Pure Nash Equilibria
complexity versus succinctness

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Natural problems related to Best response

Is best response (IsBR)
Given a strategic game Γ, a (pure) strategy profile s, and a player i, decide whether a_i is a best response to s_{-i} in Γ.

Best response (BR)
Given a strategic game Γ, a (pure) strategy profile s, and a player i, compute an strategy in $BR_i(s)$.

Best responses (BRs)
Given a strategic game Γ, a (pure) strategy profile s, and a player i, compute $BR_i(s)$.

Natural problems related to PNE

Is Nash (ISN)
Given a game Γ and a strategy profile a, decide whether a is a Nash equilibrium of Γ.

Exists Pure Nash (EPN)
Given a strategic game Γ, decide whether Γ has a Pure Nash equilibrium [and, if so, provide one].

Pure Nash with Guarantees (PNGRANT)
Given a strategic game Γ and a value v, decide whetherthere is a pure Nash equilibrium in which the first player gets payoff v or higher [and, if so, provide one].

How to represent a game?

- We are interested in fixing the representation of a game as an input to a program.
- It is natural to consider different levels of succinctness.
- In the most generic model some components of the game have to be represented by a TM/program, for example the utility functions.

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We only consider rational valued utility functions

The convention guarantees a correct and unique game definition from its description

Explicit form

Strategic games in explicit form.

- *A game is given by a tuple*

$$\Gamma = \langle 1^n, A_1, \dots, A_n, T\rangle.$$

- *It has n players,*
- *For each player i, A_i is given explicitly by listing its elements.*
- *T is a table with an entry for each strategy profile s and player i.*
- *So, $u_i(s) = T(s, i)$.*

General form

Strategic games in general form.

- *A game is given by a tuple*

$$\Gamma = \langle 1^n, A_1, \dots, A_n, M, 1^t \rangle.$$

- *It has n players,*
- *For each player i, A_i is given explicitly by listing its elements.*
- *The description of their pay-off is given by $\langle M, 1^t \rangle$.*
- *So, for each strategy profile s and player i, $u_i(s) = M(s, i)$ stopping after t steps.*

Implicit form

Strategic games in implicit form.

- *A game is given by a tuple*

$$\Gamma = \langle 1^n, 1^m, M, 1^t \rangle.$$

- *It has n players,*
- *For each player i, $A_i = \Sigma^m$*
- *The description of their pay-off is given by $\langle M, 1^t \rangle$.*
- *So, for each strategy profile s and player i, $u_i(s) = M(s, i)$ stopping after t steps.*

Forms of representation

Strategic games in explicit form. A game is descrbed by a tuple $\Gamma = \langle 1^n, A_1, \dots, A_n, T \rangle$.

Strategic games in general form. A game is described by a tuple $\Gamma = \langle 1^n, A_1, \dots, A_n, M, 1^t \rangle$.

Strategic games in implicit form. A game is described by a tuple $\Gamma = \langle 1^n, 1^m, M, 1^t \rangle$.

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Congestion games

Congestion games

A congestion game

- is defined on a finite set E of resources and
- has n players
- using a delay function d mapping $E \times \mathbb{N}$ to the integers.
- The actions for each player are subsets of E.
- The pay-off functions are the following:

$$u_i(a_1, \ldots, a_n) = -\left(\sum_{e \in a_i} d(e, f(a_1, \ldots, a_n, e))\right)$$

being $f(a_1, \ldots, a_n, e) = |\{i \mid e \in a_i\}|$.

Network congestion games

Network congestion games

A network congestion game

- is defined on a directed graph $G = (V, E)$ resources are the edges
- has n players
- using a delay function d mapping $E \times \mathbb{N}$ to the integers.
- The actions for each player are paths from s_i to t_i, for some $s_i, t_i \in V(G)$.
- The pay-off functions are the following:

$$u_i(a_1, \dots, a_n) = -\left(\sum_{e \in a_i} d(e, f(a_1, \dots, a_n, e))\right)$$

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  The condition $u_i(s) \geq u_i(s_{-i}, a_i)$ can be checked in polynomial time given Γ, s, and a_i.

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  Is this classification tight?

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A coNP complete problem?

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We have to associate to F a game Γ and a strategy profile s so that:

- F is not satisfiable iff s is a PNE of Γ
- and show that a description of Γ in implicit form and of s can be obtained in time polynomial in $|F|$.

IsPN implicit form: Hardness

Given a CNF formula F on n variables consider the game $\Gamma(F)$ which:

- Has one player and $A_1 = \{0,1\}^{n+1}$
- $u_1(0x) = 0$, for any $x \in \{0,1\}^n$
- $u_1(1x) = F(x)$, for any $x \in \{0,1\}^n$

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Thus $\Gamma(F), 0^{n+1}$ verify the first requirement.

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- $t = (n + |F|)^2$.

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The time required to obtain $\langle 1^n, 1^m, M, 1^t \rangle$, given F, is polynomial in $|F|$.

IsPN implicit form

Theorem

The IsPN problem for strategic games in implicit form is coNP-complete.

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EPN: general form

Theorem

The EPN problem for strategic games in general form is NP-complete.

We provide a reduction from SAT. Let F be a CNF formula.

- $F \to \Gamma(F) = \langle 1^n, \{0,1\} \dots \{0,1\}, M^F, 1^{(n+|F|)^2} \rangle$ where
- n is the number of variables in F and
- M^F is a TM that on input (a, i), evaluates F on assignment a and afterwards it implements the utility function of the i-th player. According to the following definition:

EPN: general form

$$u_1(a) = \begin{cases} 5 & \text{if } F(a) = 1, \\ 4 & \text{if } F(a) = 0 \wedge a_1 = 0 \wedge a_2 = 1, \\ 3 & \text{if } F(a) = 0 \wedge a_1 = 1 \wedge a_2 = 1, \\ 2 & \text{if } F(a) = 0 \wedge a_1 = 1 \wedge a_2 = 0, \\ 1 & \text{if } F(a) = 0 \wedge a_1 = 0 \wedge a_2 = 0, \end{cases}$$

$$u_2(a) = \begin{cases} 5 & \text{if } F(a) = 1, \\ 4 & \text{if } F(a) = 0 \wedge a_1 = 0 \wedge a_2 = 0, \\ 3 & \text{if } F(a) = 0 \wedge a_1 = 0 \wedge a_2 = 1, \\ 2 & \text{if } F(a) = 0 \wedge a_1 = 1 \wedge a_2 = 1, \\ 1 & \text{if } F(a) = 0 \wedge a_1 = 1 \wedge a_2 = 0. \end{cases}$$

And, for any $j > 2$

$$u_j(a) = \begin{cases} 5 & \text{if } F(a) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Reduction correctness

We have that

- Given a description of F, $\Gamma(F)$ is computable in polynomial time.

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- F is satisfiable iff $\Gamma(F)$ has a PNE?

Reduction trick

Look at the two player strategic game that can be played by the first and second players:

	0	1
0	1,4	4,3
1	2,1	3,2

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  Such a strategy profile is a PNE.
- F is a no instance of SAT.
  For any strategy profile the payoff of players $j > 2$ is always 1.
  So they cannot change strategy and improve payoff.
  However, players 1 and 2 are engaged in a game with no PNE so one of them can change strategy and increase its payoff.
  Therefore $\Gamma(F)$ has no PNE

Σ_2^p definition and a complete problem

Let $L \subseteq \Sigma^*$ be a language.
$L \in \Sigma_2^p$ if and only if there is a polynomially decidable relation R and a polynomial p such that

$$L = \{x \mid \exists z |z| \leq p(|x|) \forall y |y| \leq p(|x|) \langle x, y, z \rangle \in R\}.$$

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Q2SAT
Given $\Phi = \exists \alpha_1, \dots, \alpha_{n_1} \forall \beta_1, \dots \beta_{n_2} F$ where F is a Boolean formula over the boolean variables $\alpha_1, \dots, \alpha_{n_1}, \beta_1, \dots, \beta_{n_2}$, decide whether Φ is valid.

Q2SAT is Σ_2^p-complete.

EPN: implicit form

Theorem

The EPN problem for strategic games in implicit form is Σ_2^p-complete.

Lets provide a reduction from $\mathrm{Q2SAT}$.

EPN implicit form:reduction

For each $\Phi = \exists \alpha_1, \dots, \alpha_{n_1} \forall \beta_1, \dots \beta_{n_2} F$
we define a game $\Gamma(\Phi)$ as follows.
There are four players:

EPN implicit form:reduction

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There are four players:

- Player 1, the *existential player*, assigns truth values to the boolean variables $\alpha_1, \ldots, \alpha_{n_1}$ and
  $A_1 = \{0, 1\}^{n_1}$ and $a_1 = (\alpha_1, \ldots \alpha_{n_1}) \in A_1$.

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- Player 2, the *universal player*, assigns truth values to the boolean variables $\beta_1, \dots, \beta_{n_2}$ and
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  $A_2 = \{0,1\}^{n_2}$ and $a_2 = (\beta_1, \dots, \beta_{n_2}) \in A_2$.
- Players 3 and 4 avoid entering into a Nash equilibrium when the actions played by players 1 and 2 do not satisfy F. Their set of actions are $A_3 = A_4 = \{0,1\}$.

Let us denote by $F(a_1, a_2)$ the truth value of F under the assignment given by a_1 and a_2.

$$u_1(a_1, a_2, a_3, a_4) = \begin{cases} 1 & \text{if } F(a_1, a_2) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$u_2(a_1, a_2, a_3, a_4) = \begin{cases} 1 & \text{if } F(a_1, a_2) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$u_3(a_1,a_2,a_3,a_4)=\begin{cases}5 & \text{if } F(a_1,a_2)=1,\\ 4 & \text{if } F(a_1,a_2)=0 \wedge a_3=0 \wedge a_4=1,\\ 3 & \text{if } F(a_1,a_2)=0 \wedge a_3=1 \wedge a_4=1,\\ 2 & \text{if } F(a_1,a_2)=0 \wedge a_3=1 \wedge a_4=0,\\ 1 & \text{if } F(a_1,a_2)=0 \wedge a_3=0 \wedge a_4=0.\end{cases}$$

$$u_4(a_1,a_2,a_3,a_4)=\begin{cases}5 & \text{if } F(a_1,a_2)=1,\\ 3 & \text{if } F(a_1,a_2)=0 \wedge a_3=0 \wedge a_4=1,\\ 2 & \text{if } F(a_1,a_2)=0 \wedge a_3=1 \wedge a_4=1,\\ 1 & \text{if } F(a_1,a_2)=0 \wedge a_3=1 \wedge a_4=0,\\ 4 & \text{if } F(a_1,a_2)=0 \wedge a_3=0 \wedge a_4=0.\end{cases}$$

EPN implicit form:reduction correcteness

- Let us assume that $\Phi = \exists \alpha_1, \dots, \alpha_n \forall \beta_1, \dots, \beta_m F$, where F is a Boolean formula over the boolean variables $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m$, is true.
- Then there exists $\alpha \in \{0,1\}^n$ such that for all $\beta \in \{0,1\}^m$, $F(\alpha, \beta) = 1$.
- This means that if player 1 plays action α, for each $\beta \in \{0,1\}^m$, $a_3, a_4 \in \{0,1\}$, no player has incentive to change strategy.

EPN implicit form:reduction correcteness

- Let us assume that Φ is not valid.
- It means that for any $\alpha \in \{0,1\}^n$ there exists $\beta \in \{0,1\}^m$ such that $F(\alpha, \beta) = 0$.
- Let (α, β, a, b) be a strategy profile. We have two cases.

EPN implicit form:reduction correcteness

- Let us assume that Φ is not valid.
- It means that for any $\alpha \in \{0,1\}^n$ there exists $\beta \in \{0,1\}^m$ such that $F(\alpha, \beta) = 0$.
- Let (α, β, a, b) be a strategy profile. We have two cases.
- Case 1: $F(\alpha, \beta) = 0$, in this case players 3 an 4 engage in a no PNE game.

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- Let (α, β, a, b) be a strategy profile. We have two cases.
- Case 1: $F(\alpha, \beta) = 0$, in this case players 3 an 4 engage in a no PNE game.
- Case 2: $F(\alpha, \beta) = 1$, since Φ is not valid, there exists $\beta' \in \{0,1\}^m$ such that $F(\alpha, \beta') = 0$. Therefore player 2 has an incentive to change strategy β by β'.

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- Case 1: $F(\alpha, \beta) = 0$, in this case players 3 an 4 engage in a no PNE game.
- Case 2: $F(\alpha, \beta) = 1$, since Φ is not valid, there exists $\beta' \in \{0,1\}^m$ such that $F(\alpha, \beta') = 0$. Therefore player 2 has an incentive to change strategy β by β'.
- Therefore, the strategy profile is not a PNE.

PNGrant problem

PNGrant Given a strategic game Γ and a value v, decide whether there is a PNE s so the $u_1(s) \geq v$.

Theorem

The PNGrant problem
can be solved in polynomial time for strategic games given in explicit form but it
is NP-complete for strategic games given in general form
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Membership follows from the same arguments.
In all the reduction the utility for the first player in all PNE is constant, this provides the value of v in each reduction.

(Boolean) Circuit games

[Schoenebeck and Vadhan, EC 2006 - ACM TCT 2012]

- In a circuit game, players still control disjoint sets of variables, but each player's payoff is given by a single boolean circuit.
- The boolean circuit computes a rational value as the quotient of two integers
- Boolean circuit games are the special case of circuit games where each player controls a single boolean variable.

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TMs can be simulated by circuits and viceversa

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TMs can be simulated by circuits and viceversa

- Circuit games are equivalent to implicit form games
- Boolean circuit games are a subset of general form games.

(Boolean) weighted formula games

[Mavronicolas, Monien, Wagner, WINE 2007]

- In a formula game, players still control disjoint sets of variables, but each player's payoff is given by a weighted combination of boolean formulas.
- Boolean formula games are the special case of formula games where each player controls a single boolean variable.

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- Formulas can be casted as circuits but not viceversa as the size might grow exponentially.
- Nevertheless the utility functions of the provided reductions can be easily described in this way.
  So the problems are equivalent from the complexity point of view.

Graphical games

[Gottlob, Greco and Scarcello, JAIR 2005]

- Graphical games are a representation of multiplayer games meant to capture and exploit locality or sparsity of direct influences.
- They are most appropriate for large population games in which the payoffs of each player are determined by the actions of only a small subpopulation.
- Players' relationship is described by a graph and the payoff of a player depends only on the actions of its neighbors.

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- Provide a complementary framework to analyze complexity based on the graph parameters:

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- Players' relationship is described by a graph and the payoff of a player depends only on the actions of its neighbors.
- Provide a complementary framework to analyze complexity based on the graph parameters: bounded degree, bounded treewidth, ...

Conclusions

- We have analyzed some ways of describing strategic games with polynomial time computable utilities
- We have concentrated on the study of two computational problems.
- As expected complexity increases with succinctness.
- There are many other
  - game classes
  - and problems of interest

  with similar behavior.

References

Contents taken from a subset of the results in

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Further suggested reading (among many others)

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