

Some definitions:

A valuation function $v : \mathcal{C}_n \rightarrow \mathbb{R}$ is said to be

- *monotone* when $v(C) \leq v(D)$, for $C \subseteq D \subseteq N$.
- *superadditive* when $v(C \cup D) \geq v(C) + v(D)$, for every pair of disjoint coalitions $C, D \subseteq N$.
- *supermodular* when $v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$, for $C, D \subseteq N$.

A coalitional game (N, v) is *convex* iff v is supermodular.

5 Cooperative games

- 5.1. Consider a cooperative game which is defined on an undirected graph $G = (V, E)$. The players are the vertices in the graph and for $S \subseteq V$, $v(S) = |\{u \in V \mid N(u) \cap S \neq \emptyset\}|$. As usual $N(u) = \{v \mid (u, v) \in E\}$.
- (a) Is the valuation function monotone? superadditive? supermodular?
 - (b) Is the core empty? Can this property be decided in polynomial time?
- 5.2. Consider an undirected graph $G = (V, E)$ and two vertices s and t . Assume that $n = |V|$ and $m = |E|$. Consider the cooperative game $\mathcal{C}(s, t) = (N, v)$ where $N = E$ and, for $S \subseteq N$, $v(S)$ is m minus the length of the shortest path from s to t in the graph (V, S) , if such a path exists, and 0 otherwise. Is this game superadditive? Does the game have an empty core?
- 5.3. Consider the following game which is defined by a parameter k . Each participant has a collection of old vinyl disks, not all of them in good state, that are willing to share. A company is able to obtain an accurate re-recording of an album provided that at least k copies are provided. In this setting the value of a coalition is the number of albums for which an accurate recording is possible. You can assume that there are n participants and m different albums.
- (a) Provide an expression of the valuation function.
 - (b) Is the game convex?
 - (c) Provide an accurate expression for the Shapley values. Can those values be computed in polynomial time?

5.4. The *unanimity game* on N with respect to coalition $F \subseteq N$ is the game $\Gamma_F = (N, v_F)$ where

$$v_F(S) = \begin{cases} 1 & \text{if } F \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Can the core be empty? If so, analyze the computational complexity of deciding if the core is non-empty. When the core is nonempty, an outcome in the core can be computed in polynomial time?
- (b) Can the Shapley values be computed in polynomial time?

5.5. **The diameter game.**

Consider a cooperative game which is defined on an undirected connected graph $G = (V, E)$. The players are the edges in the graph. For $X \subseteq E$, let $G_X = (V, X)$ be the graph formed by V and the edges in X . The valuation function is the following

$$v(X) = \begin{cases} 2|X| - \text{diam}(G_X) & \text{if } G_X \text{ is connected} \\ \frac{|X|}{2} & \text{otherwise,} \end{cases}$$

where $\text{diam}(H)$ is the diameter of the graph H .

- (a) Is the valuation function monotone? superadditive? supermodular?
- (b) Are there connected graphs such that the core of the associated diameter game is non-empty?

5.6. Consider a simple game (N, \mathcal{W}) . We say that player i is a

- *passer* iff, $\forall S \subseteq N$, if $i \in S$, then $S \in \mathcal{W}$.
- *vetoer* iff, $\forall S \subseteq N$, if $i \notin S$, then $S \in \mathcal{L}$.
- *dictator* iff, $\forall S \subseteq N$, $S \in \mathcal{W}$ iff $i \in S$.

- (a) Provide a simpler characterization of those properties.
- (b) Under which of the forms of representation of simple games based on sets can those properties being decided in polynomial time?

5.7. **Games on social networks** One of the criticism to simple games is the fact of assuming that any coalition can be formed. In the context in which the players participate in a social networks, a natural restriction on a coalition to take effect is that its members should at least be able to establish some level of communication among themselves.

For simplicity you can assume that a simple game $\Gamma = (N, \mathcal{W})$ is defined and that the social network is an undirected graph $H = (N, E)$.

On top of that we can come out with different combinations for defining winning coalitions in an associated *social game*, Γ_s on N . Consider the following options:

- (a) A coalition X is winning in Γ_s iff X wins in Γ and $H[X]$ has no isolated vertices.
- (b) A coalition X is winning in Γ_s iff X wins in Γ and $H[X]$ is connected.
- (c) A coalition X is winning in Γ_s iff there is $Y \subseteq X$, so that Y wins in Γ and $H[Y]$ is connected.

Under which of the options (a), (b) or (c) is Γ_s a simple game?

5.8. Assume that a WVG is described by $\Gamma = (q; w_1, \dots, w_n)$. Analyze the computational complexity of the problemS

- Compute the smallest number of players that can form a winning coalition in Γ .
- Compute the biggest number of players that can form a losing coalition in Γ .