Auctions: An introduction to mechanism design

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Spring 2026

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- What is the right price for objects? groups of objects?
- Objectives:
  - Truth-telling
  - Efficiency: social welfare
  - Revenue: maximize profit
  - Envy-freeness ⋮

  Not all of them can be achieved at the same time.

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- How can the true preferences be revealed?
- At which cost?

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- Aim: Design and analyze the rules and properties of an auction.
- Goal: Design an auction so that in equilibrium we get the results we want.
- As in Game theory we rely on rationality.

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  - Rules for bidding and clearing.
  - A procedure to determine who wins what (allocation) and how much pays (payment) on the basis of the received information.

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- When?
- What?
- To whom?

Single item auctions

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- We analyze three such mechanisms

First price (FP) Auction

- The bidders write down a price and send it to the auctioneer.
- The auctioneer awards the good to the bidder with the highest bid.
- The winner pays the amount of his bid.

Second price (SP) Auction

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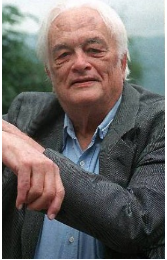

Second price auctions are also known as Vickrey auctions. defined by William Vickrey in 1961. Vickrey won the Nobel prize in Economics in 1996.

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- The auctioneer awards the good to the bidder with the highest bid.
- Everyone pays the amount of their bid regardless of whether or not they get the good.

Single item auction: model

- n bidders
- Each bidder has value v_i for the item willingness to pay.
  Known only to him – private value.
- If Bidder i wins and pays p_i, his utility is $v_i - p_i$.
  Her utility is 0 when she loses.

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- Bidders have to decide on a strategy to bid, a function applied to their valuation.

SP-Auctions: Equilibrium behaviour

Theorem

In SP-price auctions truth-telling is a dominant strategy.

SP-Auctions: Efficiency

- SP-auction is truthful.
- It is also efficient, i.e., in equilibrium, the auctioneer allocates the item to the bidder with the highest value.
  - With the actual highest value, not just the highest bid.
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- However the seller does not get maximum revenue.

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  Hard to select an strategy without some information about the others. We continue the analysis on a Bayesian setting.

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- Bidders do not know their opponent's values, i.e., there is incomplete information.
  Each bidder's strategy must maximize her expected payoff accounting for the uncertainty about opponent values.

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- What is the equilibrium in this game of incomplete information?

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- maximizing for b_1 we have: $[2b_1\,(v_1 - b_1)]' = 2v_1 - 4b_1 = 0$ which gives $b_1 = v_1/2$

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Theorem

In a FP auction with n bidders under the uniform values model, the strategy $b_i = \frac{n-1}{n} v_i$, *for* $1 \leq i \leq n$, *is a Bayesian Nash equilibrium.*

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- Thus, in the uniform value model FP is efficient.

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- Can we get a better understanding of revenue?
- Can we have a truthful auction giving maximum revenue?

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- When analyzing revenue take into account that when the item is not sold the seller gets a benefit of u.

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- So, with a single bidder,the optimal reserve price is halfway between the value of the object to the seller and the maximum possible bidder value.
- With more intricate analyses, you can determine the optimal reserve price for a second-price auction with multiple bidders

Buyer's utility

- Bidders have private values v_i for the item
- A winning bidder gets a utility of $u_i = v_i - p_i$
- A losing bidder pays nothing and gets $u_i = 0$

Seller's incentive

- Maximize social welfare: like in SP auctions
- Maximize revenue: like in SP auctions with reserve price

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- Moving from a specific example (single-item auctions) to a more general mechanism design setting.
- Objective: Design the right incentives such that the efficient outcome will be chosen.

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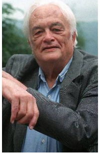
Vickrey-Clarke-Groves (VCG) mechanisms



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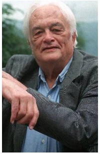  

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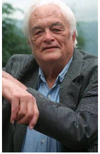

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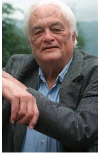

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- Solution: players should pay the damage they impose on society.

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- In a single item auction when i wins the object this payment is 2nd highest bid minus 0

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Here, again, truthfulness is a dominant strategy.

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  - Without player 1: welfare is 100.
  - $p_1 = 100 - 80$

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  - But, total payment is $20 + 30 < 100$!
    Cost is not covered!

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- In general, $p_1 = 100 - v_2$ and $p_2 = 100 - v_1$
  $p_1 + p_2 = 100 - v_1 + 100 - v_2 = 100 - (v_1 + v_2 - 100) < 100$
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  - Multiple positions, but advertisers submit only a single bid.
  - Search is highly targeted, and transaction oriented.
- Huge Google breakthrough in 2002.

A basic model: the setting

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- Advertiser i submits a bid b_i, the amount he is willing to pay for a click.

A basic model: outcome

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  The benefit per click is assumed to be independent of the slot.

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- If advertiser i is assigned to slot j at a price of p_i per click then her utility is

$$u_i = \alpha_j \gamma_i (v_i - p_i),$$

  which is the number of clicks received times profit per click.

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The social welfare is independent of the payments and the bids!

$$SW(\pi, v, \gamma)$$

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  Exercise: what would be the prices in the VCG auction?

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- The auctioneer sets $\pi(k)$ to be the advertiser with the kth highest effective bid (breaking ties arbitrarily).
- That is, the GSP mechanism assigns slots with higher click-through-rate to advertisers with higher effective bids.

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  where we take $b_{n+1} = 0$.
- In the case $\gamma_i = 1$, for each i,

  $$p_i = b_{\pi(k+1)}.$$

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- $u_i(b,\gamma)$ is the utility derived by advertiser i from the GSP mechanism when advertisers bid according to b:

$$u_i(b,\gamma) = \alpha_{\pi^{-1}(i)}\gamma_i(v_i - p_i)$$
$$= \alpha_{\pi^{-1}(i)}[\gamma_i v_i - \gamma_{\pi(\pi^{-1}(i)+1)} b_{\pi(\pi^{-1}(i)+1)}].$$

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- It is not a dominant strategy to bid "truthfully"

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GSP: Envy-free equilibrium

Given a GSP with n players defined by click-through-rates $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, quality scores $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ and valuations $v_1, \ldots, v_n$.

A bid vector b is an *envy-free* equilibrium if, for any pair j, k of players, player j would not prefer player k's allocation and payments rather than their own.

Formally

$$\alpha_j(\gamma_{\pi(j)} v_{\pi(j)} - \gamma_{\pi(j+1)} b_{\pi(j+1)}) \geq \alpha_k(\gamma_{\pi(j)} v_{\pi(j)} - \gamma_{\pi(k+1)} b_{\pi(k+1)})$$

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Theorem

The (pure) PoA of GSP in the full information setting is at most the golden ratio $\frac{1}{2}(1+\sqrt{5}) \approx 1.618$ *and at least* 1.282 .