Efficiency of Nash Equilibria

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- The players' goals can be different from those of the society.
- Fixing a social goal an optimal situation is possible.
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- To perform such an analysis for strategic games we have first to define a global function to optimize, this function is usually called the social cost or social utility.
- Society is interested in minimizing the social cost or maximizing the social utility.

Social cost

- Consider a n-player game $\Gamma = (A_1, \ldots, A_n, u_1, \ldots, u_n)$.
- Let $A = A_1 \times \cdots \times A_n$.
- Let $PNE(\Gamma)$ be the set of PNE of Γ.
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- Let $\mathcal{C} : A \to \mathbb{R}$ be a social cost function.

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- Game specific cost/utility defined by the model motivating the game.

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For social utility functions the terms are inverted in the definition.

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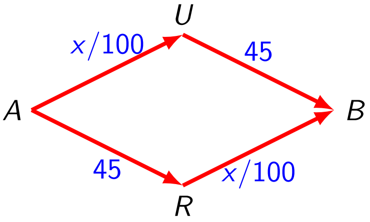
- PoA measures the worst decentralized equilibrium scenario giving the maximum system degradation.
- PoS measures the best decentralized equilibrium scenario giving the best possible degradation.

A network congestion game

- 4000 drivers drive from A to B on



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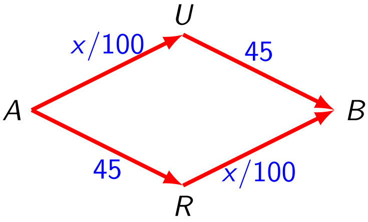

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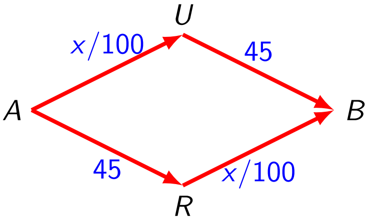

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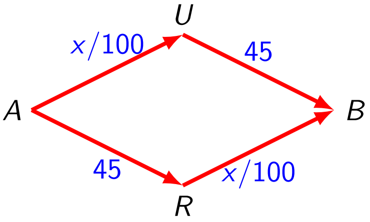

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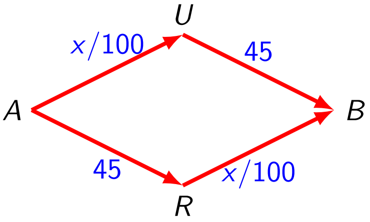

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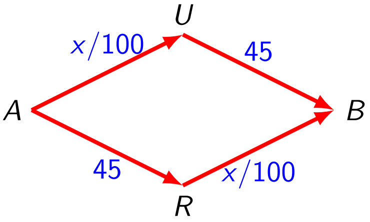

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- $PoA = PoS = 65/65 = 1$

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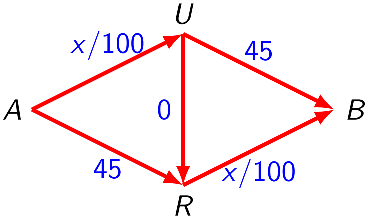

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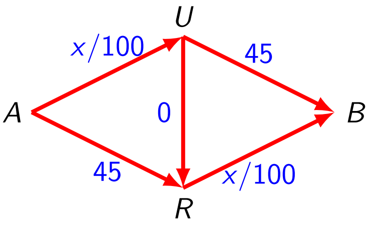

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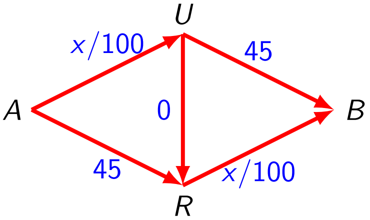

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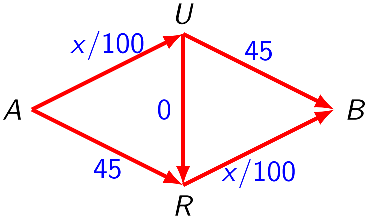

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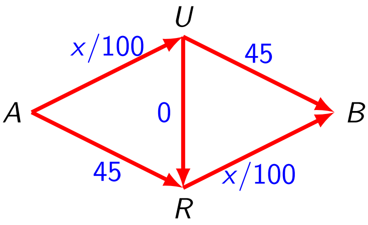

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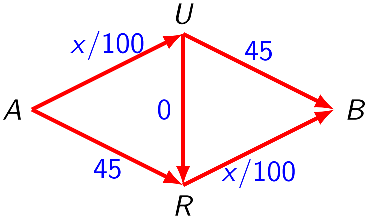

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- $PoA = PoS = 80/65 = 16/13$

Load Balancing game

- There are m servers and n jobs. Job i has load p_i.
- The game has n players, corresponding to the n jobs.
- Each player has to decide the server that will process its job. $A_i = \{1, \dots, m\}$
- The response time of server j is proportional to its load

$$L_j(s) = \sum_{i|s_i=j} p_i.$$

- Each job wants to be assigned to the server that minimizes its response time:

$$c_i(s) = L_{s_i}(s).$$

Load Balancing game: PNE?

Consider the following better response dynamic

- Start with an arbitrary state.
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- How to prove that such a process converges to a PNE?
- Seek for an adequate kind of potential function, a function that increases at each step.

Load Balancing game: PNE?

BR-inspired-algorithm analysis

- Order the servers with decreasing load (i.e., the decreasing response time):

$$L_1 \geq L_2 \geq \cdots \geq L_m.$$

- Player i moves from server j to k if $L_k + p_i < L_j$.

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- Each step of the BR algorithm defines a sorted sequence of loads.

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- So BR-inspired-algorithm terminates (although it can be rather slow).
- The load balancing game has a PNE.

Load Balancing game: Social cost

- The natural social cost is the total finish time i.e., the maximum of the server's loads

$$C(s) = \max_{j=1}^{m} L_j.$$

- How bad/good is a PNE?

Load Balancing game: PoS

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- Therefore, $PoS(\Gamma) = 1$.

Load Balancing game: PoA

Theorem

The max load of a Nash equilibrium s is within twice the max load of an optimum assignment, i.e.,.

$$C(s) \leq 2 \min_{s'} C(s').$$

Which will give $PoA(\Gamma) \leq 2$.

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$$C(s) = L_j \leq (\sum_k L_k)/m + p_i \leq (\sum_\ell p_\ell)/m + p_i$$
$$\leq C(s') + C(s').$$