Strategic games: Basic definitions and examples

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Spring 2026

Basic References non-coop game theory

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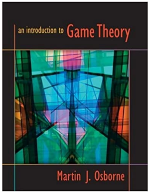

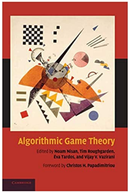

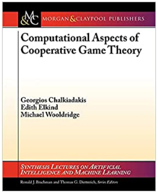

Where to use game theory?

Game theory studies

- decisions made in an environment in which players interact.
- the choice of optimal behavior when personal costs and benefits depend upon the choices of all participants.

Game theory looks for

- states of equilibrium sometimes calles solutions
- and analyzes interpretations/properties of such states

Game Theory for CS?

- Framework to analyze equilibrium states of protocols used by rational agents.
  Price of anarchy/stability.
- Tool to design protocols for internet with guarantees.
  Mechanism design.
- New concepts to analyze/justify behavior of on-line algorithms
  Give guarantees of stability to dynamic network algorithms.
- Source of new computational problems to study.
  Algorithmics and computational complexity

Types of games

- Non-cooperative games
  - strategic games
  - extensive games
  - repeated games
  - Bayesian games
  - ...
- Cooperative games
  - simple games
  - transferable utility games
  - non-transferable utility games
  - ...

One example: Strategic games

The story

- Two people choose, simultaneously, whether to show the head or tail of a coin.
- If they show same side, person 2 pays person 1 1eur, otherwise person 1 pays person 2 1eur.
- Payoff are equal to the amounts of money involved.

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utility	Head	Tail
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Tail	-1,1	1,-1

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  Type 2 costs \$9, contains 750g
  Type 3 costs \$11, contains 1kg

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- The payoff of each group is the maximum quantity of ice-cream the members of the group can buy by pooling all their money.
- The ice-cream can be shared arbitrarily within the group.

Strategic game

A strategic game Γ (with ordinal preferences) consists of:

- A finite set $N = \{1, \ldots, n\}$ of players.
- For each player $i \in N$, a nonempty (finite) set of actions A_i.
- Each player chooses his action once. Players choose actions simultaneously.
  No player is informed, when he chooses his action, of the actions chosen by others.
- After individual choice, a strategy profile is formed. The outcome of the game is an element in $A = A_1 \times \cdots \times A_n$.
- For each player $i \in N$, a preference relation (a complete, transitive, reflexive binary relation) $\preceq_i$ over the set A.

It is frequent to specify the players' preferences by giving either utility (pay-off) functions $u_i(a_1, \ldots a_n)$ or cost functions $c_i(a_1, \ldots a_n)$.

Example: Prisoner's Dilemma

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The story

- Two suspects in a major crime are held in separate cells.
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The penalties

- If both stay quiet, be convicted for a minor offense (1 year each).
- If only one finks, he will be freed (and used as a witness) and the other will be convicted for a major offense (4 years).
- If both fink, each one will be convicted for a major offense with a reward for cooperation (3 years each).

Prisoner's Dilemma: Benefits?

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The Prisoner's Dilemma models a situation in which

- there is a gain from cooperation,
- but each player has an incentive to free ride.

Prisoner's Dilemma: rules and preferences

Rules

- Players $N = \{\text{Suspect 1}, \text{Suspect 2}\}$.
- Actions $A_1 = A_2 = \{\text{Quiet}, \text{Fink}\}$.
- Strategy profiles $A = A_1 \times A_2 =$ $\{(\text{Quiet}, \text{Quiet}), (\text{Quiet}, \text{Fink}), (\text{Fink}, \text{Quiet}), (\text{Fink}, \text{Fink})\}$

Preferences

- Player 1

$$(\text{Fink}, \text{Quiet}) \preceq_1 (\text{Quiet}, \text{Quiet}) \preceq_1 (\text{Fink}, \text{Fink}) \preceq_1 (\text{Quiet}, \text{Fink})$$

- Player 2

$$(\text{Quiet}, \text{Fink}), \preceq_2 (\text{Quiet}, \text{Quiet}) \preceq_2 (\text{Fink}, \text{Fink}) \preceq_2 (\text{Fink}, \text{Quiet})$$

Prisoner's Dilemma: rules and costs

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  $\{(\text{Quiet}, \text{Quiet}), (\text{Quiet}, \text{Fink}), (\text{Fink}, \text{Quiet}), (\text{Fink}, \text{Fink})\}$

profile	c_1	c_2
(Fink, Quiet)	0	3
(Quiet, Quiet)	1	1
(Fink, Fink)	2	2
(Quiet, Fink)	3	0

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Rationality: Players choose actions in order to minimize personal cost

Prisoner's Dilemma: rules and utilities

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profile	u_1	u_2
(Fink, Quiet)	3	0
(Quiet, Quiet)	2	2
(Fink, Fink)	1	1
(Quiet, Fink)	0	3

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- Players $N = \{\text{Suspect 1}, \text{Suspect 2}\}$.
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(Quiet, Quiet)	2	2
(Fink, Fink)	1	1
(Quiet, Fink)	0	3

Rationality: Players choose actions in order to maximize personal utility

Prisoner's Dilemma: bi-matrix representation

We can represent the game in a compact way on a bi-matrix.

utility	Quiet	Fink
Quiet	2,2	0,3
Fink	3,0	1,1

cost	Quiet	Fink
Quiet	1,1	3,0
Fink	0,3	2,2

Example: Matching Pennies

The story

- Two people choose, simultaneously, whether to show the head or tail of a coin.
- If they show same side, person 2 pays person 1 1eur, otherwise person 1 pays person 2 1eur.
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This is an example of a zero-sum game

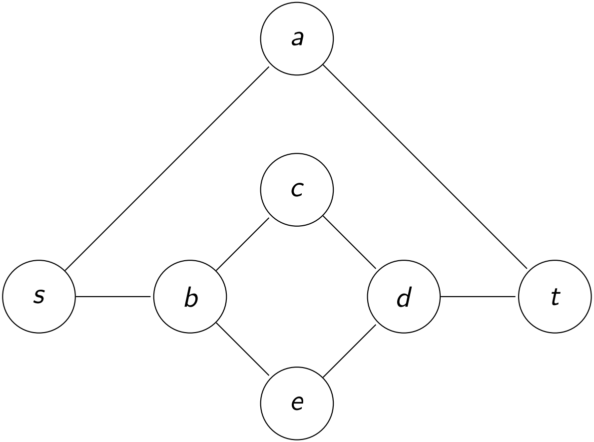
Example: Sending from s to t

The story

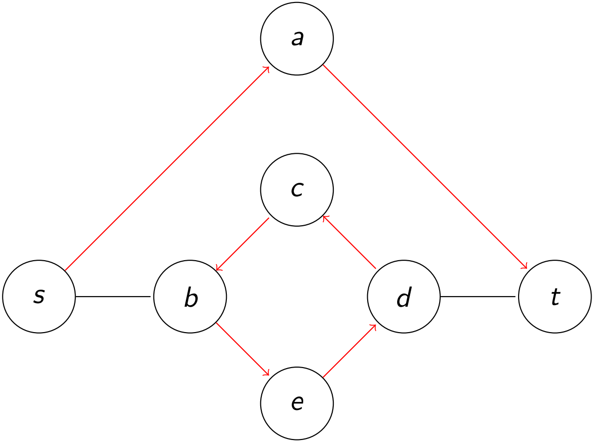
- We have a graph $G = (V, E)$ with n vertices and two vertices $s, t \in V$.
- There is one player for each vertex $u \in V$, $u \neq t$.
- The set of actions for player u is $N_G(u)$.
- A strategy profile associates a neighbor v_u to any vertex $u \in V$, $u \neq t$. This is a set of $n - 1$ vertices.
- Pay-offs are defined as follows:
  player u gets 1 if a shortest path joining s to t in the digraph $(V, \{(u, v_u) \mid u \neq t\})$ contains (u, v_u), otherwise u gets 0.

Players are selfish but the system can get a different reward/cost. For example the length of a shortest path.

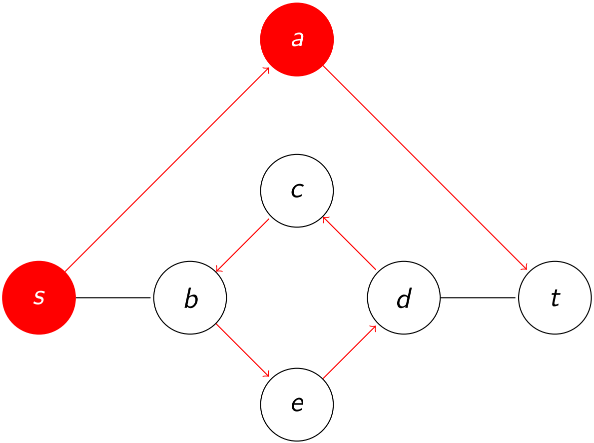
Sending from s to t: example



Sending from s to t: strategy profile (1)

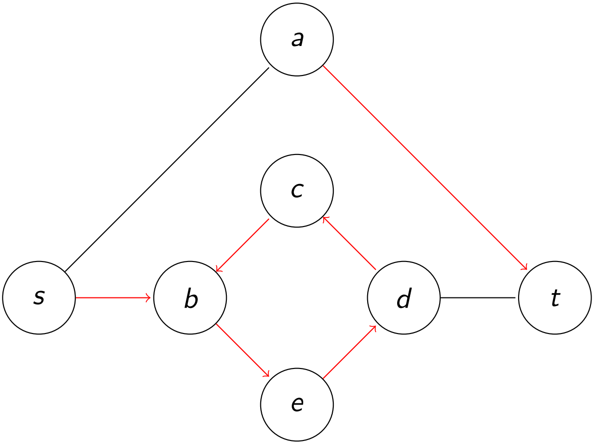


Sending from s to t: pay-offs (1)



Red nodes get pay-off 1, other nodes get pay-off 0.

Sending from s to t: strategy profile (2)



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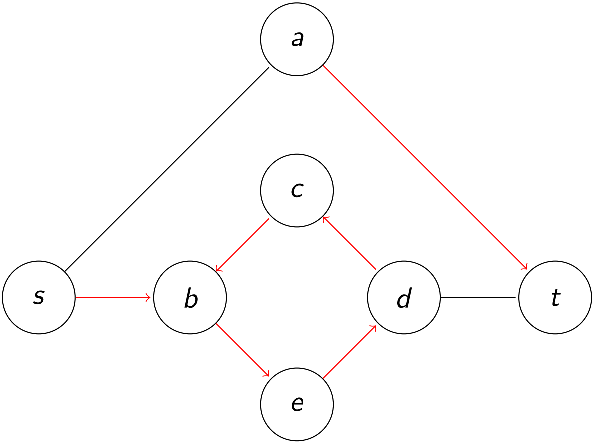

All nodes get pay-off 0.

Strategies: Notation

A strategy of player $i \in N$ in a strategic game Γ is an action $a_i \in A_i$.
A strategy profile $s = (s_1, \ldots, s_n)$ consists of a strategy for each player.

For each $s = (s_1, \ldots, s_n)$ and $s'_i \in A_i$ we denote by

$(s_{-i}, s'_i) = (s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)$

$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$
is not a strategy profile but can be seen as a strategy for the other players with respect to player i.

Best response

Let Γ be a strategic game defined through pay-off functions
The set of best responses for player i to strategy profile s is

$$BR_i(s) = \{a_i \in A_i \mid u_i(s_{-i}, a_i) = \max_{a'_i \in A_i} u_i(s_{-i}, a'_i)\}$$

Those are the actions that give maximum pay-off provided the other players do not change their strategies.
Best responses to s depend only in s_{-i}.

More games

utility	Quiet	Fink
Quiet	2,2	0,3
Fink	3,0	1,1

utility	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

utility	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

utility	swerve	don't sw
swerve	3,3	2,4
don't sw	4,2	1,1

What are the best responses of the row player to a strategy of the column player?

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What are the best responses of a vertex to a strategy of the others?

Solution concepts

- Dominant strategies
- Pure Nash equilibrium
- (Mixed) Nash equilibrium
- Strong Nash equilibrium
- Correlated equilibrium

Dominant strategies

A dominant strategy for player i is a strategy s_i^* if regardless of what other players do the outcome is better for player i.
Formally, for every strategy profile $s = (s_1, \ldots, s_n)$, $u_i(s) \leq u_i(s_{-i}, s_i^*)$.

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Formally, for every strategy profile $s = (s_1, \dots, s_n)$, $u_i(s) \leq u_i(s_{-i}, s_i^*)$.

If player i has a dominant strategy, player i will use it!

In such a a case the game can be simplified until reaching a game in which no player has a dominant strategy.

Pure Nash equilibrium

A pure Nash equilibrium is an strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ such that no player i can do better choosing an action different from s_i^*, given that every other player j adheres to s_j^*:

for every player i and for every action $a_i \in A_i$ it holds $u_i(s_{-i}^*, s_i^*) \geq u_i(s_{-i}^*, a_i)$.

Equivalently, for every player i and for every action $a_i \in A_i$ it holds $s_i^* \in BR_i(s^*)$.

Pure Nash Equilibrium

- Is a strategy profile in which all players are happy.
- Identified with a fixed point of an iterative process of computing a best response.
- However, the game is played only once!
- GT deals with the existence and analysis of equilibria assuming rational behavior.
  players try to maximize their benefit
- GT does not provide algorithmic tools for computing such equilibrium if one exists.

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Bach	2,1	0,0
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swerve	3,3	2,4
don't sw	4,2	1,1

Dominant strategies? Nash equilibria?

Examples of Nash equilibrium

- Prisoner's Dilemma, (Fink, Fink).
- Bach or Stravinsky, (Bach, Bach), (Stravinsky, Stravinsky).
- Matching Pennies, none.
- Chicken, (swerve, don't sw), (don't sw, swerve).

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- A strategy profile is a set of vertices $(v_1, \ldots, v_{n-1})$.
- Pay-offs are defined as follows:
  player u gets 1 if the shortest path joining s to t in the digraph induced by $v_1, \ldots, v_{n-1}$ contains (u, v_u), otherwise gets 0.

Exercise: Nash equilibria?

Pure Nash equilibrium

- First notion of equilibrium for non-cooperative games.
- There are strategic games with no pure Nash equilibrium.
- There are games with more than one pure Nash equilibrium.
- How to compute a Nash equilibrium if there is one?