shortest path from s to t in the graph (V, S), if such a path exists, and 0 otherwise. Is this game supperadditive? Does the game have an empty core?

5.3. Consider the following game which is defined by a parameter k. Each participant has a collection of old vinyl disks, not all of them in good state, that are willing to share. A company is able to obtain an accurate re-recording of an album provided that at least k copies are provided. In this setting the value of a coalition is the number of albums for which an accurate recording is possible. You can assume that there are n participants and m different albums.

(a) Provide an expression of the valuation function.

(b) Is the game convex?

(c) Provide an accurate expression for the Shapley values. Can those values be computed in polynomial time?

5.4. The *unanimity game on N with respect to coalition $F \subseteq N$* is the game $\Gamma_F = (N, v_F)$ where

$$v_F(S) = \begin{cases} 1 & \text{if } F \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

(a) Can the core be empty? If so, analyze the computational complexity of deciding if the core is non-empty. When the core is nonempty, an outcome in the core can be computed in polynomial time?

(b) Can the Shapley values be computed in polynomial time?

5.5. **The diameter game.**

Consider a cooperative game which is defined on an undirected connected graph $G = (V, E)$. The players are the edges in the graph. For $X \subseteq E$, let $G_X = (V, X)$ be the graph formed by V and the edges in X. The valuation function is the following

$$v(X) = \begin{cases} 2|X| - diam(G_X) & \text{if } G_X \text{ is connected} \\ \frac{|X|}{2} & \text{otherwise,} \end{cases}$$

where $diam(H)$ is the diameter of the graph H.

(a) Is the valuation function monotone? supperadditive? supermodular?

(b) Are there connected graphs such that the core of the associated diameter game is non-empty?

5.6. Consider a simple game $(N, \mathcal{W})$. We say that player i is a

- *passer* iff, $\forall S \subseteq N$, if $i \in S$, then $S \in \mathcal{W}$.
- *vetoer* iff, $\forall S \subseteq N$, if $i \notin S$, then $S \in \mathcal{L}$.
- *dictator* iff, $\forall S \subseteq N$, $S \in \mathcal{W}$ iff $i \in S$.

(a) Provide a simpler characterization of those properties.

(b) Under which of the forms of representation of simple games based on sets can those properties being decided in polynomial time?

5.7. **Games on social networks** One of the criticism to simple games is the fact of assuming that any coalition can be formed. In the context in which the players participate in a social networks, a natural restriction on a coalition to take effect is that its members should at least be able to establish some level of communication among themselves.

For simplicity you can assume that a simple game $\Gamma = (N, \mathcal{W})$ is defined and that the social network is an undirected graph $H = (N, E)$.

On top of that we can came out with different combinations for defining winning coalitions in an associated *social game*, Γ_s on N. Consider the following options:

(a) A coalition X is winning in Γ_s iff X wins in Γ and $H[X]$ has no isolated vertices.

(b) A coalition X is winning in Γ_s iff X wins in Γ and $H[X]$ is connected.

(c) A coalition X is winning in Γ_s iff there is $Y \subseteq X$, so that Y wins in Γ and $H[Y]$ is connected.

Under which of the options (a), (b) or (c) is Γ_s a simple game?

5.8. Assume that a WVG is described by $\Gamma = (q; w_1, \ldots, w_n)$. Analyze the computational complexity of the problemS

- Compute the smallest number of players that can form a wining coalition in Γ.
- Compute the biggest number of players that can form a losing coalition in Γ.