1 Strategic games

For the families of strategic games defined below:

- Provide a formal characterization of the best response sets, for a player $i \in N$ and a (pure) strategy profile s.
- Provide a formal characterization of the strategy profiles that are pure Nash equilibrium of the game.
- Analyze the computational complexity of the problems related to Best responses and pure Nash equilibria.

1.1. (**Exact cooperation**)

The *cooperation* game is defined as follows. There is a group N of n people and a task to be performed. To perform correctly the task requires that exactly k persons cooperate. Each player can decide whether to cooperate (1) or not (0). The utility of a strategy profile $x \in \{1,0\}^n$ for player i is defined as

$$u_i(x) = \begin{cases} 1 & \text{the task is performed and } x_i = 1. \\ 0 & \text{otherwise} \end{cases}$$

1.2. (**Weak cooperation**)

The *weak cooperation* game is defined as follows. There is a group N of n people and a task to be performed. To perform correctly the task requires that at least k persons cooperate. Each player can decide whether to cooperate (1) or not (0). The utility of a strategy profile $x \in \{1,0\}^n$ for player i is defined as

$$u_i(x) = \begin{cases} 1 & \text{the task is performed and } x_i = 1. \\ 0 & \text{otherwise} \end{cases}$$

1.3. (**Split cooperation**)

The *split cooperation* game is defined as follows. There is a group N of n people and a task to be performed. To perform correctly the task requires that at least k persons cooperate. Each player can decide whether to cooperate (1) or not (0). The utility of a strategy profile $x \in \{1,0\}^n$ for player i is defined as

$$u_i(x) = \begin{cases} \frac{k}{|x|_1} & \text{the task is performed and } x_i = 1. \\ 0 & \text{otherwise} \end{cases}$$

where $|x|_1 = |\{i \mid x_i = 1\}|$.

1.4. (**Matching**)

The *matching* game is played in a bipartite graph $G = (V_1, V_2, E)$ in which edges are connect only vertices V_1 to vertices in V_2. The players are the vertices in the graph that is $V_1 \cup V_2$. Each player has to select one of its neighbors. Player i gets utility 1 when the selection is mutual (player i selects j and player j selects i) otherwise he gets 0.

1.5. (**List coloring**)

Assume that we have fixed a finite set K of k colors. Consider a graph $G = (V, E)$ with a labeling function $\ell : V \to 2^K \setminus \{\emptyset\}$, associating to each vertex a subset of colors. The l *list coloring game*, $\Gamma(G, \ell)$, is defined as follows

- the players are $V(G)$,
- the set of strategies for player v is $\ell(v)$,
- the payoff function of player v is $u_v(s) = |\{u \in N(v) \mid s_u = s_v\}|$.

1.6. (**Cover**)

In the **cover game** the players are the vertices in an undirected graph $G = (V, E)$ on a set of n vertices. The goal of the game is to select a set of vertices X that covers a lot of edges. An edge is covered by a set X if at least one of its ends points belongs to X.

Formally, the set of actions allowed to player i is $A_i = \{0, 1\}$. Those players playing 1 will form the set. Let $s = (s_1, \dots, s_n)$, $s_i \in \{0, 1\}$, be an strategy profile, and let $X(s) = \{i \mid s_i = 1\}$.

The cost function for player $i \in V$ is defined as follows

$$c_i(s) = s_i + |\{(a, b) \in E \mid a, b \notin X(s)\}|.$$

1.7. (**Splitting**)

Consider a set of n players that must be partitioned into two groups. However, there is a set of bad pairings and the two players in such a pair do not want to be in the same group. Moreover, each player is free to choose which of the two groups to be in. We can model this by a graph $G = (V, E)$ where each player i is a vertex. There is an edge (i, j) if i and j form a bad pair. The private objective of player i is to maximize the number of its neighbors that are in the other group.

1.8. (**Connected network**)

Consider a network formation game where players are interested only in creating a connected network. In such a game we are given a connected graph G in which each edge e has a cost $c(e)$. Consider a game with one player per vertex in G. Player u can select any subset s_u of the edges incident with u in G. The cost for each player is ∞ if the subgraph resulting from the union of the selected edges is not connected, otherwise the cost for player u is the sum of the costs of the edges in s_u.