SAT modulo the theory of linear arithmetic: Exact, inexact and commercial solvers

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Overview of the talk

- SAT Modulo Theories (SMT)
  - DPLL(T) = Boolean engine + T-Solver
  - What is needed from T-Solver?
- Use of OR solvers for DPLL(LA)
  - Existing and non-existing functionalities
  - Adapting OR solvers
- Experimental evaluation
- New prospects and conclusions

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SAT Modulo Theories (SMT)

- Some problems are more naturally expressed in other logics than propositional logic, e.g:
  - Software verification needs reasoning about equality, arithmetic, data structures, ...
- SMT consists of deciding the satisfiability of a (ground) FO formula with respect to a background theory
- Example (Equality with Uninterpreted Functions – EUF):

$$g(a)=c \;\wedge\; (\, f(g(a))\neq f(c) \vee g(a)=d \,) \;\wedge\; c\neq d$$

- Wide range of applications:
  - Predicate abstraction
  - Model checking
  - Equivalence checking
  - Static analysis
  - Scheduling
  - ...

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  - Difference logic ($x - y \leq 4$)
    / UTVPI constraints ($x - y \leq 2$, $x + y \leq 7$)
    / General linear constraints (e.g., $2x + y - z \leq 3$)

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THIS TALK: general quantifier-free formulas in $\mathbb{R}$

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Solving SMT with DPLL(T)

Methodology:

$$\underbrace{x \leq 2}_{1} \;\wedge\; (\underbrace{x + y \geq 10}_{2} \vee \underbrace{2x + 3y \geq 30}_{3}) \;\wedge\; \underbrace{y \leq 4}_{4}$$

- SAT solver returns model $[1, 2, 4]$

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Two components: Boolean engine DPLL(X) + T-Solver

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Solving SMT with DPLL(T) (2)

Several optimizations for enhancing efficiency:

- Check T-consistency only of full prop. models (at a leaf)

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THIS TALK: obtain LA-solvers that are incremental, backtrackable and produce inconsistency explanations

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Current state of LA-solvers in SMT

- Different techniques have been tried:
  - Simplex-based procedures:
    ARGOLIB, BARCELOGIC, CVC3, MATHSAT, YICES, Z3
  - Fourier-Motzkin:
    CVC, CVCLITE, SVC
  - Automata
- So far, clear winner is Simplex
- SMT-oriented Simplex implementations are recent (3 years)
- Why not trying mature OR Linear Programming tools?

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Use of OR tools - Good news

- Provide incremental addition / removal of constraints
- Provide support for explanations of inconsistencies
- All these functionalities available through an API
- Can handle millions of constraints, millions of vars
- Many years of application to real-life problems
- These features both in commercial and publicly available solvers, e.g.:
  - Commercial: CPLEX 11
  - Publicly available: GLPK 4.25

Use of OR tools - Bad news

- Take the set of constraints:

$$\begin{array}{rcl} -x-y+u & \leq & 0 \\ -u+z & \leq & 0 \\ -t+y & \leq & 0 \end{array} \qquad \begin{array}{rcl} -11z+v+11t & \leq & 0 \\ 11x-v & \leq & -10^{-5} \\ x & \geq & 10^{-5} \end{array}$$

and give it to CPLEX 11. Answer: **SAT**.

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- Now, multiply the 3 constraints on the left by 11

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- Now, multiply the 3 constraints on the left by 11
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- We obtain $0 \leq -10^{-5}$. Hence, answer should be **UNSAT!!!!**.
- What is the PROBLEM here?

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Use of OR tools - Bad news (2)

The problem is that

- Most LP solvers work with floating-point arithmetic

In addition,

- They cannot handle natively strict inequalities
- They cannot handle natively disequalities

Last two problems, have en easy “fix”:

- Transform $< k$ into $\leq k - \varepsilon$ (ε “small”)
- Split input disequalities $x \neq y$ into $x < y \vee x > y$ (formula level)

at the expense of introducing more inaccuracies

Use of OR tools - Bad news (3)

How to fix the inaccuracies due to floating-point arithmetic?
Use an exact T-Solver (Ex-Solver in the following)

- Only two critical situations:
  1. We are at a leaf an T-Solver says model is consistent
  2. T-Solver says assignment is inconsistent

- They both have a successful solution:
  1. Check whether solution satisfies all literals.
     If not, send model to an exact solver and proceed.
  2. Ask T-Solver for an explanation and send it to Ex-Solver.
     If inconsistent **then** continue.
     Else, **If** in a leaf **then** check assignment with Ex-Solver.
     Else continue as if assignment was consistent.

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Experimental evaluation

Setting used:

- Used BARCELOGIC as SMT solver
- Integrated both CPLEX 11 and GLPK 4.25 (the former substantially faster)
- Used our own Simplex-like T-Solver as exact solver
- Ran experiments over the QF_LRA division of SMT-LIB (500 problems)

Questions to answer:

1. How frequent are wrong results?
2. How expensive is result checking?
3. Which is the overall gain in performance?

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Experimental evaluation - Analysis

QUESTION: How frequent are wrong results?

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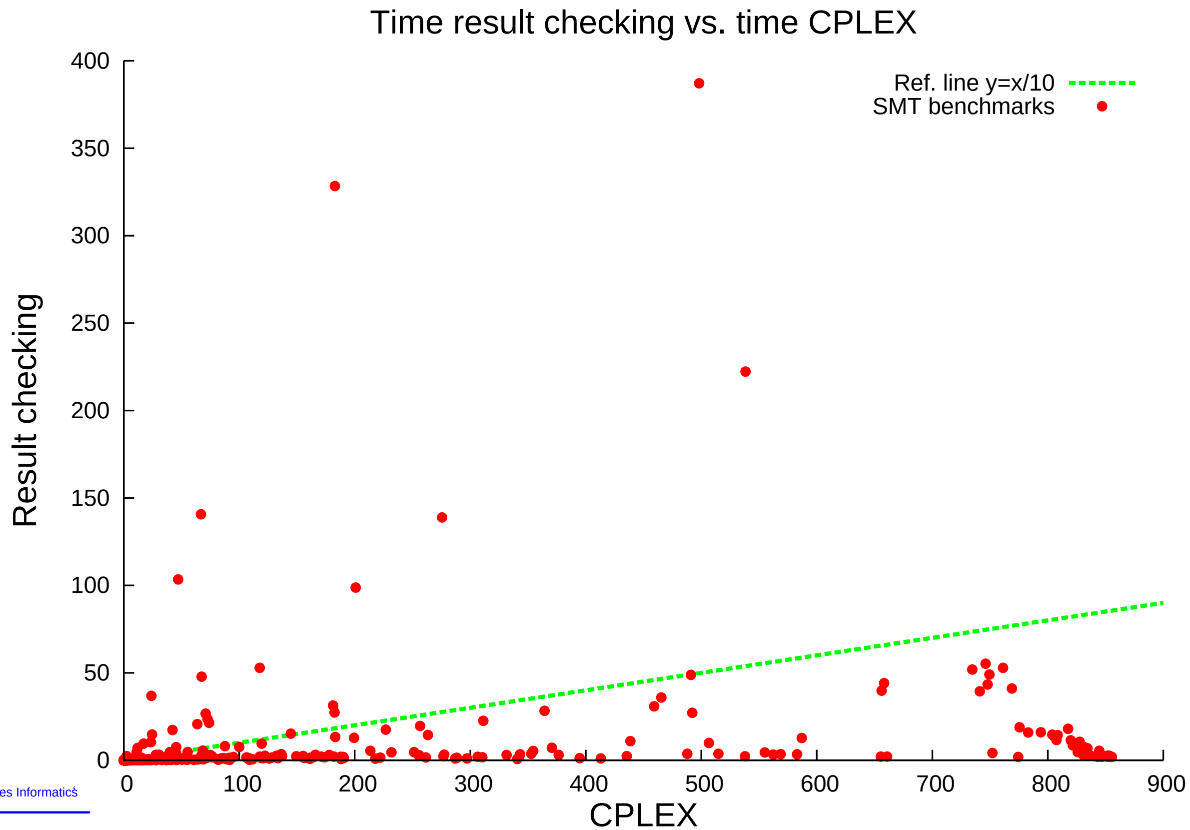

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Experimental evaluation - Analysis (2)

- So far, experiments reveal that:
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  - Result checking is cheap comparing to CPLEX time
- Conclusion: replacing our exact T-Solver by CPLEX + result checking will improve performance

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- So far, experiments reveal that:
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  - Result checking is cheap comparing to CPLEX time
- Conclusion: replacing our exact T-Solver by CPLEX + result checking will improve performance
- Reality: no speed up is obtained. Sometimes even slower!!!!
- Analysis: could it be due to luck or parameter tuning?

  ANSWER: try fair comparison, i.e.

  1. Simultaneously call our exact solver and CPLEX at consistency checks
  2. Measure time taken by each solver
  3. Let exact solver guide the search

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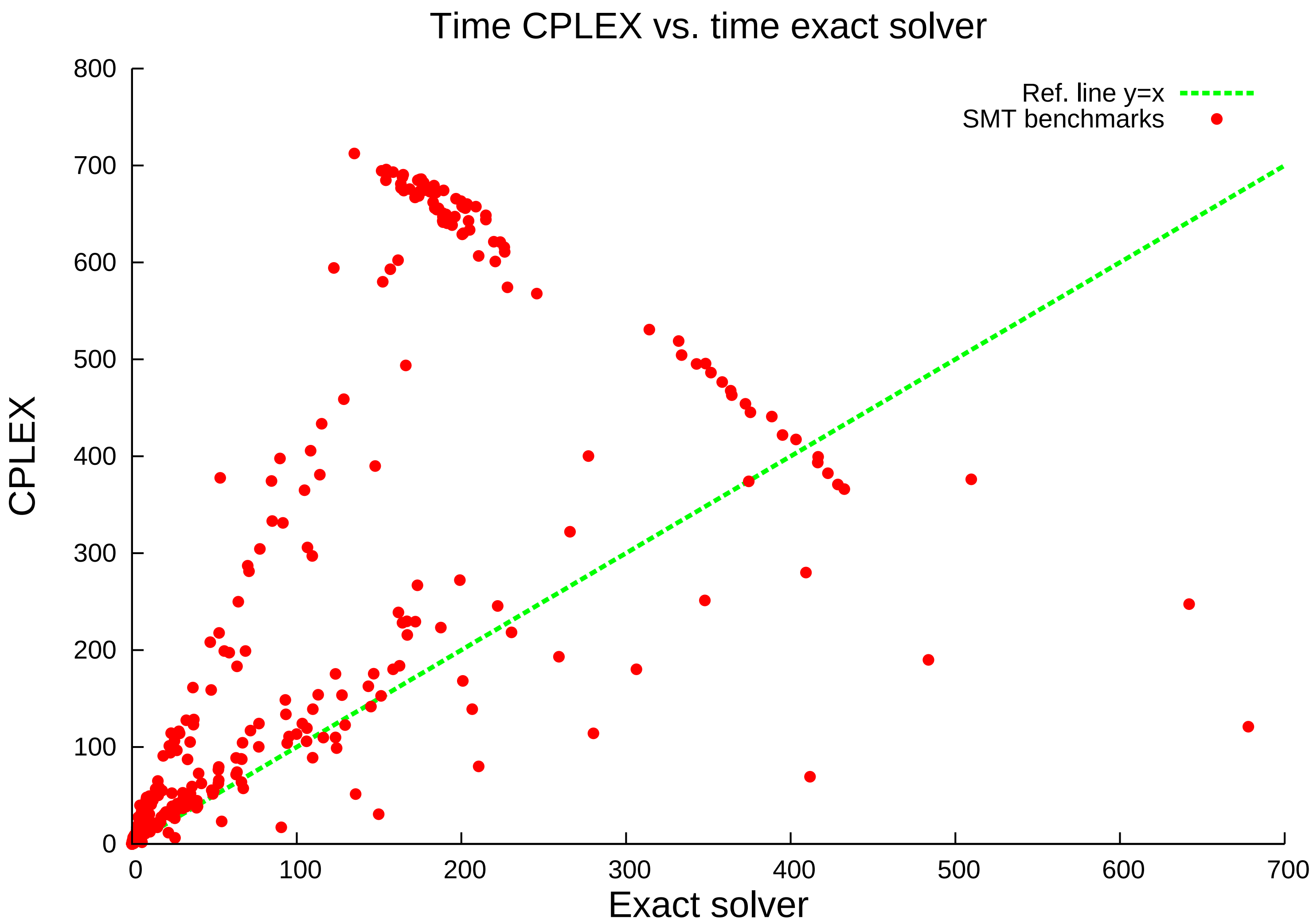

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Experimental evaluation - Analysis (4)

How to explain these results?

- We have tried various CPLEX settings, algorithms, parameters
- We have worked on many hypothesis (see paper)
- OUR CONCLUSION: CPLEX is not designed for being used as a solver in DPLL(T) nor the kind of problems arising in SMT
  - SMT problems are small (thousands constraints, hundreds variables)
  - Adding/removing constraints is not determinant in OR
  - Inconsistency explanations are not crucial in OR
  - Unlike with OR problems, SMT problems are easy and can be solved with few iterations of the simplex

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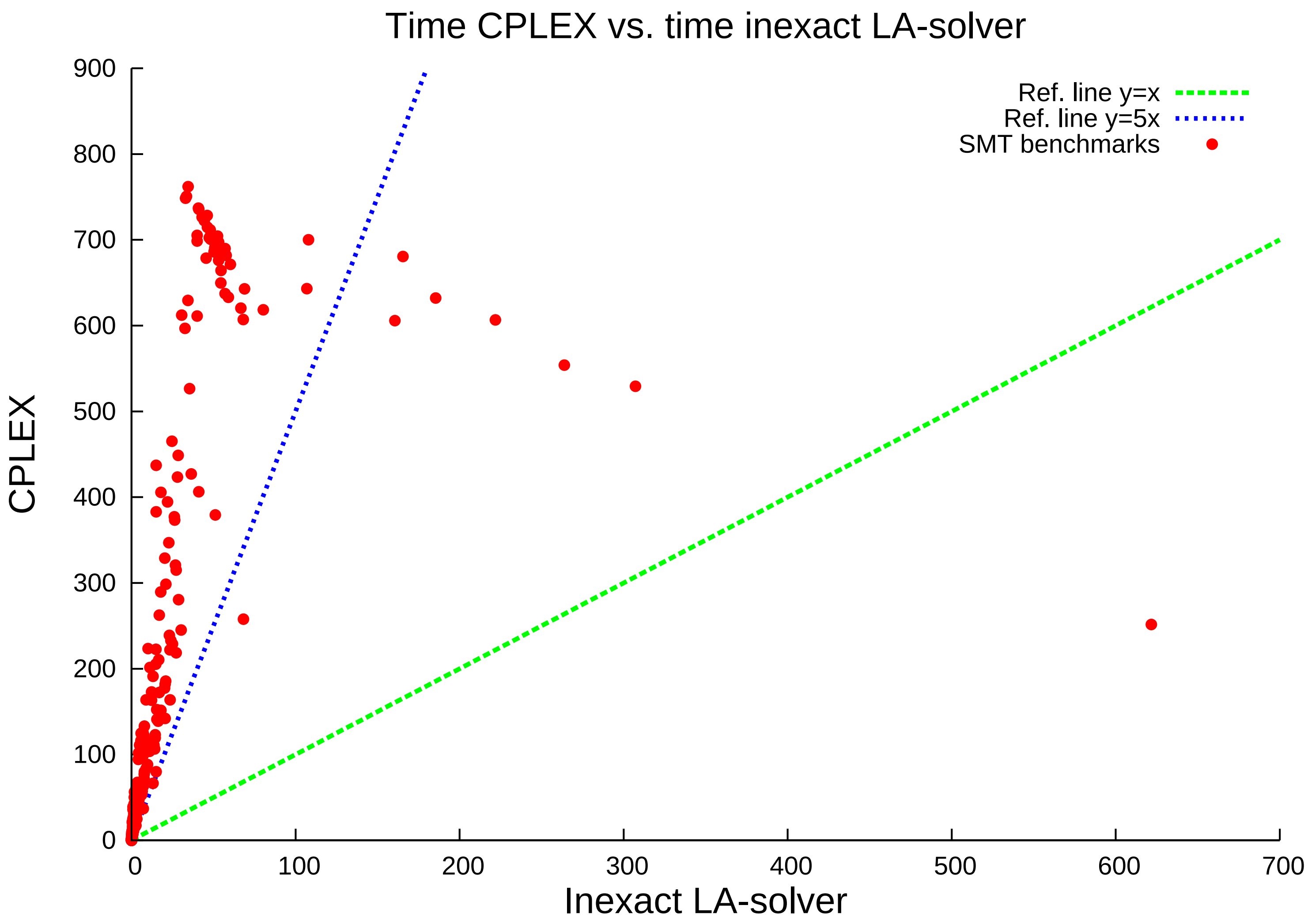
New prospects

- Experiments reveal that OR solvers are not competitive in SMT problems
- This negative result suggests new line of research:

> Combine result checking with implementations of our SMT T-Solver's using floating-point arithmetic

- Our initial experiments are promising, but result checking could be further improved
- The following figure shows a comparison of using CPLEX and a floating-point version of our T-Solver on the same sequence of calls.

New prospects (2)



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Conclusions

- Use of OR solvers inside an SMT system
- Result checking policies are cheap
- OR solvers are slower than SMT-dedicated solvers
- Possibility of improving performance by SMT-dedicated floating-point solvers

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Thank you!

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