Mixed strategies and Nash Equilibria

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1. Nash equilibrium
2. Linear Algebra formulation
3. Zero-sum games
4. The complexity of finding a NE
5. An exact algorithm to compute NE
6. NE algorithms

Mixed strategies

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- Consider an strategic game $\Gamma = (n, A_1, \ldots, A_n, u_1, \ldots, u_n)$.
- A mixed strategy for player i in Γ is a distribution (lottery) σ_i on the set of actions A_i.
- The utility function for player i is the expected utility under the joint distribution $\sigma = (\sigma_1, \ldots, \sigma_n)$ assuming independence:

$$U_i(\sigma) = \sum_{(a_1,\ldots,a_n)\in A} \sigma_1(a_1)\cdots\sigma_n(a_n)\cdot u_i(a_1\ldots,a_n)$$

Mixed Nash equilibrium

A mixed Nash equilibrium (NE) is a profile $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ such that no player i can get better utility choosing a distribution different from σ_i^*, given that every other player j adheres to σ_j^*.

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Theorem (Nash, 1950)

Every strategic game has a mixed Nash equilibrium.

From a computational point of view, mixed strategies present an additional representation problem.
In CS we can store only rational numbers. It is known

- For any two player strategic game with rational utilities there is always a mixed Nash equilibrium with rational probabilities.
- There are three player strategic games with rational utilities without rational mixed Nash equilibrium. [Schoenebeck and Vadhan: eccc 51, 2005]

NE in the Matching pennies game

utility	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

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- Is $((.5, .5), (.5, .5))$ a NE?

Checking for a Nash equilibrium

Given a distribution σ_i on A_i define the support of σ_i to be the set

$$\text{supp}(\sigma_i) = \{a_i \mid \sigma_i(a_i) \neq 0\}$$

Theorem

A mixed strategy profile σ is a Nash equilibrium iff, for any player i and any action $a_i \in supp(\sigma_i)$, a_i is a best response to σ_{-i}

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Consider a 2-player game $\Gamma = (A_1, A_2, u_1, u_2)$.
Let $X = \Delta(A_1)$ and $Y = \Delta(A_2)$.

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A Nash equilibrium is a mixed strategy profile $\sigma = (x, y) \in X \times Y$ such that, for every $x' \in X$, $y' \in Y$, it holds

$$U_1(x, y) \geq U_1(x', y) \text{ and } U_2(x, y) \geq U_2(x, y')$$

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$$U_1(x, y) = x^T R\, y \text{ and } U_2(x, y) = x^T C\, y$$

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A best response can be computed in polynomial time for 2-player games with rational utilities.

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- In terms of matrices we have $C = -R$.

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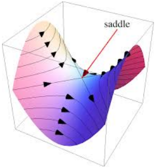

i.e., (x^*, y^*) is a saddle point
of the function $x^T R\, y$ defined over $X \times Y$.

Minimax inequality

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Theorem

For any function $\Phi : X \times Y :\to \mathbb{R}$, *we have*

$$\sup_{x \in X} \inf_{y \in Y} \Phi(x, y) \leq \inf_{y \in Y} \sup_{x \in X} \Phi(x, y).$$

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Taking the supremum over $x' \in X$ on the left hand-side we get the inequality. □

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We refer to $\inf_{y \in Y} \sup_{x \in X} x^T R\, y$ as the value of the game.

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$$\min v$$
$$v\mathbf{1}_n \geq Ry, y \in Y.$$

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- LP can be solved efficiently, thus there is a polynomial time algorithm for computing NE for zero-sum games.

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- Such problems are defined by an implicitly defined directed graph G and an unbalanced node u of G
  and the objective is finding another unbalanced node.
- Usually G is huge but implicitly defined as the graphs defining solutions in local search algorithms.

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- The class PPAD contains interesting computational problems not known to be in P
  has complete problems.
- But not a clear complexity cut.

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- We know that
  Every directed graph with in/outdegree 1 and a source, has a sink.

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- Which guarantees that
  the End-of-Line problem has always a solution.

End-of-Line: graph representation

- G is given implicitly by a circuit C
- C provides a predecessor and successor pair for each given vertex in G, i.e. $C(u) = (v, w)$.
- A special label indicates that a node has no predecessor/successor.

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- C. Daskalakis, P-W. Goldberg, C.H. Papadimitriou: The complexity of computing a Nash equilibrium. SIAM J. Comput. 39(1): 195-259 (2009) first version STOC 2006
- X. Chen, X. Deng, S-H. Teng: Settling the complexity of computing two-player Nash equilibria. J. ACM 56(3) (2009) first version FOCS 2006

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NE characterization

Theorem

In a strategic game in which each player has finitely many actions a mixed strategy profile σ^* is a NE iff, for each player i,

- *the expected payoff, given σ_{-i}, to every action in the support of σ_i^* is the same*
- *the expected payoff, given σ_{-i}, to every action not in the support of σ_i^* is at most the expected payoff on an action in the support of σ_i^*.*

NE conditions given support

Let $A \subseteq \{1, \dots n\}$ and $B \subseteq \{1, \dots m\}$.
The conditions for having a NE on this particular support can be written as follows:

$$\max \lambda_1 + \lambda_2$$

Subject to:

$$[R\, y]_i = \lambda_1, \text{ for } i \in A$$
$$[R\, y]_i \leq \lambda_1, \text{ for } i \notin A$$
$${}_j[C\, x] = \lambda_2, \text{ for } j \in B$$
$${}_j[C\, x] \leq \lambda_2, \text{ for } j \notin B$$

Iterating over all supports

- For every possible combination of supports $A \subseteq \{1, \dots n\}$ and $B \subseteq \{1, \dots m\}$.
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- This is an exact exponential time algorithm as the number of supports can be exponential.
- The same algorithm can be applied to a multiplayer game. We would be able to compute a NE on rationals if such a NE exists.

1 Nash equilibrium

2 Linear Algebra formulation

3 Zero-sum games

4 The complexity of finding a NE

5 An exact algorithm to compute NE

6 NE algorithms

NE algorithms

- Lemke-Howson (1964) algorithm defines a polytope based on best response conditions and membership to the support and uses ideas similar to Simplex with an ad-hoc pivoting rule.
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- Mixed-Integer Programming formulations [Sandholm, Gilpin and Conitzer, AAAI-05]