Automatic Generation of Polynomial Invariants for System Verification

Enric Rodríguez-Carbonell

Plan of the Talk

- Introduction
  - Need for program verification
  - Invariants and abstract interpretation
  - Polynomial invariants

Plan of the Talk

- Introduction
- Generation of Invariant Polynomial Equalities (with D. Kapur: ISSAC'04, SAS'04)
  - Related work
  - Abstract domain of ideals
  - Particular case: loops without nesting

Plan of the Talk

- Introduction
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
  - Imperative programs
    (with D. Kapur: ICTAC'04)
  - Petri nets
    (with R. Clarisó, J. Cortadella: ATPN'05)
  - Hybrid systems
    (with A. Tiwari: HSCC'05)

Plan of the Talk

- Introduction
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
- Generation of Invariant Polynomial Inequalities (with R. Bagnara, E. Zaffanella: SAS'05)
  - Abstract domain of polynomial cones

Plan of the Talk

- Introduction
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
- Generation of Invariant Polynomial Inequalities
- Conclusions and Future Work

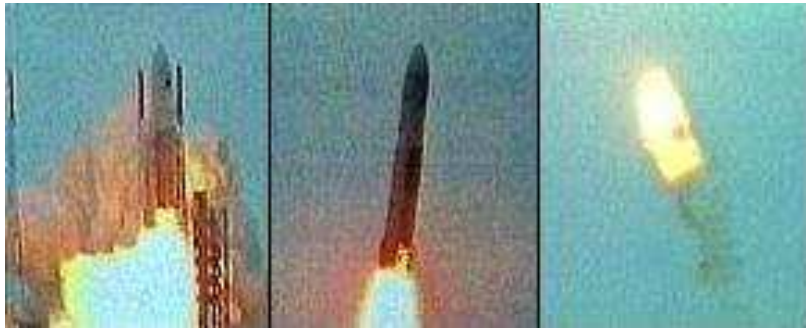
- Introduction
  - Need for program verification
  - Invariants and abstract interpretation
  - Polynomial invariants
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
- Generation of Invariant Polynomial Inequalities
- Conclusions and Future Work

Need for Software Verification

- Critical systems
  - safety
  - security
  - economy

Need for Software Verification

- Critical systems
  - safety
  - security
  - economy



Failure of the Ariane 5 launcher in 1996

Need for Software Verification

- Critical systems
  - safety
  - security
  - economy
- Fundamental finding errors asap. See Microsoft's Software Productivity Tools group: verification pays off

Need for Software Verification

- Critical systems
  - safety
  - security
  - economy
- Fundamental finding errors asap. See Microsoft's Software Productivity Tools group: verification pays off
- Invariants are crucial for program verification!

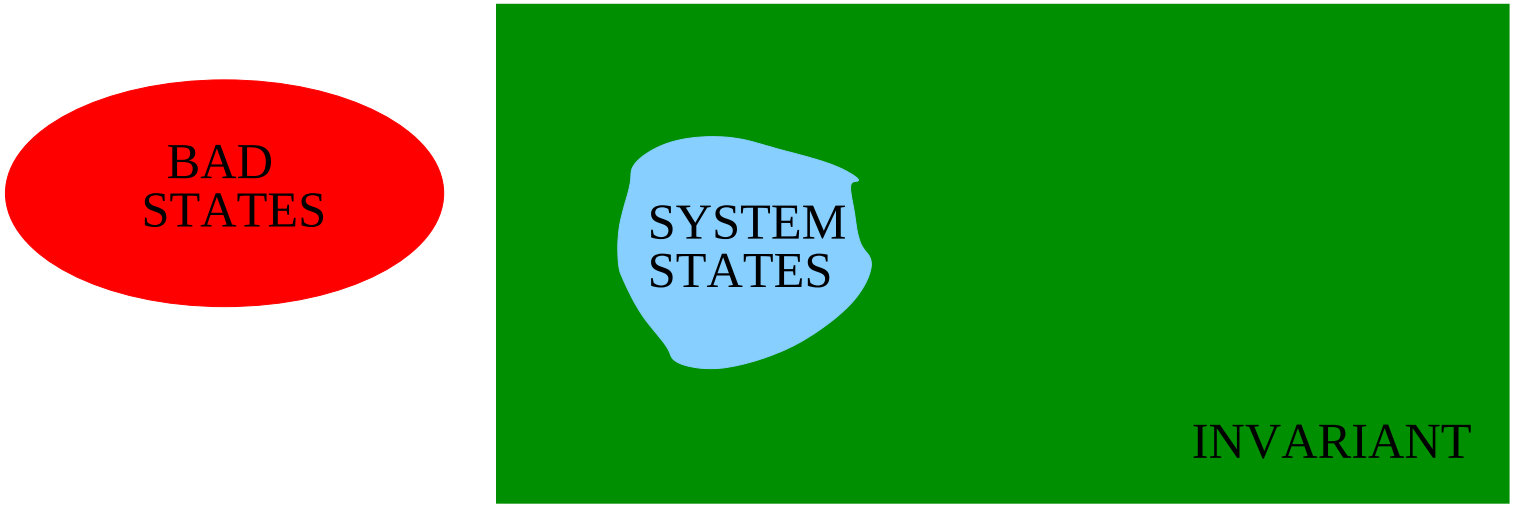
- Introduction
  - Need for program verification
  - Invariants and abstract interpretation
  - Polynomial invariants
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
- Generation of Invariant Polynomial Inequalities
- Conclusions and Future Work

Invariants in Verification

CORRECTNESS OF THE SYSTEM:

$$\text{SYSTEM STATES} \cap \text{BAD STATES} = \emptyset$$

Invariants in Verification



CORRECTNESS OF THE SYSTEM:

$$\text{SYSTEM STATES} \cap \text{BAD STATES} = \varnothing$$

SUFFICIENT CONDITION:

$$\text{INVARIANT} \cap \text{BAD STATES} = \varnothing$$

Overview of Abstract Interpretation

Abstract interpretation allows computing invariants:

Overview of Abstract Interpretation

Abstract interpretation allows computing invariants:

- intervals (Cousot & Cousot 1976, Harrison 1977)

$$x \in [0, 1] \ \wedge \ y \in [0, \infty)$$

Overview of Abstract Interpretation

Abstract interpretation allows computing invariants:

- intervals (Cousot & Cousot 1976, Harrison 1977)

$$x \in [0, 1] \ \wedge \ y \in [0, \infty)$$

- congruences (Granger 1991)

$$x \equiv y \ \mathsf{mod}(2)$$

Overview of Abstract Interpretation

Abstract interpretation allows computing invariants:

- intervals (Cousot & Cousot 1976, Harrison 1977)

$$x \in [0, 1] \ \wedge \ y \in [0, \infty)$$

- congruences (Granger 1991)

$$x \equiv y \ \mathsf{mod}(2)$$

- linear inequalities (Cousot & Halbwachs 1978, Colón & Sankaranarayanan & Sipma 2003)

$$x + 2y - 3z \leq 3$$

Overview of Abstract Interpretation

- octagonal inequalities (Mine 2001)

$$x - y \leq 3$$

Overview of Abstract Interpretation

- octagonal inequalities (Mine 2001)

$$x - y \leq 3$$

- octahedral inequalities (Clariso & Cortadella 2004)

$$x - y + z \leq 2$$

Overview of Abstract Interpretation

- octagonal inequalities (Mine 2001)

$$x - y \leq 3$$

- octahedral inequalities (Clariso & Cortadella 2004)

$$x - y + z \leq 2$$

- ...

Overview of Abstract Interpretation

- octagonal inequalities (Mine 2001)

$$x - y \leq 3$$

- octahedral inequalities (Clariso & Cortadella 2004)

$$x - y + z \leq 2$$

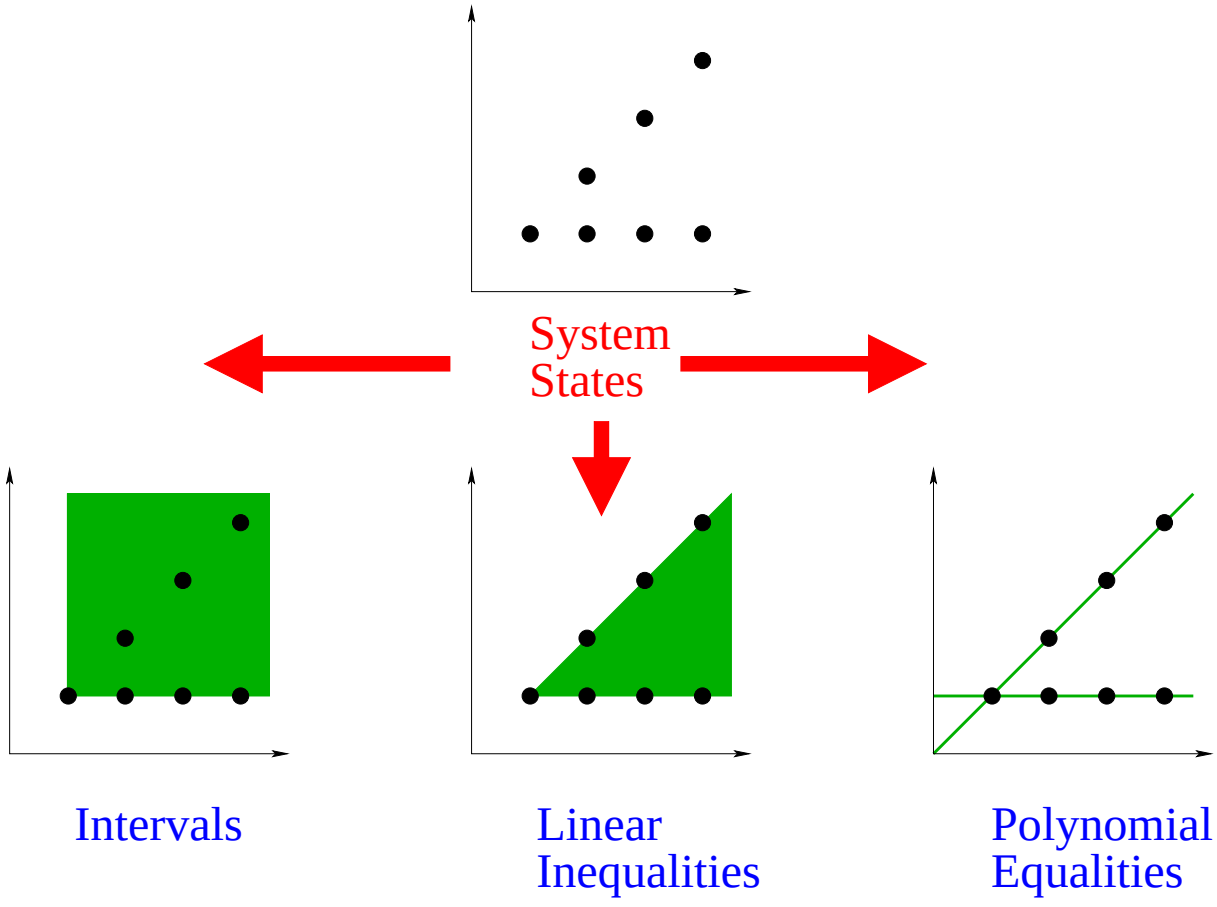
- ...
- polynomial equalities and inequalities

$$x = y^2$$

$$(a + 1)^2 > b^2 \geq a^2$$

Abstract Interpretation: Overapproximation

Sets of variable values overapproximated by abstract values



Abstract Interpretation: Operations

- Invariants generated by symbolic execution of system using abstract values
- Symbolic execution requires abstracting concrete operations on states:

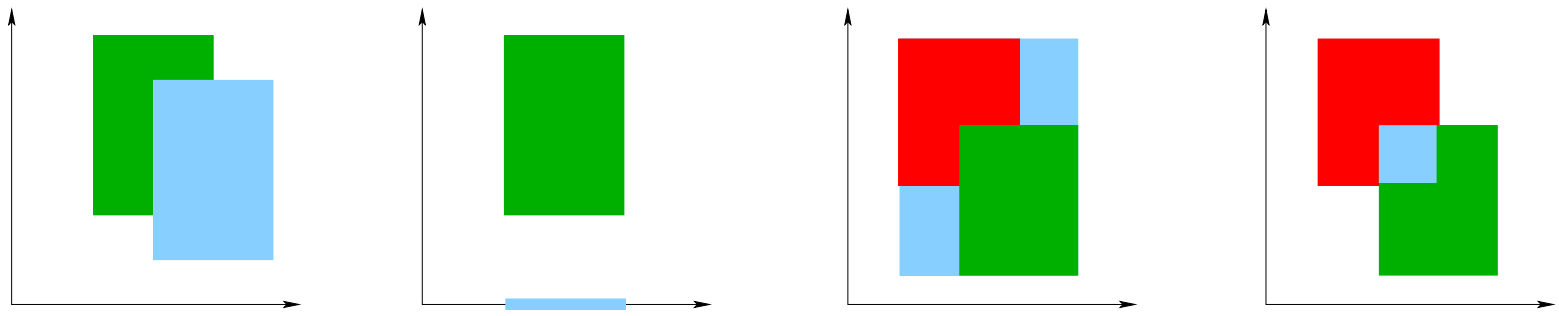


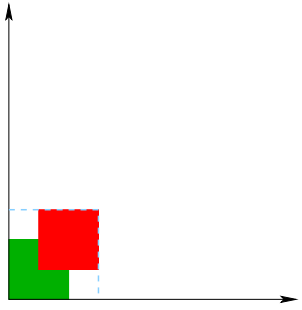
Image	Projection	Union	Intersection
↓	↓	↓	↓
assignments	*assignments*	*merging in loops and conditionals*	*guards in loops and conditionals*

Abstract Interpretation: Extrapolation

- Termination is not guaranteeed in general

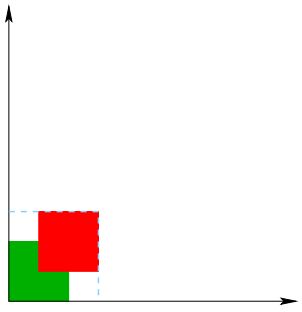
Abstract Interpretation: Extrapolation

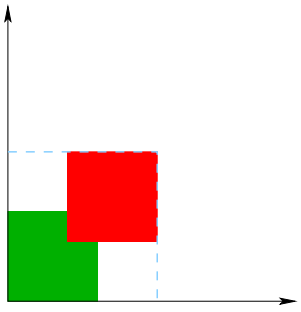
- Termination is not guaranteeed in general



Abstract Interpretation: Extrapolation

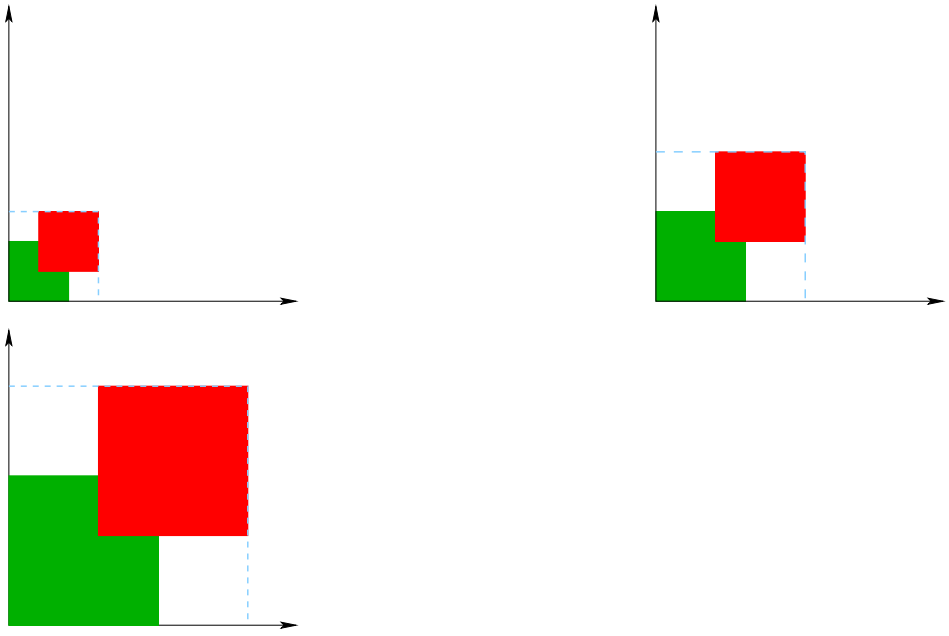
- Termination is not guaranteeed in general





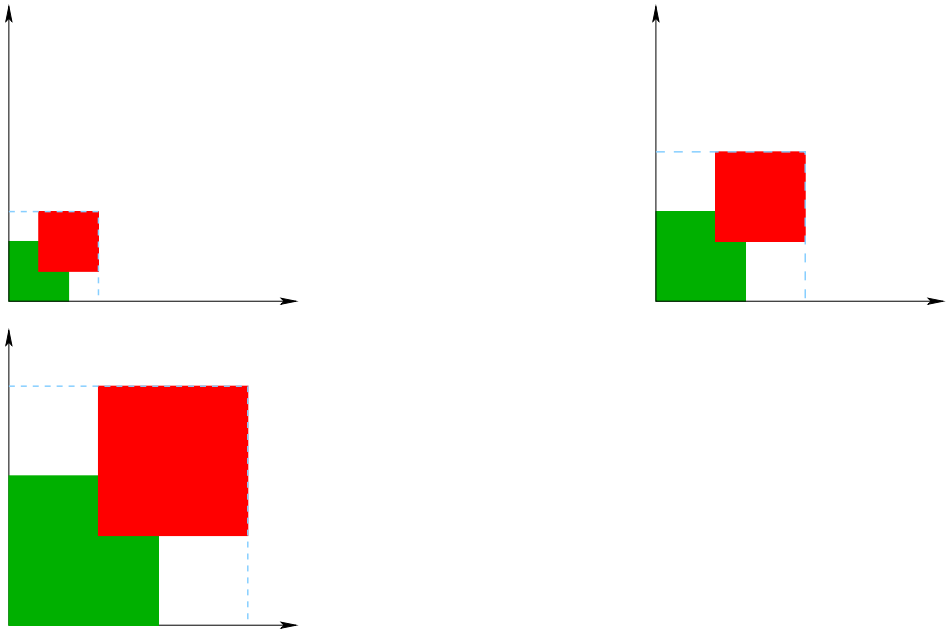
Abstract Interpretation: Extrapolation

- Termination is not guaranteeed in general



Abstract Interpretation: Extrapolation

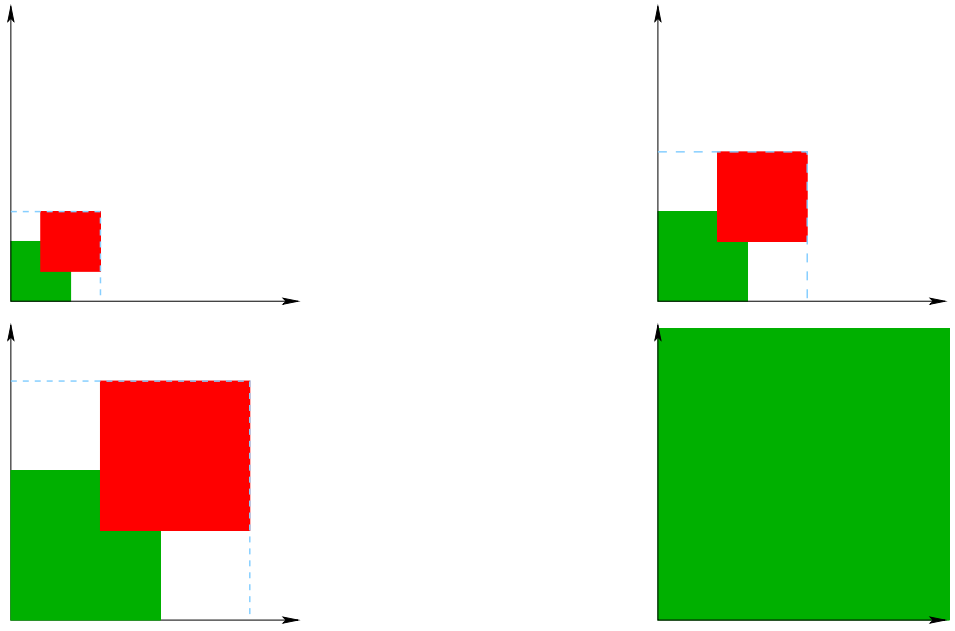
- Termination is not guaranteeed in general



- Union in loops must be extrapolated:
  widening operator introduced to ensure termination

Abstract Interpretation: Extrapolation

- Termination is not guaranteeed in general



- Union in loops must be extrapolated:
  widening operator introduced to ensure termination

- Introduction
  - Need for program verification
  - Invariants and abstract interpretation
  - Polynomial invariants
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
- Generation of Invariant Polynomial Inequalities
- Conclusions and Future Work

Why Care about Polynomial Invariants?

- Linear invariants used to verify many classes of systems:
  - Imperative programs
  - Logic programs
  - Hybrid systems
  - ...

Why Care about Polynomial Invariants?

- Linear invariants used to verify many classes of systems:
  - Imperative programs
  - Logic programs
  - Hybrid systems
  - ...
- But some applications require polynomial invariants:

The abstract interpreter ASTRÉE employs polynomial invariants to verify absence of run-time errors in flight control software

- Introduction
- Generation of Invariant Polynomial Equalities
  - Related work
  - Abstract domain of ideals
  - Particular case: loops without nesting
- Applications of Polynomial Equality Invariants
- Generation of Invariant Polynomial Inequalities
- Conclusions and Future Work

Related Work (1)

- Iterative fixpoint approaches
  - Forward propagation
    - Rodríguez-Carbonell & Kapur 2004
    - Colón 2004
  - Backward propagation
    - Müller-Olm & Seidl 2004

Related Work (1)

- Iterative fixpoint approaches
  - Forward propagation
    - Rodríguez-Carbonell & Kapur 2004
    - Colón 2004
  - Backward propagation
    - Müller-Olm & Seidl 2004
- Constraint-based approaches
  - Sankaranarayanan & Sipma & Manna 2004

Related Work (2)

Work	Restrictions	Conds $=$	Conds $\neq$	Complete
MOS, POPL'04	bounded deg	no	no	yes
SSM, POPL'04	fixed form	yes	no	no
MOS, IPL'04	fixed form	no	yes	yes
COL, SAS'04	bounded deg	yes	no	no
RCK, SAS'04	bounded deg	yes	yes	yes*
RCK, ISSAC'04	no restriction	no	no	yes

- Introduction
- Generation of Invariant Polynomial Equalities
  - Related work
  - Abstract domain of ideals
  - Particular case: loops without nesting
- Applications of Polynomial Equality Invariants
- Generation of Invariant Polynomial Inequalities
- Conclusions and Future Work

Overview of our Method

- States abstracted to ideal of polynomials evaluating to 0

Overview of our Method

- States abstracted to ideal of polynomials evaluating to 0
- Programming language admits
  - Polynomial assignments: $variable := polynomial$
  - Polynomial equalities and disequalities in conditions:

$$polynomial = 0 \quad , \quad polynomial \neq 0$$

Overview of our Method

- States abstracted to ideal of polynomials evaluating to 0
- Programming language admits
  - Polynomial assignments: $variable := polynomial$
  - Polynomial equalities and disequalities in conditions:

$$polynomial = 0 \quad , \quad polynomial \neq 0$$

- Implementation successfully applied to many programs

Overview of our Method

- States abstracted to ideal of polynomials evaluating to 0
- Programming language admits
  - Polynomial assignments: $variable := polynomial$
  - Polynomial equalities and disequalities in conditions:

$$polynomial = 0 \quad , \quad polynomial \neq 0$$

- Implementation successfully applied to many programs
- Ideals of polynomials represented by special finite bases of generators: Gröbner bases

Overview of our Method

- States abstracted to ideal of polynomials evaluating to 0
- Programming language admits
  - Polynomial assignments: $variable := polynomial$
  - Polynomial equalities and disequalities in conditions:

$$polynomial = 0 \quad , \quad polynomial \neq 0$$

- Implementation successfully applied to many programs
- Ideals of polynomials represented by special finite bases of generators: Gröbner bases
- Many tools available manipulating ideals, Gröbner bases, e.g. Macaulay 2, Maple

Ideals of Polynomials (1)

- Intuitively, an ideal is a set of polynomials and all their consequences

Ideals of Polynomials (1)

- Intuitively, an ideal is a set of polynomials and all their consequences
- An ideal is a set of polynomials I such that
  - $0 \in I$
  - If $p, q \in I$, then $p + q \in I$
  - If $p \in I$ and q any polynomial, $pq \in I$

Ideals of Polynomials (2)

- E.g. polynomials evaluating to 0 on a set of points S

Ideals of Polynomials (2)

- E.g. polynomials evaluating to 0 on a set of points S
  - 0 evaluates to 0 everywhere

$$\forall \omega \in S, \quad 0(\omega) = 0$$

Ideals of Polynomials (2)

- E.g. polynomials evaluating to 0 on a set of points S
  - 0 evaluates to 0 everywhere

$$\forall \omega \in S, \quad 0(\omega) = 0$$

  - If p, q evaluate to 0 on S, then $p + q$ evaluates to 0 on S

$$\forall \omega \in S, \quad p(\omega) = q(\omega) = 0 \Longrightarrow p(\omega) + q(\omega) = 0$$

Ideals of Polynomials (2)

- E.g. polynomials evaluating to 0 on a set of points S
  - 0 evaluates to 0 everywhere

$$\forall \omega \in S, \quad 0(\omega) = 0$$

  - If p, q evaluate to 0 on S, then $p + q$ evaluates to 0 on S

$$\forall \omega \in S, \quad p(\omega) = q(\omega) = 0 \Longrightarrow p(\omega) + q(\omega) = 0$$

  - If p evaluates to 0 on S, then pq evaluates to 0 on S

$$\forall \omega \in S, \quad p(\omega) = 0 \Longrightarrow p(\omega) \cdot q(\omega) = 0$$

Ideals of Polynomials (3)

- E.g. multiples of a polynomial p, $\langle p \rangle$
  - $0 = 0 \cdot p \in \langle p \rangle$
  - $q_1 \cdot p + q_2 \cdot p = (q_1 + q_2)p \in \langle p \rangle$
  - If q_2 is any polynomial, then $q_2 \cdot q_1 \cdot p \in \langle p \rangle$

Ideals of Polynomials (3)

- E.g. multiples of a polynomial p, $\langle p \rangle$
  - $0 = 0 \cdot p \in \langle p \rangle$
  - $q_1 \cdot p + q_2 \cdot p = (q_1 + q_2)p \in \langle p \rangle$
  - If q_2 is any polynomial, then $q_2 \cdot q_1 \cdot p \in \langle p \rangle$
- In general, ideal generated by p_1,...,p_k:

$$\langle p_1, ..., p_k \rangle = \{\sum_{j=1}^{k} q_j \cdot p_j \text{ for arbitrary } q_j\}$$

Ideals of Polynomials (3)

- E.g. multiples of a polynomial p, $\langle p \rangle$
  - $0 = 0 \cdot p \in \langle p \rangle$
  - $q_1 \cdot p + q_2 \cdot p = (q_1 + q_2)p \in \langle p \rangle$
  - If q_2 is any polynomial, then $q_2 \cdot q_1 \cdot p \in \langle p \rangle$
- In general, ideal generated by p_1,...,p_k:

$$\langle p_1, ..., p_k \rangle = \{\sum_{j=1}^{k} q_j \cdot p_j \text{ for arbitrary } q_j\}$$

- Hilbert's basis theorem: all ideals are finitely generated $\longrightarrow$ there is finite representation for ideals

Operations with Ideals

- Several operations available. Given ideals I, J in the variables x_1, ..., x_n:

Operations with Ideals

- Several operations available. Given ideals I, J in the variables x_1, ..., x_n:
  - projection: $I \cap \mathbb{C}[x_1, ..., x_{i-1}, x_{i+1}, ..., x_n]$

Operations with Ideals

- Several operations available. Given ideals I, J in the variables x_1, ..., x_n:
  - projection: $I \cap \mathbb{C}[x_1, ..., x_{i-1}, x_{i+1}, ..., x_n]$
  - addition: $I + J = \{p + q \mid p \in I, q \in J\}$

Operations with Ideals

- Several operations available. Given ideals I, J in the variables x_1, ..., x_n:
  - projection: $I \cap \mathbb{C}[x_1, ..., x_{i-1}, x_{i+1}, ..., x_n]$
  - addition: $I + J = \{p + q \mid p \in I, q \in J\}$
  - quotient: $I : J = \{p \mid \forall q \in J, p \cdot q \in I\}$

Operations with Ideals

- Several operations available. Given ideals I, J in the variables x_1, ..., x_n:
  - projection: $I \cap \mathbb{C}[x_1, ..., x_{i-1}, x_{i+1}, ..., x_n]$
  - addition: $I + J = \{p + q \mid p \in I, q \in J\}$
  - quotient: $I : J = \{p \mid \forall q \in J, p \cdot q \in I\}$
  - intersection: $I \cap J$

Operations with Ideals

- Several operations available. Given ideals I, J in the variables x_1, ..., x_n:
  - projection: $I \cap \mathbb{C}[x_1, ..., x_{i-1}, x_{i+1}, ..., x_n]$
  - addition: $I + J = \{p + q \mid p \in I, q \in J\}$
  - quotient: $I : J = \{p \mid \forall q \in J, p \cdot q \in I\}$
  - intersection: $I \cap J$
- All operations implemented using Gröbner bases
- These operations are used in abstract semantics

Our Widening Operator

- Parametric widening $I \nabla_d J$
- Based on taking polynomials of $I \cap J$ of degree $\leq d$

Our Widening Operator

- Parametric widening $I \nabla_d J$
- Based on taking polynomials of $I \cap J$ of degree $\leq d$
- Definition uses Gröbner bases:

$$I \nabla_d J := \mathrm{V}(\mathrm{I}(\{p \in GB(I \cap J) \mid \deg(p) \leq d\}))$$

Our Widening Operator

- Parametric widening $I \nabla_d J$
- Based on taking polynomials of $I \cap J$ of degree $\leq d$
- Definition uses Gröbner bases:

$$I \nabla_d J := \mathrm{V}(\mathrm{I}(\{p \in GB(I \cap J) \mid \deg(p) \leq d\}))$$

- Termination guaranteed

Example

$a := 0; b := 0;$

while $b \neq c$ **do**

$\quad a := a + 2b + 1; b := b + 1;$

end while

Example

$a := 0; b := 0;$

while $b \neq c$ **do**

$\quad a := a + 2b + 1; b := b + 1;$

end while

$F_0(I) = \langle 0 \rangle$

$F_1(I) = (\langle a \rangle + \langle I_0(a \leftarrow a') \rangle) \cap \mathbb{C}[a, b, c]$

$F_2(I) = (\langle b \rangle + \langle I_1(b \leftarrow b') \rangle) \cap \mathbb{C}[a, b, c]$

$F_3(I) = I_3 \nabla_2 (I_2 \cap I_6)$

$F_4(I) = \langle I_3 \rangle : \langle b - c \rangle$

$F_5(I) = I_4(a \leftarrow a - 2b - 1)$

$F_6(I) = I_5(b \leftarrow b - 1)$

$F_7(I) = \mathrm{I}(\mathrm{V}(I_3 + \langle b - c \rangle))$

Example

$a := 0; b := 0;$

while $b \neq c$ **do**

$\quad a := a + 2b + 1; b := b + 1;$

end while

$F_0(I) = \langle 0 \rangle$

$F_1(I) = (\langle a \rangle + \langle I_0(a \leftarrow a') \rangle) \cap \mathbb{C}[a, b, c]$

$F_2(I) = (\langle b \rangle + \langle I_1(b \leftarrow b') \rangle) \cap \mathbb{C}[a, b, c]$

$F_3(I) = I_3 \nabla_2 (I_2 \cap I_6)$

$F_4(I) = \langle I_3 \rangle : \langle b - c \rangle$

$F_5(I) = I_4(a \leftarrow a - 2b - 1)$

$F_6(I) = I_5(b \leftarrow b - 1)$

$F_7(I) = \mathrm{I}(\mathrm{V}(I_3 + \langle b - c \rangle))$

In 6 steps found loop invariant:

$$a = b^2$$

- Introduction
- Generation of Invariant Polynomial Equalities
  - Related work
  - Abstract domain of ideals
  - Particular case: loops without nesting
- Applications of Polynomial Equality Invariants
- Generation of Invariant Polynomial Inequalities
- Conclusions and Future Work

Particular case: loops without nesting

- Are there programs for which no widening is required?

Particular case: loops without nesting

- Are there programs for which no widening is required?
- Yes: unnested loops with solvable assignments with eigenvalues in $\mathbb{Q}^+$

Particular case: loops without nesting

- Are there programs for which no widening is required?
- Yes: unnested loops with solvable assignments with eigenvalues in $\mathbb{Q}^+$
- Solvable assignments generalize linear assignments

Particular case: loops without nesting

- Are there programs for which no widening is required?
- Yes: unnested loops with solvable assignments with eigenvalues in $\mathbb{Q}^+$
- Solvable assignments generalize linear assignments

$a := 0$;
$b := 0$;
while $b \neq c$ **do**
    $a := a + 2b + 1$;
    $b := b + 1$;
end while

Overview of the Method

- $(a_n, b_n, c) \equiv$ program state after n loop iterations

$$\begin{cases} a_{n+1} & = & a_n + 2b_n + 1 \\ b_{n+1} & = & b_n + 1 \end{cases}, \quad \begin{cases} a_0 & = & 0 \\ s_0 & = & 0 \end{cases}$$

Overview of the Method

- $(a_n, b_n, c) \equiv$ program state after n loop iterations

$$\begin{cases} a_{n+1} & = & a_n + 2b_n + 1 \\ b_{n+1} & = & b_n + 1 \end{cases}, \quad \begin{cases} a_0 & = & 0 \\ s_0 & = & 0 \end{cases}$$

- Solution to recurrence: $\begin{cases} a_n & = & n^2 \\ b_n & = & n \end{cases}$
- Program states characterized by $\exists n(a = n^2 \ \wedge \ b = n)$

Overview of the Method

- $(a_n, b_n, c) \equiv$ program state after n loop iterations

$$\begin{cases} a_{n+1} &= a_n + 2b_n + 1 \\ b_{n+1} &= b_n + 1 \end{cases}, \quad \begin{cases} a_0 &= 0 \\ s_0 &= 0 \end{cases}$$

- Solution to recurrence: $\begin{cases} a_n &= n^2 \\ b_n &= n \end{cases}$
- Program states characterized by $\exists n(a = n^2 \ \wedge \ b = n)$
- Quantifier elimination: $b = n \Longrightarrow a = b^2$ is loop invariant

Overview of the Method

- $(a_n, b_n, c) \equiv$ program state after n loop iterations

$$\begin{cases} a_{n+1} &= a_n + 2b_n + 1 \\ b_{n+1} &= b_n + 1 \end{cases}, \quad \begin{cases} a_0 &= 0 \\ s_0 &= 0 \end{cases}$$

- Solution to recurrence: $\begin{cases} a_n &= n^2 \\ b_n &= n \end{cases}$
- Program states characterized by $\exists n(a = n^2 \ \wedge \ b = n)$
- Quantifier elimination: $b = n \Longrightarrow a = b^2$ is loop invariant
- Gröbner bases can be used to eliminate loop counters

Our Handling of Conditional Statements (1)

$x := R;$
$y := 0;$
$r := R^2 - N;$
while $?$ **do**
    if $?$ **then**
        $r := r + 2x + 1;$
        $x := x + 1;$
    else
        $r := r - 2y - 1;$
        $y := y + 1;$
    end if
end while

Our Handling of Conditional Statements (2)

- 1st idea:

Our Handling of Conditional Statements (2)

- 1st idea:

  1. Compute invariants for two distinct loops:

while $?$ **do**	**while** $?$ **do**
$r := r + 2x + 1;$	$r := r - 2y - 1;$
$x := x + 1;$	$y := y + 1;$
end while	**end while**

Our Handling of Conditional Statements (2)

- 1st idea:

1. Compute invariants for two distinct loops:

while $?$ **do**	**while** $?$ **do**
$r := r + 2x + 1;$	$r := r - 2y - 1;$
$x := x + 1;$	$y := y + 1;$
end while	**end while**

2. Compute common invariants for both loops

Our Handling of Conditional Statements (2)

- 1st idea:

  1. Compute invariants for two distinct loops:

while $?$ **do**	**while** $?$ **do**
$r := r + 2x + 1;$	$r := r - 2y - 1;$
$x := x + 1;$	$y := y + 1;$
end while	**end while**

  2. Compute common invariants for both loops

- Finding common invariants $\equiv$
  Finding intersection of invariant ideals

Our Handling of Conditional Statements (2)

- 1st idea:

  1. Compute invariants for two distinct loops:

while $?$ **do**	**while** $?$ **do**
$r := r + 2x + 1;$	$r := r - 2y - 1;$
$x := x + 1;$	$y := y + 1;$
end while	**end while**

  2. Compute common invariants for both loops

- Finding common invariants $\equiv$
  Finding intersection of invariant ideals
- But this is not sound!

Our Handling of Conditional Statements (3)

- 2nd idea: take intersection as initial condition and repeat

Our Handling of Conditional Statements (3)

- 2nd idea: take intersection as initial condition and repeat

Program	Algorithm
$\bar{x} := \bar{\alpha};$	$I' := \langle 1 \rangle; I := \langle x_1 - \alpha_1, \cdots, x_m - \alpha_m \rangle;$
while $?$ **do**	**while** $I' \neq I$ **do**
$\bar{x} := f(\bar{x});$	$I' := I;$
or	$I := \bigcap_{n=0}^{\infty} [\, I(\bar{x} \leftarrow f^{-n}(\bar{x}))$
$\bar{x} := g(\bar{x});$	$\bigcap I(\bar{x} \leftarrow g^{-n}(\bar{x}))\,];$
end while	**end while**

Properties of our Algorithm

- No widening employed!
- Termination in $n+1$ steps, where n = number of variables

Properties of our Algorithm

- No widening employed!
- Termination in $n+1$ steps, where n = number of variables
- Correct and complete:
  finds all polynomial equality invariants

Properties of our Algorithm

- No widening employed!
- Termination in $n+1$ steps, where n = number of variables
- Correct and complete:
  finds all polynomial equality invariants
- Implemented in Maple:

1. Solving recurrences
2. Eliminating variables
3. Intersecting ideals

$\left.\begin{array}{l}\text{2. Eliminating variables}\\ \text{3. Intersecting ideals}\end{array}\right\}$ Gröbner bases

Proof of Termination (1)

$x := 0; y := 0;$

while $?$ **do**

$x := x + 1;$

or

$y := y + 1;$

end while

Proof of Termination (1)

$x := 0; y := 0;$

while $?$ **do**

$x := x + 1;$

or

$y := y + 1;$

end while

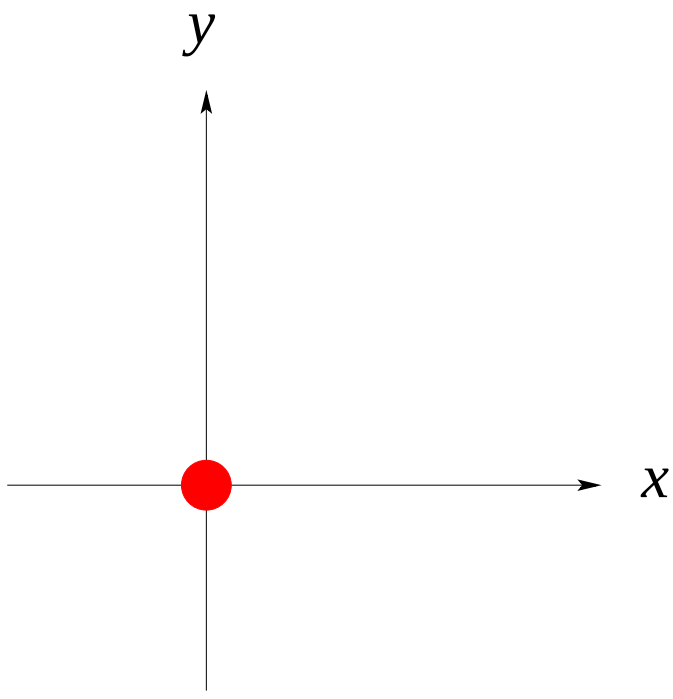
- Program states $\equiv \mathbb{N} \times \mathbb{N}$

Proof of Termination (1)

$x := 0; y := 0;$
while $?$ **do**
$\quad x := x + 1;$
$\quad$ **or**
$\quad y := y + 1;$
end while

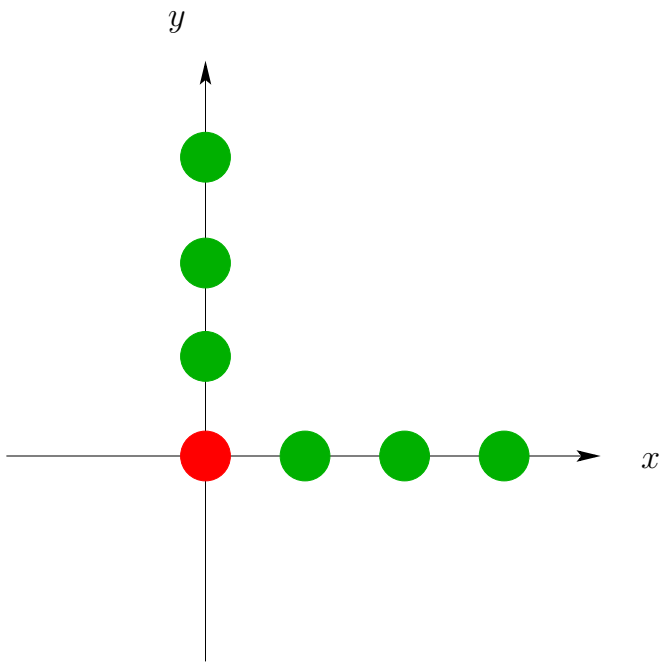
- Program states $\equiv \mathbb{N} \times \mathbb{N}$
- Initial state $(x, y) = (0, 0) \longrightarrow$ initial ideal $\langle x, y \rangle$

Proof of Termination (2)



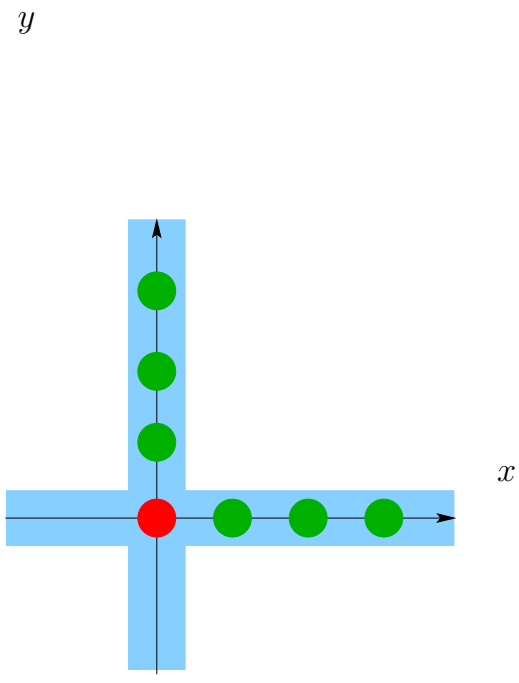
Step 0: $\langle x, y \rangle \quad \longrightarrow \quad \{(0,0)\}$ dimension 0

Proof of Termination (2)



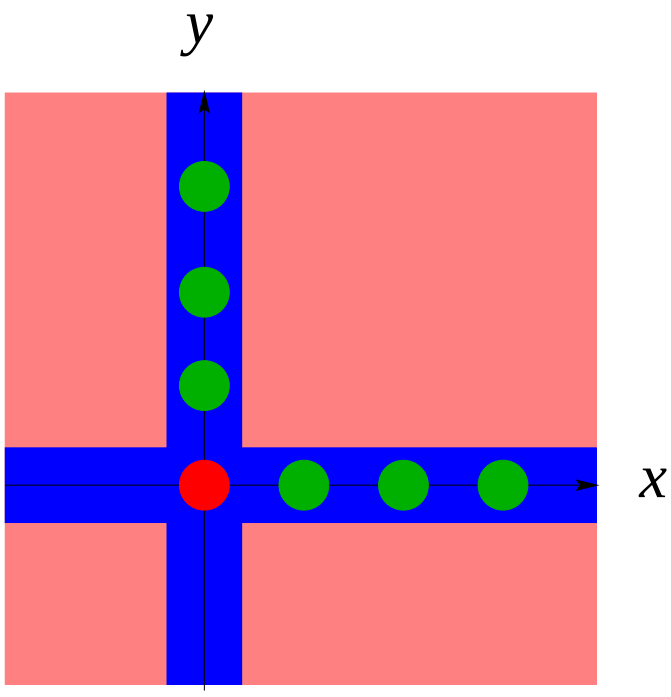
Step 0: $\langle x, y \rangle \quad \longrightarrow \quad \{(0,0)\}$ dimension 0

Proof of Termination (2)



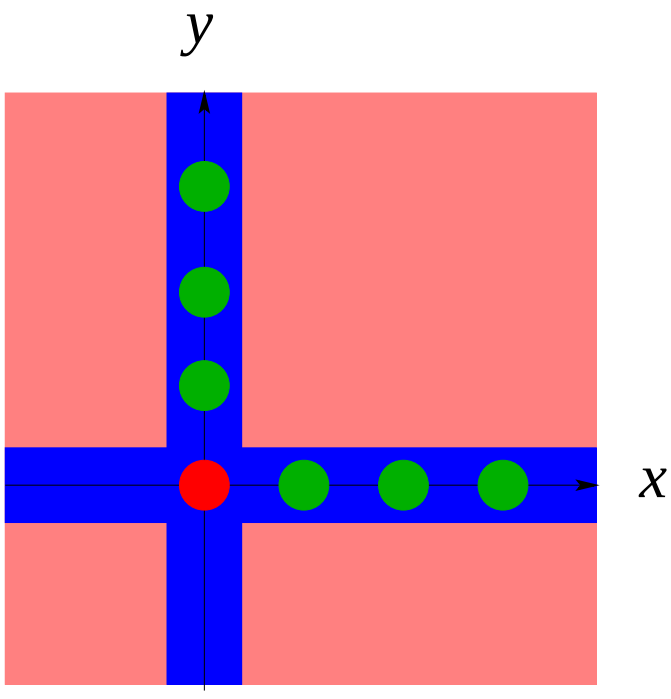
Step 0:	$\langle x, y \rangle$	$\longrightarrow$	$\{(0,0)\}$	dimension 0
Step 1:	$\langle xy \rangle$	$\longrightarrow$	$\{(\alpha,0)\} \cup \{(0,\alpha)\}$	dimension 1

Proof of Termination (2)



Step 0:	$\langle x, y \rangle$	$\longrightarrow$	$\{(0,0)\}$	dimension 0
Step 1:	$\langle xy \rangle$	$\longrightarrow$	$\{(\alpha,0)\} \cup \{(0,\alpha)\}$	dimension 1
Step 2:	$\langle 0 \rangle$	$\longrightarrow$	$\mathbb{R}^2$	dimension 2

Proof of Termination (2)



The dimension has increased at every step,
and there is a finite number of variables!

Example

$x := R;$
$y := 0;$
$r := R^2 - N;$
while $?$ **do**
    if $?$ **then**
        $r := r + 2x + 1;$
        $x := x + 1;$
    else
        $r := r - 2y - 1;$
        $y := y + 1;$
    end if
end while

Example

$x := R;$
$y := 0;$
$r := R^2 - N;$
while $?$ **do**
  if $?$ **then**
    $r := r + 2x + 1;$
    $x := x + 1;$
  else
    $r := r - 2y - 1;$
    $y := y + 1;$
  end if
end while

Invariant polynomial equality:

$$x^2 - y^2 = r + N$$

- Introduction
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
  - Imperative programs
  - Petri nets
  - Hybrid systems
- Generation of Invariant Polynomial Inequalities
- Conclusions and Future Work

Imperative Programs

Pre: $\{ N \geq 1 \}$

$x := R;\ y := 0;\ r := R^2 - N;$

Inv: $\{ N \geq 1 \ \wedge \ x^2 - y^2 = r + N \}$

while $r \neq 0$ **do**

  if $r < 0$ **then**

    $r := r + 2x + 1;$

    $x := x + 1;$

  else

    $r := r - 2y - 1;$

    $y := y + 1;$

  end if

end while

Post: $\{x \neq y \wedge N \bmod (x - y) = 0\}$

Imperative Programs

Pre: $\{ N \geq 1 \}$

$x := R;\ y := 0;\ r := R^2 - N;$

Inv: $\{ N \geq 1 \ \wedge \ x^2 - y^2 = r + N \}$

while $r \neq 0$ **do**

  if $r < 0$ **then**

    $r := r + 2x + 1;$

    $x := x + 1;$

  else

    $r := r - 2y - 1;$

    $y := y + 1;$

  end if

end while

Post: $\{x \neq y \wedge N \bmod (x - y) = 0\}$

- $N \geq 1 \implies$
  $R^2 - 0^2 = (R^2 - N) + N$

Imperative Programs

Pre: $\{ N \geq 1 \}$

$x := R;\ y := 0;\ r := R^2 - N;$

Inv: $\{ N \geq 1 \ \wedge \ x^2 - y^2 = r + N \}$

while $r \neq 0$ **do**

  if $r < 0$ **then**

    $r := r + 2x + 1;$

    $x := x + 1;$

  else

    $r := r - 2y - 1;$

    $y := y + 1;$

  end if

end while

Post: $\{x \neq y \wedge N \bmod (x - y) = 0\}$

- $N \geq 1 \implies$
  $R^2 - 0^2 = (R^2 - N) + N$
- $x^2 - y^2 = r + N \ \wedge \ r < 0 \implies$
  $(x+1)^2 - y^2 = (r + 2x + 1) + N$

Imperative Programs

Pre: $\{ N \geq 1 \}$

$x := R;\ y := 0;\ r := R^2 - N;$

Inv: $\{ N \geq 1 \ \wedge \ x^2 - y^2 = r + N \}$

while $r \neq 0$ **do**

  if $r < 0$ **then**

    $r := r + 2x + 1;$

    $x := x + 1;$

  else

    $r := r - 2y - 1;$

    $y := y + 1;$

  end if

end while

Post: $\{x \neq y \wedge N \bmod (x - y) = 0\}$

- $N \geq 1 \implies$
  $R^2 - 0^2 = (R^2 - N) + N$
- $x^2 - y^2 = r + N \ \wedge \ r < 0 \implies$
  $(x+1)^2 - y^2 = (r + 2x + 1) + N$
- $x^2 - y^2 = r + N \ \wedge \ r > 0 \implies$
  $x^2 - (y+1)^2 = (r - 2y - 1) + N$

Imperative Programs

Pre: $\{ N \geq 1 \}$

$x := R;\ y := 0;\ r := R^2 - N;$

Inv: $\{ N \geq 1 \ \wedge \ x^2 - y^2 = r + N \}$

while $r \neq 0$ **do**

  if $r < 0$ **then**

    $r := r + 2x + 1;$

    $x := x + 1;$

  else

    $r := r - 2y - 1;$

    $y := y + 1;$

  end if

end while

Post: $\{x \neq y \wedge N \bmod (x - y) = 0\}$

- $N \geq 1 \implies$
  $R^2 - 0^2 = (R^2 - N) + N$
- $x^2 - y^2 = r + N \ \wedge \ r < 0 \implies$
  $(x+1)^2 - y^2 = (r + 2x + 1) + N$
- $x^2 - y^2 = r + N \ \wedge \ r > 0 \implies$
  $x^2 - (y+1)^2 = (r - 2y - 1) + N$
- $N \geq 1 \ \wedge \ x^2 - y^2 = r + N \implies$
  $x \neq y \wedge N \bmod (x - y) = 0$

- Introduction
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
  - Imperative programs
  - Petri nets
  - Hybrid systems
- Generation of Invariant Polynomial Inequalities
- Conclusions and Future Work

Petri Nets: Introduction

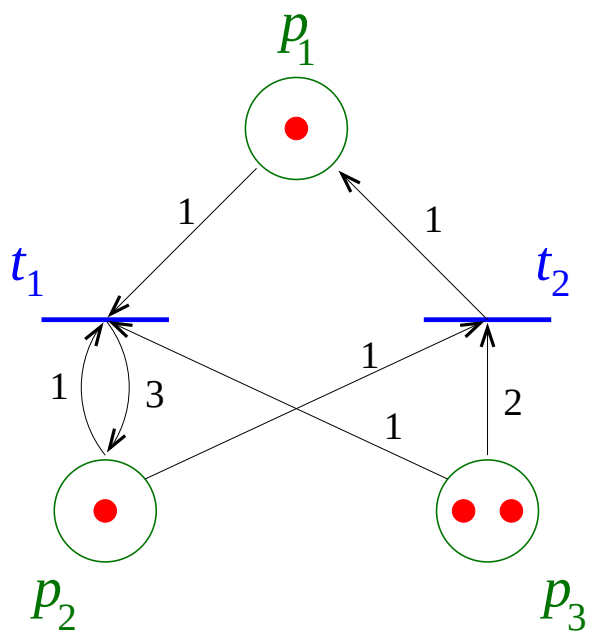
- Petri nets: mathematical model for studying systems
  - concurrency
  - parallelism
  - non-determinism

Petri Nets: Introduction

- Petri nets: mathematical model for studying systems
  - concurrency
  - parallelism
  - non-determinism
- Applications:
  - Manufacturing and Task Planning
  - Communication Networks
  - Hardware Design

Definitions

- A Petri net is a bipartite directed graph where:
  - Nodes partitioned into places (◯) and transitions (|)
  - Arcs are labelled with a natural number
- A marking maps a number of tokens to each place



$p_1 \longrightarrow 1$

$p_2 \longrightarrow 1$

$p_3 \longrightarrow 2$

Dynamics (1)

- Dynamics of a Petri net described by
  - initial marking
  - firing of transitions

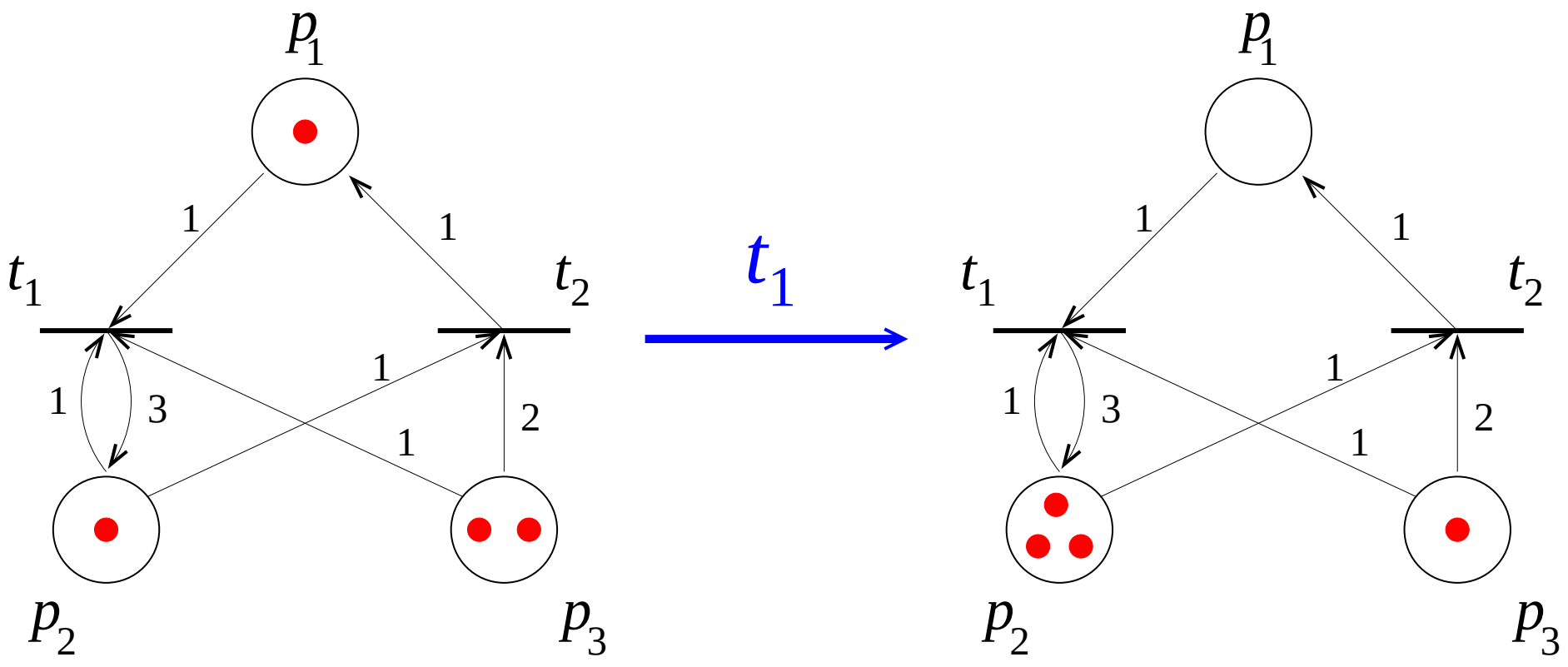
Dynamics (1)

- Dynamics of a Petri net described by
  - initial marking
  - firing of transitions
- A transition is enabled if there are $\geq$ tokens in each input place than indicated in the arcs

Dynamics (1)

- Dynamics of a Petri net described by
  - initial marking
  - firing of transitions
- A transition is enabled if there are $\geq$ tokens in each input place than indicated in the arcs
- When a transition is enabled, it can fire:
  1. the number of tokens indicated in the arcs is removed from input places
  2. tokens are produced in output places according to arcs

Dynamics (2)

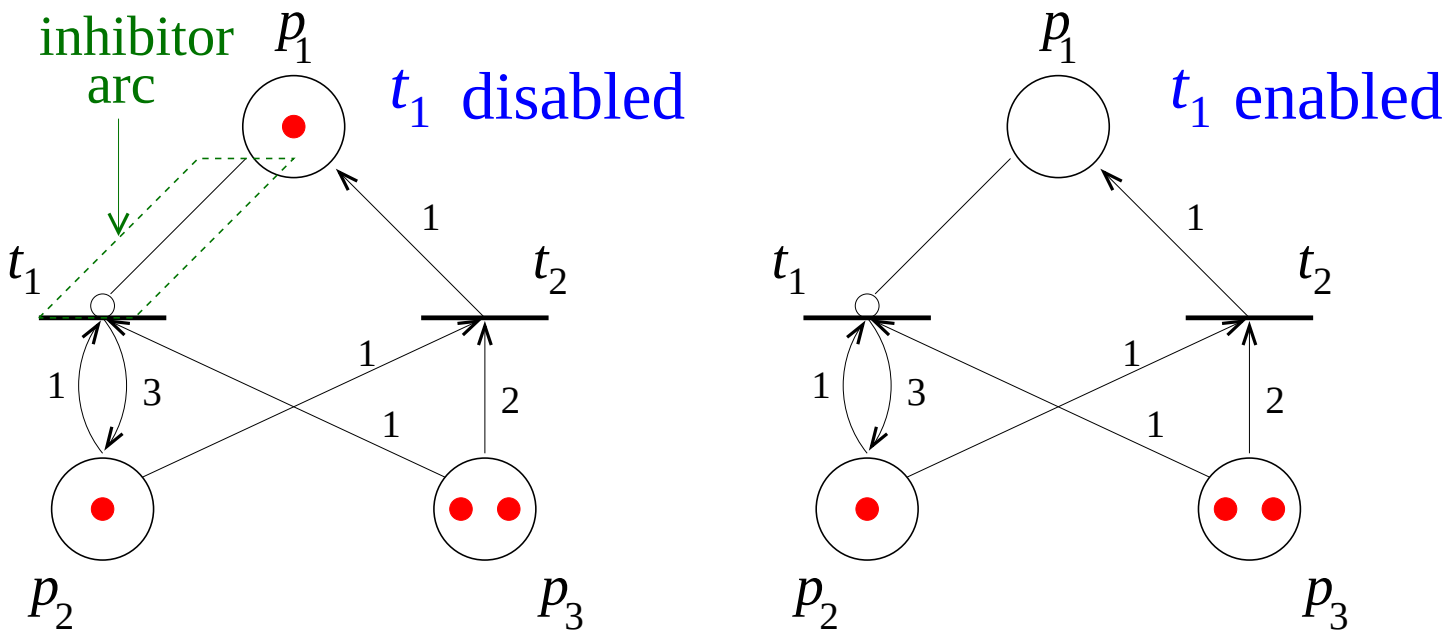


Dynamics (3)

- Enabling of transitions may also depend on inhibitor arcs
- An inhibitor arc is an arc connecting place p to transition t so that there cannot be tokens in p for t to be enabled

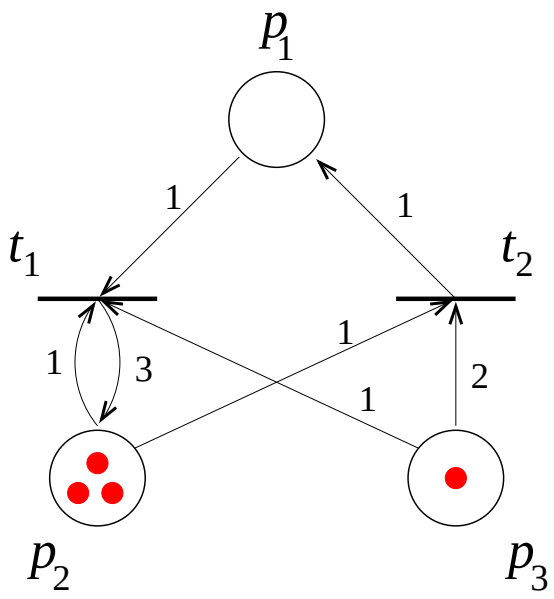
Dynamics (3)

- Enabling of transitions may also depend on inhibitor arcs
- An inhibitor arc is an arc connecting place p to transition t so that there cannot be tokens in p for t to be enabled



Dynamics (4)

- Deadlocks are markings for which all transitions are disabled



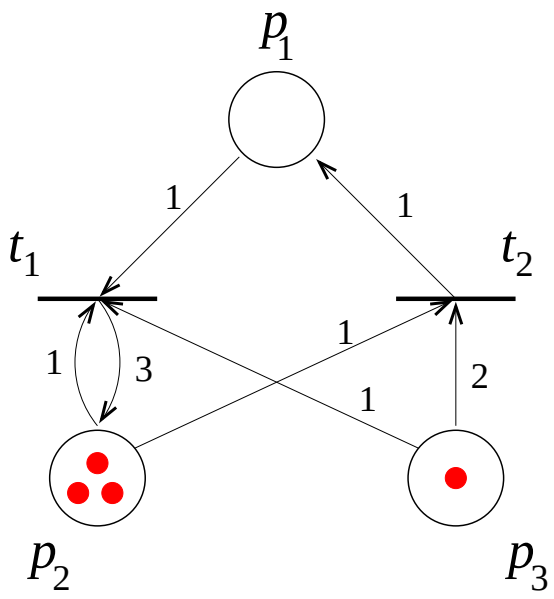
t_1 disabled

t_2 disabled

$\longrightarrow$ DEADLOCK !!

Dynamics (4)

- Deadlocks are markings for which all transitions are disabled



t_1 disabled

t_2 disabled

$\longrightarrow$ DEADLOCK !!

- Given a Petri net with an initial marking:
  - Invariant properties of reachable states ?
  - Any deadlocks ?

Translation into Loop Programs

- Define variable x_i meaning number of tokens at place p_i

Translation into Loop Programs

- Define variable x_i meaning number of tokens at place p_i
- Initial marking transformed into sequence of initializing assignments

Translation into Loop Programs

- Define variable x_i meaning number of tokens at place p_i
- Initial marking transformed into sequence of initializing assignments
- Transitions transformed into conditional statements

Translation into Loop Programs

- Define variable x_i meaning number of tokens at place p_i
- Initial marking transformed into sequence of initializing assignments
- Transitions transformed into conditional statements
- Enabling of a transition with input place p_i and label c_i:

$$\cdots (x_i \neq 0) \wedge (x_i \neq 1) \wedge \cdots \wedge (x_i \neq c_i - 1) \cdots$$

Translation into Loop Programs

- Define variable x_i meaning number of tokens at place p_i
- Initial marking transformed into sequence of initializing assignments
- Transitions transformed into conditional statements
- Enabling of a transition with input place p_i and label c_i:

$$\cdots (x_i \neq 0) \wedge (x_i \neq 1) \wedge \cdots \wedge (x_i \neq c_i - 1) \cdots$$

- Enabling of a transition with inhibitor place p_i: $x_i = 0$

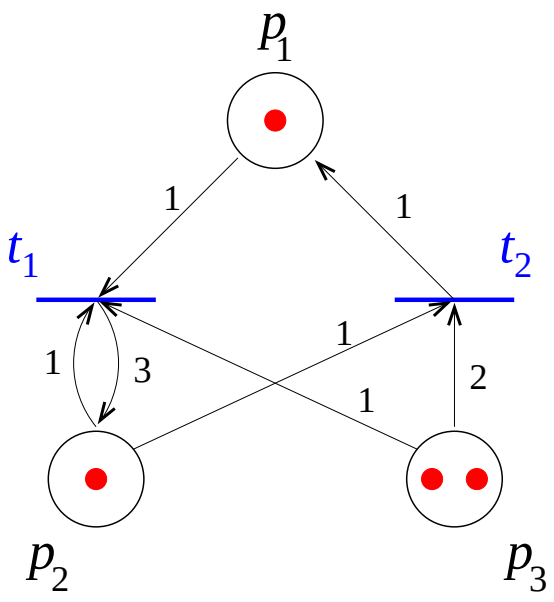
Translation into Loop Programs

- Define variable x_i meaning number of tokens at place p_i
- Initial marking transformed into sequence of initializing assignments
- Transitions transformed into conditional statements
- Enabling of a transition with input place p_i and label c_i:

$$\cdots (x_i \neq 0) \wedge (x_i \neq 1) \wedge \cdots \wedge (x_i \neq c_i - 1) \cdots$$

- Enabling of a transition with inhibitor place p_i: $x_i = 0$
- Firing of a transition
  - with input place p_i and label c_i: $x_i := x_i - c_i;$
  - with output place p_i and label c_i: $x_i := x_i + c_i;$

Translation into Programs (2)



$x_1 := 1; x_2 := 1; x_3 := 2;$

while $?$ **do**

t_1 : **if** $x_1 \neq 0 \wedge x_2 \neq 0 \wedge x_3 \neq 0 \rightarrow$

$x_1 := x_1 - 1;$

$x_2 := x_2 + 2;$

$x_3 := x_3 - 1;$

t_2 : $[]$ $x_2 \neq 0 \wedge x_3 \neq 0 \wedge x_3 \neq 1 \rightarrow$

$x_1 := x_1 + 1;$

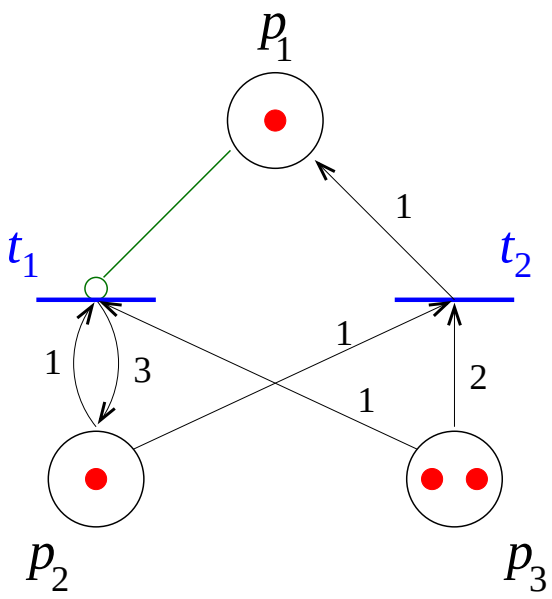
$x_2 := x_2 - 1;$

$x_3 := x_3 - 2;$

end if

end while

Translation into Programs (3)



$x_1 := 1; x_2 := 1; x_3 := 2;$

while $?$ **do**

t_1 : **if** $x_1 = 0 \wedge x_2 \neq 0 \wedge x_3 \neq 0 \rightarrow$

$x_1 := x_1 - 1;$

$x_2 := x_2 + 2;$

$x_3 := x_3 - 1;$

t_2 : $[]$ $x_2 \neq 0 \wedge x_3 \neq 0 \wedge x_3 \neq 1 \rightarrow$

$x_1 := x_1 + 1;$

$x_2 := x_2 - 1;$

$x_3 := x_3 - 2;$

end if

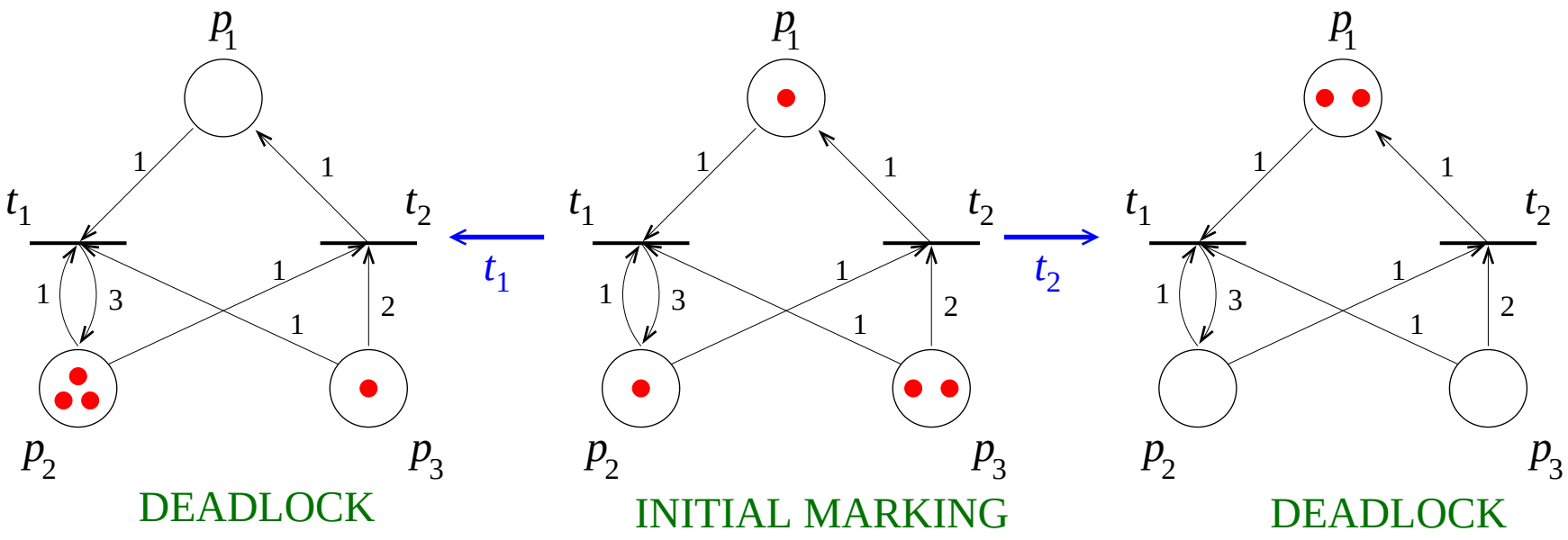
end while

Generating Polynomial Invariants (1)

- Abstract interpretation is applied to the loop program to obtain polynomial invariants of the Petri net

Generating Polynomial Invariants (1)

- Abstract interpretation is applied to the loop program to obtain polynomial invariants of the Petri net
- Example:



Generating Polynomial Invariants (2)

- Polynomial invariants obtained:

$$Inv = \begin{cases} 5x_1 + 3x_2 + x_3 - 10 & = 0 \\ 5x_3^2 + 2x_2 - 11x_3 & = 0 \\ x_2x_3 + 2x_3^2 - 5x_3 & = 0 \\ 5x_2^2 - 17x_2 + 6x_3 & = 0 \end{cases}$$

Generating Polynomial Invariants (2)

- Polynomial invariants obtained:

$$Inv = \begin{cases} 5x_1 + 3x_2 + x_3 - 10 & = 0 \\ 5x_3^2 + 2x_2 - 11x_3 & = 0 \\ x_2x_3 + 2x_3^2 - 5x_3 & = 0 \\ 5x_2^2 - 17x_2 + 6x_3 & = 0 \end{cases}$$

- In this example invariants characterize reachability set

$$Inv \Leftrightarrow (x_1, x_2, x_3) \in \{(0,3,1), (1,1,2), (2,0,0)\}$$

Generating Polynomial Invariants (2)

- Polynomial invariants obtained:

$$Inv = \begin{cases} 5x_1 + 3x_2 + x_3 - 10 & = 0 \\ 5x_3^2 + 2x_2 - 11x_3 & = 0 \\ x_2x_3 + 2x_3^2 - 5x_3 & = 0 \\ 5x_2^2 - 17x_2 + 6x_3 & = 0 \end{cases}$$

- In this example invariants characterize reachability set

$$Inv \Leftrightarrow (x_1, x_2, x_3) \in \{(0,3,1), (1,1,2), (2,0,0)\}$$

- In general overapproximation of reach set is obtained

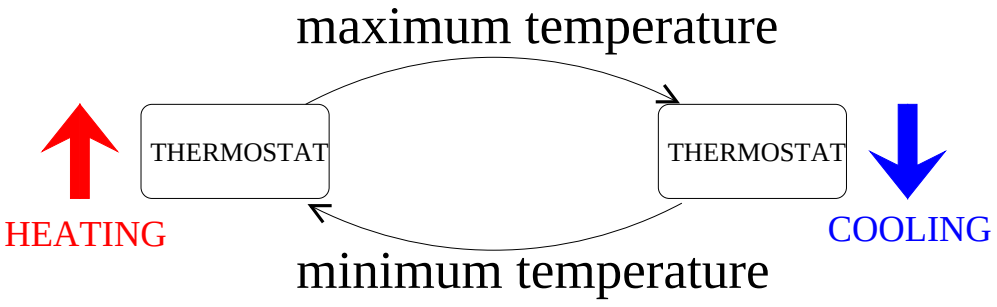
- Introduction
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
  - Imperative programs
  - Petri nets
  - Hybrid systems
- Generation of Invariant Polynomial Inequalities
- Conclusions and Future Work

Hybrid Systems: Introduction

- Hybrid System: discrete system in analog environment

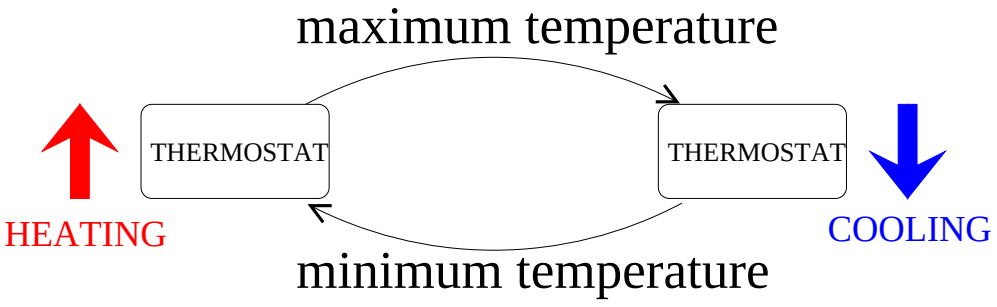
Hybrid Systems: Introduction

- Hybrid System: discrete system in analog environment
- Examples:
  - A thermostat that heats/cools depending on the temperature in the room



Hybrid Systems: Introduction

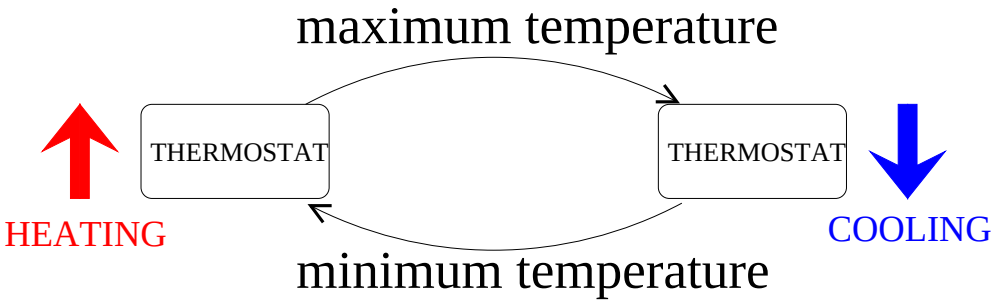
- Hybrid System: discrete system in analog environment
- Examples:
  - A thermostat that heats/cools depending on the temperature in the room



  - A robot controller that changes the direction of movement if the robot is too close to a wall.

Hybrid Systems: Introduction

- Hybrid System: discrete system in analog environment
- Examples:
  - A thermostat that heats/cools depending on the temperature in the room



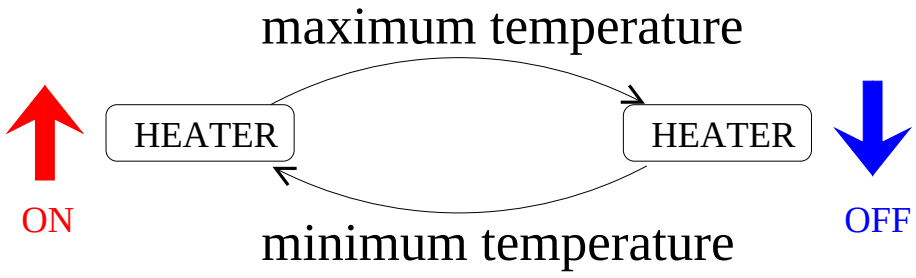
  - A robot controller that changes the direction of movement if the robot is too close to a wall.
  - A biochemical reaction whose behaviour depends on concentration of substances in environment

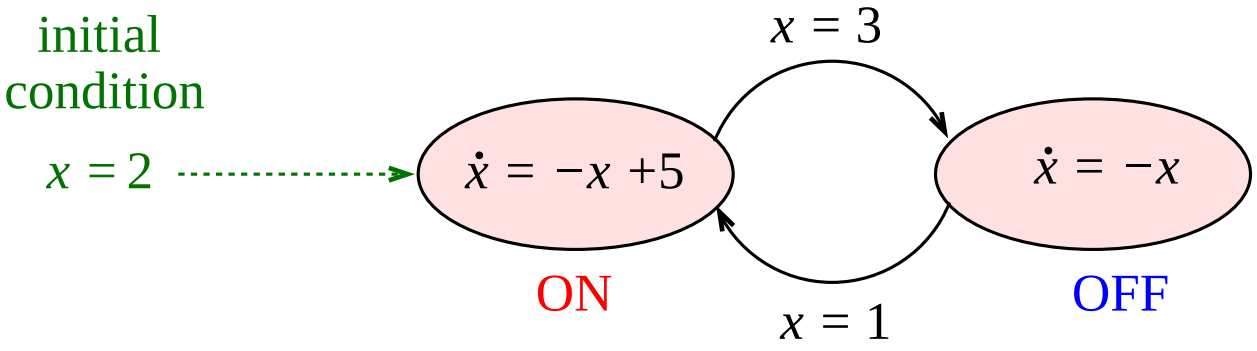
Definition

- A hybrid system is a finite automaton with real-valued variables that change continuously according to a system of differential equations at each location

Definition

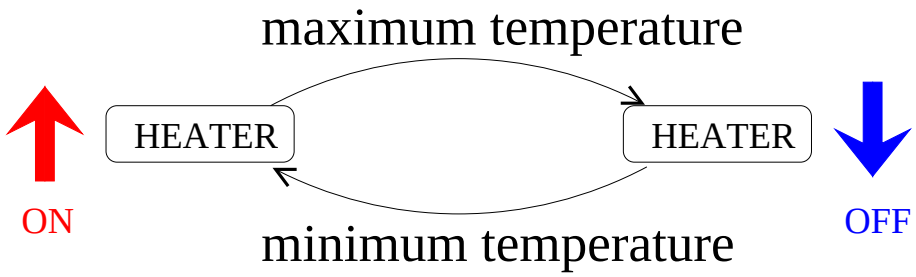
- A hybrid system is a finite automaton with real-valued variables that change continuously according to a system of differential equations at each location

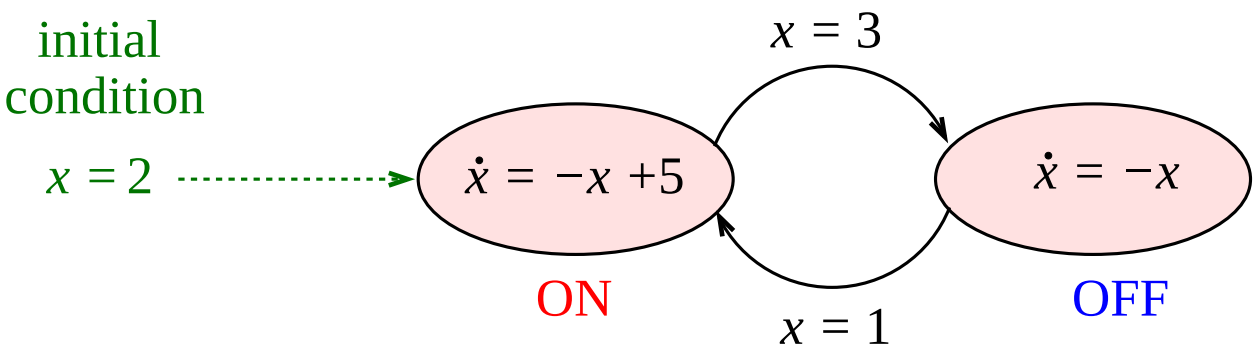




Definition

- A hybrid system is a finite automaton with real-valued variables that change continuously according to a system of differential equations at each location





- Restricted to linear differential equations at locations

Dynamics (1)

- A computation is a sequence of states (discrete location, valuation of variables)

$$(l_0, x_0), (l_1, x_1), (l_2, x_2), ...$$

such that

Dynamics (1)

- A computation is a sequence of states (discrete location, valuation of variables)

$$(l_0, x_0), (l_1, x_1), (l_2, x_2), ...$$

such that

1. Initial state (l_0, x_0) satisfies the initial condition

Dynamics (1)

- A computation is a sequence of states (discrete location, valuation of variables)

$$(l_0, x_0), (l_1, x_1), (l_2, x_2), ...$$

such that

1. Initial state (l_0, x_0) satisfies the initial condition
2. For each consecutive pair of states $(l_i, x_i), (l_{i+1}, x_{i+1})$:
   - Discrete transition: there is a transition of the automaton (l_i, l_{i+1}, ρ) such that $(x_i, x_{i+1}) \models \rho$

Dynamics (1)

- A computation is a sequence of states (discrete location, valuation of variables)

$$(l_0, x_0), (l_1, x_1), (l_2, x_2), ...$$

such that

1. Initial state (l_0, x_0) satisfies the initial condition
2. For each consecutive pair of states $(l_i, x_i), (l_{i+1}, x_{i+1})$:
   - Discrete transition: there is a transition of the automaton (l_i, l_{i+1}, ρ) such that $(x_i, x_{i+1}) \models \rho$
   - Continuous evolution: there is a trajectory going from x_i to x_{i+1} along the flow determined by the differential equation $\dot{x} = Ax + B$ at location $l_i = l_{i+1}$

Dynamics (2)

- Goal: generate invariant polynomial equalities

Dynamics (2)

- Goal: generate invariant polynomial equalities
  - We know how to deal with discrete systems
  - How to handle continuous evolution?

Dynamics (2)

- Goal: generate invariant polynomial equalities
  - We know how to deal with discrete systems
  - How to handle continuous evolution?
- Problem:
  computing polynomial invariants of linear systems of differential equations

Form of the Solution

Solution to $\dot{x} = Ax + B$ can be expressed as polynomials in t, $e^{\pm at}$, $\cos(bt)$, $\sin(bt)$, where $\lambda = a + bi$ are eigenvalues of matrix A.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{v_x} \\ \dot{v_y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix}$$

$$\begin{cases} x &= x^* + 2\sin(t/2)\, v_x^* + (2\cos(t/2) - 2)\, v_y^* \\ y &= y^* + (-2\cos(t/2) + 2)\, v_x^* + 2\sin(t/2)\, v_y^* \\ v_x &= \cos(t/2)\, v_x^* - \sin(t/2)\, v_y^* \\ v_y &= \sin(t/2)\, v_x^* + \cos(t/2)\, v_y^* \end{cases}$$

Elimination of Time

Idea: eliminate terms depending on t from solution:

- transform solution into polynomials using new variables
- eliminate by means of Gröbner bases using auxiliary equations

Elimination of Time

Idea: eliminate terms depending on t from solution:

- transform solution into polynomials using new variables
- eliminate by means of Gröbner bases using auxiliary equations

SOLUTION

$$\begin{cases} x &= x^* + 2zv_x^* + (2w-2)v_y^* \\ y &= y^* + (-2w+2)\,v_x^* + 2zv_y^* \\ v_x &= wv_x^* - zv_y^* \\ v_y &= zv_x^* + wv_y^* \end{cases}$$

INITIAL CONDITIONS

$$\begin{cases} v_x^* &= 2 \\ v_y^* &= -2 \end{cases}$$

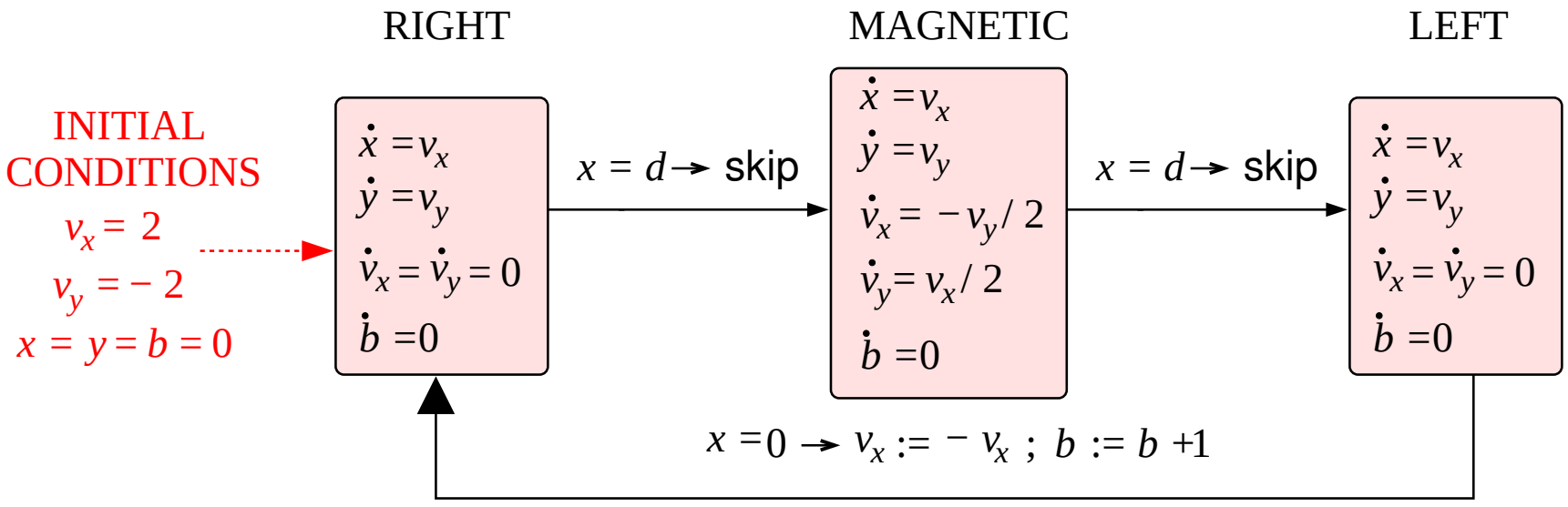
AUXILIARY EQUATIONS

$$\left\{ w^2 + z^2 = 1 \right.$$

$$\Downarrow$$

$v_x^2 + v_y^2 = 8$ (conservation of energy)

Example



RIGHT $\rightarrow$ $v_y = -2 \;\wedge\; v_x = 2 \;\wedge\; 2db - 8b + y + x = 0$

MAGNETIC $\rightarrow$ $x - 2v_y - d = 4 \wedge v_x^2 + v_y^2 = 8 \wedge 2v_x + y + 2db - 8b + d = 4$

LEFT $\rightarrow$ $v_y = -2 \;\wedge\; v_x = -2 \;\wedge\; 2db - 8b + y - x = 8$

- Introduction
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
- Generation of Invariant Polynomial Inequalities
- Conclusions and Future Work

Drawing a Parallel from Equalities

Linear equalities

[Karr'76]

Polynomial equalities

[Colon'04]

Drawing a Parallel from Equalities

Linear equalities
[Karr'76]

↓

Polynomial equalities
[Colon'04]

Linear inequalities
[Cousot & Halbwachs'78]

↓

Polynomial inequalities
[Bagnara & Rodríguez-Carbonell & Zaffanella'05]

From Linear to Polynomial Equalities

$a := 0$;
$b := 0$;
$c := 1$;

while $?$ **do**

$a := a + 1$;
$b := b + c$;
$c := c + 2$;

end while

From Linear to Polynomial Equalities

$a := 0$;
$b := 0$;
$c := 1$;

$\{\ a = 0 \wedge b = 0 \wedge c = 1\ \}$
while $?$ **do**

$a := a + 1$;
$b := b + c$;
$c := c + 2$;

end while

From Linear to Polynomial Equalities

$a := 0$;
$b := 0$;
$c := 1$;

$\{ \ a = b \wedge c = 2a + 1 \ \}$
while $?$ **do**

$a := a + 1$;
$b := b + c$;
$c := c + 2$;

end while

From Linear to Polynomial Equalities

$a := 0$;
$b := 0$;
$c := 1$;

$\{ \ c = 2a + 1 \ \}$
while $?$ **do**

$a := a + 1$;
$b := b + c$;
$c := c + 2$;

end while

Loop invariant
$\{ \ c = 2a + 1 \ \}$

From Linear to Polynomial Equalities

$a := 0$;
$b := 0$;
$c := 1$;

Introduce new variable s standing for a^2

while $?$ **do**

$a := a + 1$;
$b := b + c$;
$c := c + 2$;

end while

From Linear to Polynomial Equalities

$a := 0$;
$b := 0$;
$c := 1$;
$s := 0$; ←

while $?$ **do**

$a := a + 1$;
$b := b + c$;
$c := c + 2$;
$s := s + 2a + 1$; ←

end while

Introduce new variable s standing for a^2

Extend program with new variable s

$$a := 0 \quad \rightarrow \quad s := 0$$

$$a := a + 1 \quad \rightarrow \quad s := s + 2a + 1$$

From Linear to Polynomial Equalities

$a := 0$;
$b := 0$;
$c := 1$;
$s := 0$;
$\{\ a = 0 \wedge b = 0 \wedge c = 1 \wedge s = 0\ \}$
while $?$ **do**

$a := a + 1$;
$b := b + c$;
$c := c + 2$;
$s := s + 2a + 1$;
end while

From Linear to Polynomial Equalities

$a := 0$;
$b := 0$;
$c := 1$;
$s := 0$;
$\{\ a = b \wedge b = s \wedge c = 2a + 1\ \}$
while $?$ **do**

$a := a + 1$;
$b := b + c$;
$c := c + 2$;
$s := s + 2a + 1$;

end while

From Linear to Polynomial Equalities

$a := 0$;
$b := 0$;
$c := 1$;
$s := 0$;
$\{\ b = s \wedge c = 2a + 1\ \}$
while $?$ **do**

$a := a + 1$;
$b := b + c$;
$c := c + 2$;
$s := s + 2a + 1$;

end while

Loop invariant

$\{\ b = a^2 \wedge c = 2a + 1\ \}$

is more precise

From Linear to Polynomial Inequalities

$\{ \text{Pre}: \ b \geq 0 \}$

$a := 0 \ ;$

while $(a+1)^2 \leq b$ **do**

$\quad a := a + 1 \ ;$

end while

$\{ \text{Post}: \ (a+1)^2 > b \wedge b \geq a^2 \}$

From Linear to Polynomial Inequalities

$\{ \text{Pre}: \ b \geq 0 \}$

$a := 0 \ ;$

while $(a+1)^2 \leq b$ **do**

$a := a + 1 \ ;$

end while

$\{ \text{Post}: \ (a+1)^2 > b \wedge b \geq a^2 \}$

Linear analysis cannot deal with the quadratic condition

$(a+1)^2 \leq b$

From Linear to Polynomial Inequalities

$\{ \text{Pre}: \ b \geq 0 \}$

$a := 0 \ ;$

$\{ \ a \geq 0 \wedge b \geq 0 \ \}$

while $(a+1)^2 \leq b$ **do**

$\quad a := a + 1 \ ;$

end while

$\{ \text{Post}: \ (a+1)^2 > b \wedge b \geq a^2 \ \}$

Loop invariant $\{ \ a \geq 0 \wedge b \geq 0 \ \}$

not precise enough

From Linear to Polynomial Inequalities

$\{ \text{Pre}: \ b \geq 0 \}$

$a := 0 \ ;$
$s := 0 \ ;$ ←

while $(a+1)^2 \leq b$ **do**

$a := a + 1 \ ;$
$s := s + 2a + 1 \ ;$ ←

end while

$\{ \text{Post}: \ (a+1)^2 > b \wedge b \geq a^2 \}$

Introduce new variable s standing for a^2

Extend program with new variable s

$$a := 0 \quad \rightarrow \quad s := 0$$

$$a := a + 1 \quad \rightarrow \quad s := s + 2a + 1$$

From Linear to Polynomial Inequalities

$\{ \text{Pre}: \ b \geq 0 \}$

$a := 0 \ ;$
$s := 0 \ ;$
$\{ \ b \geq s \wedge \cdots \ \}$
while $(a+1)^2 \leq b$ **do**

$a := a + 1 \ ;$
$s := s + 2a + 1 \ ;$
end while

$\{ \text{Post}: \ (a+1)^2 > b \wedge b \geq a^2 \}$

Loop invariant
$\{ \ b \geq a^2 \wedge \cdots \ \}$
enough to prove partial correctness

Linearization of Polynomial Constraints

- Abstract values = sets of constraints
- Given a degree bound d, all terms x^α with $\deg(x^\alpha) \leq d$ are considered as different and independent variables

Vector Spaces $\leftrightarrow$ Polynomial Cones

polynomial $= 0$

- $\forall$ polynomial p, $p \sim p = 0$
- Vector space =
  set of polynomials V s.t.
  - $0 \in V$
  - $\forall p, q \in V$ and $\lambda, \mu \in \mathbb{R}$,
    $\lambda p + \mu q \in V$

$$\frac{}{0 = 0}$$

$$\frac{p = 0 \quad q = 0 \quad \lambda, \mu \in \mathbb{R}}{\lambda p + \mu q = 0}$$

Vector Spaces $\leftrightarrow$ Polynomial Cones

polynomial $= 0$

- $\forall$ polynomial p, $p \sim p = 0$
- Vector space = set of polynomials V s.t.
  - $0 \in V$
  - $\forall p, q \in V$ and $\lambda, \mu \in \mathbb{R}$, $\lambda p + \mu q \in V$

$$\frac{}{0 = 0}$$

$$\frac{p = 0 \quad q = 0 \quad \lambda, \mu \in \mathbb{R}}{\lambda p + \mu q = 0}$$

polynomial ≥ 0

- $\forall$ polynomial p, $p \sim p \geq 0$
- Polynomial cone = set of polynomials C s.t.
  - $1 \in C$
  - $\forall p, q \in C$ and $\lambda, \mu \in \mathbb{R}_+$, $\lambda p + \mu q \in C$

$$\frac{}{1 \geq 0}$$

$$\frac{p \geq 0 \quad q \geq 0 \quad \lambda, \mu \in \mathbb{R}_+}{\lambda p + \mu q \geq 0}$$

Explicitly Adding Other Inference Rules

$$\text{polynomial} = 0$$

$$\frac{p = 0 \qquad \deg(pq) \leq d}{pq = 0}$$

Explicitly Adding Other Inference Rules

polynomial $= 0$

$$\frac{p = 0 \quad \deg(pq) \leq d}{pq = 0}$$

polynomial ≥ 0

$$\frac{p \geq 0 \quad p \leq 0 \quad \deg(pq) \leq d}{pq = 0}$$

$$\frac{p \geq 0 \quad q \geq 0 \quad \deg(pq) \leq d}{pq \geq 0}$$

- Introduction
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
- Generation of Invariant Polynomial Inequalities
- Conclusions and Future Work

Conclusions

- Designed a new abstract domain for generating invariant polynomial equalities based on ideals of polynomials

Conclusions

- Designed a new abstract domain for generating invariant polynomial equalities based on ideals of polynomials
- Identified a class of programs for which all polynomial equality invariants can be generated

Conclusions

- Designed a new abstract domain for generating invariant polynomial equalities based on ideals of polynomials
- Identified a class of programs for which all polynomial equality invariants can be generated
- Applied polynomial equality invariants to verifying imperative programs, Petri nets and hybrid systems

Conclusions

- Designed a new abstract domain for generating invariant polynomial equalities based on ideals of polynomials
- Identified a class of programs for which all polynomial equality invariants can be generated
- Applied polynomial equality invariants to verifying imperative programs, Petri nets and hybrid systems
- Designed a new abstract domain for generating invariant polynomial inequalities based on polynomial cones

Future Work

- Extend the techniques to interprocedural analyses

Future Work

- Extend the techniques to interprocedural analyses
- Develop methods for tuning the precision/efficiency trade-off

Future Work

- Extend the techniques to interprocedural analyses
- Develop methods for tuning the precision/efficiency trade-off
- Find new areas of application for polynomial invariants

Future Work

- Extend the techniques to interprocedural analyses
- Develop methods for tuning the precision/efficiency trade-off
- Find new areas of application for polynomial invariants
- Design specific widening operators for the context of polynomial invariants

Publications

- E. Rodríguez-Carbonell, D. Kapur. An abstract interpretation approach for automatic generation of polynomial invariants. SAS'04.
- E. Rodríguez-Carbonell, D. Kapur. Automatic generation of polynomial loop invariants: algebraic foundations. ISSAC'04.
- E. Rodríguez-Carbonell, D. Kapur. Program verification using automatic generation of polynomial invariants. ICTAC'04.
- E. Rodríguez-Carbonell, A. Tiwari. Generating polynomial invariants for hybrid systems. HSCC'05.
- R. Clarisó, E. Rodríguez-Carbonell, J. Cortadella. Derivation of non-structural invariants of Petri nets using abstract interpretation. ATPN'05.
- R. Bagnara, E. Rodríguez-Carbonell, E. Zaffanella. Generation of basic semi-algebraic invariants using convex polyhedra. SAS'05.
- E. Rodríguez-Carbonell, D. Kapur. Automatic generation of polynomial invariants of bounded degree using abstract interpretation. Science of Computer Programming, 2006. Accepted for publication.