An introduction to Computational Social Choice

Spring 2026

1. Social Choice
2. Scoring protocols
3. Some properties of voting rules
4. Manipulation, bribery and control
5. More voting rules
6. Basic manipulation

Social Choice Theory

- Mathematical theory for aggregating individual preferences into collective decisions
- Originated in ancient Greece. Formal foundations:
  - 18th Century (Condorcet and Borda)
  - 19th Century: Charles Dodgson (a.k.a. Lewis Carroll)
  - 20th Century: Nobel prizes to Arrow and Sen
- Objective: Methods to select a collective outcome based on (possibly different) individual preferences.

Social Choice Theory

- Set of voters $N = \{1, \ldots, n\}$
- Set of alternatives $A = \{1, \ldots, m\}$
- Voter i has a preference ranking over alternatives $\succ_i$
- Preference ranking $\succ$ is the collection of all voters' rankings

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Ply	1	2	3
	a	c	b
	b	a	c
	c	b	a

Social Choice Theory: Objective

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- voting rule = social choice function

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Voting rules: Plurality

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- Each voter awards one point to her top alternative
- Alternative with the most point wins

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	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

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Winner
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- Most frequently used voting rule
- Many political elections use plurality

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Problems?

Voting rules: Borda count

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N	1	2	3	4	5	pnts
	a	a	a	b	b	4
	b	b	b	c	c	3
	c	c	c	d	d	2
	d	d	d	e	e	1
	e	e	e	a	a	0

Total
a: 12
b: 17
c: 12
d: 7
e: 2

Winner
b

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- Used for elections to the national assembly of Slovenia
- A modified Borda Count is used in Eurovision Song Contest

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	d	d	d	e	e
	e	e	e	a	a

$k = 3$ Total
a: 3
b: 5
c: 5
d: 2
e: 0

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	a	a	a	b	b
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$k = 3$ Total
a: 3
b: 5
c: 5
d: 2
e: 0

Winner
b or c

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$k = 3$ Total
a: 3
b: 5
c: 5
d: 2
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Winner
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- Approval voting was used for papal conclaves between 1294 and 1621.
- A modified k-approval is used in FIB committee elections

Voting rules: Positional Scoring Rules

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- The family include many rules
  - Plurality $s = (1, 0, \dots, 0)$
  - Borda $s = (m-1, m-2, \dots, 0)$
  - k-aproval $s = (1, \dots, 1, 0, \dots, 0)$
  - Veto $s = (0, \dots, 0, 1)$
  - ...

Which rule to use?

- We just introduced infinitely many rules, and there are more
- How do we know which is the “right” rule to use?
  Axioms, Characterization theorems, Impossibility Theorems
- Impossibility versus Computational hardness

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Condorcet winner

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Marie Jean Antoine
Nicolas de Caritat,
marqués de Condorcet
(1743-1794)

- x beats y in a pairwise election if a strict majority of voters prefer x to y.

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- A Condorcet paradox happens when the majority preference has a cycle.

Condorcet Paradox: Example

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Majority Pref
$a \succ b$
$b \succ c$
$c \succ a$

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Also known as Dodgson's Paradox (Alice in Wonderland by Charles L. Dodgson alias Lewis Carroll)

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- Computational problems:
  - Existence of Condorcet winner.
  - Obtaining the Condorcet winner, when it exists

Three alternatives $\{A, B, C\}$ and 19 voters

		Pairwise comparison		
#	Vote	$A?B$	$A?C$	$B?C$
6	$A > B > C$	A	A	B
4	$B > A > C$	B	A	B
2	$B > C > A$	B	C	B
4	$C > A > B$	A	C	C
3	$C > B > A$	B	C	C
Result		A	A	B

A is the Condorcet winner

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A is the Condorcet winner

Borda		
A	B	C
12	6	0
4	8	0
0	4	2
4	0	8
0	3	6
20	21	16

B is the winner

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4	$C > A > B$	A	C	C	0	0	4
3	$C > B > A$	B	C	C	0	0	3
Result		A	A	B	6	6	7

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Plurality A	B	C
6	0	0
0	4	0
0	2	0
0	0	4
0	0	3
6	6	7

A is the Condorcet winner

C is the winner

Plurality = 1-aproval. None of the three rules is Condorcet consistent.

Dodgson Voting

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- The Dodgson score of candidate c is the minimum number of swaps in the votes required to make c the Condorcet winner.
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- Of course, the rule is Condorcet consistent.

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- The Dodgson score of candidate c is the minimum number of swaps in the votes required to make c the Condorcet winner.
- All candidates with the lowest Dodgson score win.

- Of course, the rule is Condorcet consistent.
- However, determining the winner is NP-hard:
  J. Bartholdi, III, C. A. Tovey and M. A. Trick, *Voting Schemes for which It Can Be Difficult to Tell Who Won the Election*, Social Choice and Welfare Vol. 6, No. 2 (1989), pp. 157-165

Strategy-proofness

- Would any of the voting rules ensure that voters report their preferences truthfully?

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- Assume $\succ = (\succ_1, \ldots, \succ_n)$ is a preference profile so that, $\succ_i$ is the true preferences of voter i.
- A voting rule F is strategy-proof if for every preference profile $\succ' = (\succ_{-i}, \succ'_i)$, the profile differs from $\succ$ only in the i-th component, it is not the case that $F(\succ') \succ_i F(\succ)$

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- Informally, a voting rule is strategy-proof if there exists no profile where some voter can obtain a preferred outcome by changing her preferences.
- Which voting rules are strategy-proof?
- Do they have good properties?
- When they are not, can the manipulation be computed easily?

Strategy-proofness: Borda count

- Borda with true preferences

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	c	c	c	1
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Winner
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- Voter 3 can make *a* win by

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Which rules are strategy-proof?

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Strategy-proof rules?

- Are there strategy-proof rules?
- Dictatorial voting rule
  - The winner is always the most preferred alternative of voter 1
- Constant voting rule
  - The winner is always the same
- Yes, both are strategy-proof, but not very satisfactory!

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- Onto: Every alternative can win under some preference profile.
- Non-dictatorial: There is no voter i such that $F(\succ)$ is always the top alternative for voter i.

Gibbard-Satterthwaite

Theorem

For $m \geq 3$, no deterministic social choice function can be strategyproof, onto, and nondictatorial simultaneously.

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In ☺*words,*

Gibbard-Satterthwaite

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In ☺*words, for $m \geq 3$, any deterministic social choice function must be at least one of the following:*

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In ☺words, for $m \geq 3$, any deterministic social choice function must be at least one of the following:

- *dictatorial: there exists a single fixed voter whose most-preferred alternative is chosen for every profile;*
- *imposing: there is at least one alternative that does not win under any profile;*
- *manipulable (i.e., not strategyproof).*

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The first two properties are not acceptable. We need to assume that the voters have an incentive to misreport true preferences.

Circumventing G-S: Complexity approach

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  For once NP-hardness can be good!!

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Manipulation/Bribery

E-manipulation: Given a set C of candidates, a set V of nonmanipulative voters, a set S of manipulative voters, with $S \cap V = \emptyset$, and a candidate $c \in C$. Is there a way to set the preference lists of the voters in S such that, under election system E, c is the (a) winner?

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E-Bribery: Given a set C of candidates, a set V of voters, a candidate $c \in C$, and a nonnegative integer k. Is there a way to set the preference lists of at most k voters such that, under election system E, c is the (a) winner?

Control

E-Control under additive candidates: Given a set C of candidates, a pool D of potential additional candidates, a candidate $c \in C$, and a set of voters V with preferences over $C \cup D$. Is there a set $D' \subseteq D$, such that setting the set of candidates to $C \cup D'$, under election system E, c is the (a) winner?

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E-Destructive control under additive candidates: Given a set C of candidates, a pool D of potential additional candidates, a candidate $c \in C$, and a set of voters V with preferences over $C \cup D$. Is there a set $D' \subseteq D$, such that setting the set of candidates to $C \cup D'$, under election system E, c is not a winner?

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	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

Voting rules: Plurality with runoff

- First round: two alternatives with the highest plurality scores survive
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- Similar to the French presidential election system
  - Problem: vote division
  - Happened in the 2002 French presidential election

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- Social choice: The top alternative in σ^*

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Condorcet consistency

- Among the rules we just saw
  - All positional scoring rules (plurality, Borda, . . .), plurality with runoff, STV, are NOT Condorcet consistent.
  - Kemeny, Copeland, Maximin ARE Condorcet consistent.

1. Social Choice
2. Scoring protocols
3. Some properties of voting rules
4. Manipulation, bribery and control
5. More voting rules
6. **Basic manipulation**

Manipulation Problem

We consider the simplest version of manipulation.

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- For plurality, this problem is computationally trivial as the only sensible manipulation is to put a as your most preferred candidate.

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- For every profile Π and every alternative a, a score $S(\Pi, a)$ can be defined so that it is,
  - Responsive: the candidate with the largest score wins (in the voting under the joint profile)
  - Monotone: for any two profiles Π and Π' and for any alternative a, if for each voter i, $\{b \mid a \succ_i b\} \subseteq \{b \mid a \succ_i' b\}$, then $S(\Pi, a) \leq S(\Pi', a)$.

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  Determine whether a candidate b can be placed in the next lower position (independent of remaining choices) without preventing c from winning.
  If so, place b in the next position, otherwise terminate claiming that such order does not exist.

Manipulable rules by Greedy

Theorem

*For any voting rule F satisfying the BTT conditions, G-Man solves the F-*MANIPULATION *problem.*

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- If Greedy-Manipulation succeeds, it constructs a preference order that guarantees that under the joint profile c wins.
- Assume that such an order exists and that Greedy-Manipulation terminates without providing an ordering. Let us reach a contradiction.

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- Consider any completion P of the preference order started by G-Man that places u in the first unassigned place, and let Π be the corresponding joint profile. W.l.o.g., n is the manipulative player.

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- So, $S(\Pi, c) \geq S(\Pi, u)$.
- But G-Man did not assign u, so $S(\Pi, c) < S(\Pi, u)$ since u cannot go in the assignes slot and we get the contradiction.

Manipulable rules by Greedy

Corollary

For any voting rule F satisfying the BTT conditions, and for which the scoring rule can be computed in polynomial time G-Man solves the F-MANIPULATION *problem in polynomial time.*

Proof

The iterative step can be checked by

- Placing a candidate b at the next position and completing arbitrarily the rest of the order to profile Π
- Compute $S(\Pi, c)$ and $S(\Pi, b)$.
- $S(\Pi, c) > S(\Pi, b)$ iff b can placed in the next position in the order without preventing
  By monotonicity, the score cannot change by modifying the remaining alternatives.

With this implementation, a step of the greedy algorithm requires $O(n)$ score computations.

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- Both, Copeland vote and the score can be computed in polynomial time.
- Copeland is polynomial time manipulable.

Maximin

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  - $Score(x) = min_y n_{x \succ y}$
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- Working in a similar way, Maximin is polynomial time manipulable.

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STV-Manipulation is NP-hard (Bartholdi III and Orlin, Social Choice and Welfare, 1991)

- The NP-hardness follows by a reduction from the 3-cover problems which is NP-complete problem (3-Cover).
- The basic idea is to build a large election instance introducing all sorts of constraints on the ballot of the manipulator, such that finding a ballot meeting those constraints solves a given instance of 3-Cover.