Non-linear Arithmetic Solving for Termination Analysis

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Overview of the Talk

- **Non-linear constraint solving**
  - Review of [JAR'12]
  - Alternative Max-SMT approach
- **Constraint-based termination analysis**
  - Review of program termination and constraint-based program analysis
  - Using Max-SMT for termination analysis
  - Implementation and experiments
- **Conclusions & future work**

Non-linear Constraint Solving

- Problem: Given a quantifier-free formula F containing polynomial inequality atoms, is F satisfiable?
- Applications: system analysis and verification, ...
  Here, focus will be on termination of imperative programs
- In $\mathbb{Z}$: undecidable (Hilbert's 10th problem)
- In $\mathbb{R}$: decidable, even with quantifiers (Tarski)
  But algorithms have prohibitive complexity
- Goal: Can we have a procedure that works "well" in practice?

Review of [JAR'12]

- Our method is aimed at proving satisfiability in the integers (as opposed to finding non-integer solutions, or proving unsatisfiability)
- Basic idea: use bounds on integer variables to linearize the formula
- Refinement: analyze unsatisfiable cores to enlarge bounds (and sometimes even prove unsatisfiability)

Translating into Linear Arithmetic

- For any formula there is an equisatisfiable one of the form

$$F \wedge (\bigwedge_i y_i = M_i)$$

where F is linear and each M_i is non-linear

- Example

$$u^4 v^2 + 2u^2 vw + w^2 \leq 4 \ \wedge \ 1 \leq u, v, w \leq 2$$

$$x_{u^4 v^2} + 2x_{u^2 vw} + x_{w^2} \leq 4 \ \wedge \ 1 \leq u, v, w \leq 2 \ \wedge$$
$$x_{u^4 v^2} = u^4 v^2 \ \wedge \ x_{u^2 vw} = u^2 vw \ \wedge \ x_{w^2} = w^2$$

Translating into Linear Arithmetic

- Idea: linearize non-linear monomials with case analysis on some of the variables with finite domain
- Assume variables are in $\mathbb{Z}$
- $F \ \wedge \ \ x_{u^4v^2} = u^4v^2 \ \ \wedge \ \ x_{u^2vw} = u^2vw \ \ \wedge \ \ x_{w^2} = w^2$
  where F is $x_{u^4v^2} + 2x_{u^2vw} + x_{w^2} \leq 4 \ \wedge \ 1 \leq u, v, w \leq 2$
- Since $1 \leq w \leq 2$, add $x_{u^2v} = u^2v$ and
  $w = 1 \ \rightarrow \ x_{u^2vw} = x_{u^2v}$
  $w = 2 \ \rightarrow \ x_{u^2vw} = 2x_{u^2v}$

Translating into Linear Arithmetic

Applying the same idea recursively, the following linear formula is obtained:

$$
\begin{aligned}
& x_{u^4v^2} + 2x_{u^2vw} + x_{w^2} \leq 4 \\
& \wedge\ 1 \leq u, v, w \leq 2 \\
& \wedge\ w = 1 \ \rightarrow\ x_{u^2vw} = x_{u^2v} \\
& \wedge\ w = 2 \ \rightarrow\ x_{u^2vw} = 2x_{u^2v} \\
& \wedge\ u = 1 \ \rightarrow\ x_{u^2v} = v \\
& \wedge\ u = 2 \ \rightarrow\ x_{u^2v} = 4v \\
& \wedge\ w = 1 \ \rightarrow\ x_{w^2} = 1 \\
& \wedge\ w = 2 \ \rightarrow\ x_{w^2} = 4 \\
& \wedge\ v = 1 \ \rightarrow\ x_{u^4v^2} = x_{u^4} \\
& \wedge\ v = 2 \ \rightarrow\ x_{u^4v^2} = 4x_{u^4} \\
& \wedge\ u = 1 \ \rightarrow\ x_{u^4} = 1 \\
& \wedge\ u = 2 \ \rightarrow\ x_{u^4} = 16
\end{aligned}
$$

A model can be computed:

$$
\begin{aligned}
& u = 1 \\
& v = 1 \\
& w = 1 \\
& x_{u^4v^2} = 1 \\
& x_{u^4} = 1 \\
& x_{u^2vw} = 1 \\
& x_{u^2v} = 1 \\
& x_{w^2} = 1
\end{aligned}
$$

Unsatisfiable Core Analysis

- If linearization achieves a linear formula then we have a sound and complete decision procedure

- If we don't have enough variables with finite domain...
  ... we can add bounds at cost of losing completeness
  We cannot trust UNSAT answers!

- But we can analyze why the CNF is UNSAT:
  an unsatisfiable core (= unsatisfiable subset of clauses) can be obtained from the trace of the DPLL execution [Zhang & Malik'03]

- If core contains no extra bound: truly UNSAT
  If core contains extra bound: guide to enlarge domains

Unsatisfiable Core Analysis

- $u^4v^2 + 2u^2vw + w^2 \leq 3$ cannot be linearized
- Consider $u^4v^2 + 2u^2vw + w^2 \leq 3 \;\wedge 1 \leq u, v, w \leq 2$
- The linearization is unsatisfiable:

$$
\begin{array}{l}
x_{u^4v^2} + 2x_{u^2vw} + x_{w^2} \leq 3 \\
\;\wedge 1 \leq x_{u^4v^2} \;\; \wedge x_{u^4v^2} \leq 64 \\
\;\wedge 1 \leq x_{u^2vw} \;\; \wedge x_{u^2vw} \leq 16 \\
\;\wedge 1 \leq x_{w^2} \;\; \wedge x_{w^2} \leq 4 \\
\;\wedge 1 \leq u \;\; \wedge u \leq 2 \\
\;\wedge 1 \leq v \;\; \wedge v \leq 2 \\
\;\wedge 1 \leq w \;\; \wedge w \leq 2 \\
\ldots
\end{array}
$$

- Should decrease lower bounds for u, v, w

An Alternative Max-SMT Approach

- Max-STM(T): Given a set of weighted clauses, find a T-consistent assignment that minimizes cost ($=$ sum of weights) of falsified clauses

- Assume we are given a non-linear formula and have computed a linearization (possibly with extra bounds).

  Then we transform the linear formula into a weighted one as follows:

  - Clauses C of extra bounds are given finite weights ω_C (soft clauses)
  - Rest of clauses are given weight ∞ (hard clauses)

- So we have a Max-SMT(LIA) problem, instead of an SMT(LIA) one

- If found model with null cost, we have a solution
- Else falsified soft clauses show bounds to relax

An Alternative Max-SMT Approach

- There exist simple Branch & Bound algorithms for Max-SMT [Nieuwenhuis & Oliveras, SAT'06], [Cimatti et al., TACAS'10]

- Advantages over the analysis of unsatisfiable cores
  - Max-SMT approach is easier to implement and maintain
  - Leads naturally to an extension to Max-SMT(NIA):
    Given a set of weighted clauses in NIA, linearize as usual but
    - Original clauses keep their weight
    - Clauses of case splits are given weight ∞
    - Clauses of extra bounds are given weights $\omega > W$, where W is the sum of the weights of the original soft clauses

    So models that violate original clauses are preferred over those violating case splits (that ensure a true model for NA can be reconstructed)

An Alternative Max-SMT Approach

- Example revisited
- $u^4v^2 + 2u^2vw + w^2 \leq 3$ cannot be linearized
- Consider $u^4v^2 + 2u^2vw + w^2 \leq 3 \;\wedge 1 \leq u, v, w \leq 2$, with extra bounds having weight 1
- Linearization does not have 0-cost solution:
  optimal solutions have weight 1, e.g. falsifying $1 \leq w$
- Should decrease lower bound of w

Termination

Current set of targeted programs:

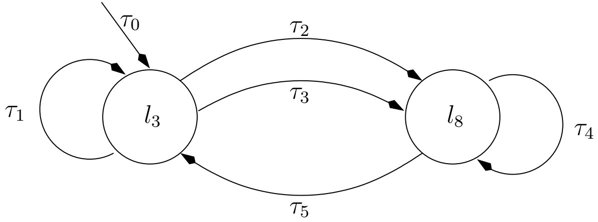
- Imperative programs: iterative and recursive (ignoring return values)
- Integer variables and linear expressions
  (other constructions considered unknowns)

Example

```
int gcd ( int a, int b ) {
   int tmp;
   while ( a >= 0 && b > 0 ) {
      tmp = b;
      if (a == b)   b = 0;
      else {
         int z = a;
         while ( z > b )   z -= b;
         b = z;
      }
      a = tmp;
   }
   return a;
}
```

Example

As a transition system:



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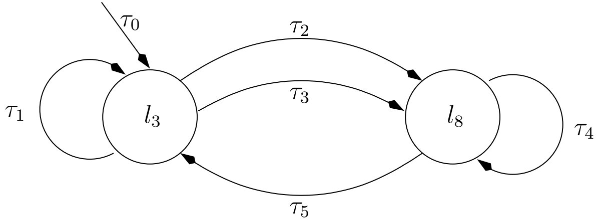

$$
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\tau_0: & & & & a' = ?, & b' = ?, & tmp' = ?, & z' = ? \\
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Proving Termination

- Idea: prove that no transition can be executed infinitely many times.
- In order to discard a transition τ_i we need either:
  - an unfeasibility argument, or
  - a ranking function f over $\mathbb{Z}$ such that
    1. $\tau_i \Longrightarrow f(x_1, \ldots, x_n) \geq 0$ **(bounded)**
    2. $\tau_i \Longrightarrow f(x_1, \ldots, x_n) > f(x'_1, \ldots, x'_n)$ **(strict-decreasing)**
    3. $\tau_j \Longrightarrow f(x_1, \ldots, x_n) \geq f(x'_1, \ldots, x'_n)$ for all j **(non-increasing)**

Auxiliary Assertions: Invariants

- We may need invariant assertions to build our termination argument
- We consider inductive invariants:
  - Initiation condition
    (it holds the first time the location is reached)
  - Consecution condition
    (it is preserved under every cycle back to the location)

Constraint-based Program Analysis

Introduced in [Colon,Sankaranarayanan & Sipma, CAV'03]

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- Fix a template for candidate invariants

$$c_1 x_1 + \ldots + c_n x_n + d \leq 0$$

where $c_1, \ldots, c_n, d$ are unknowns

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- Transform with Farkas' Lemma into $\exists$ problem over non-linear arith.

- Constraints can be solved with SMT(NA) solver, e.g. Barcelogic.

Constraint-based Program Analysis

Following the ideas in [Bradley, Manna & Sipma, CAV'05]:
constraint-based invariant gen. (IG) + linear ranking function gen. (RG)

Assume a single location:

- Templates
  - For the invariant: $I = c_1x_1 + \ldots + c_nx_n + d \leq 0$
  - For the ranking function: $R = r_0 + r_1x_1 + \ldots + r_nx_nx$
- Constraints
  - Initiation condition on I
  - Consecution condition on I
  - R is non-increasing for all transitions
  - Some transition τ_i can be discarded
    - $I \implies$ unfeasibility of τ_i, or
    - $I \implies$ strict decreasingness and boundedness of τ_i

Constraint-based Program Analysis

Although this looks like the way to work, it is not that good in practice:

- Sometimes several invariants needed to generate ranking function

  Then the problem is unsatisfiable (no solution for ranking function)

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We can do it with Max-SMT

Using Max-SMT to combine IG and RG

We can assign weights to the termination conditions:

1. $I \ \wedge \ \tau_i \implies R \geq 0$
2. $I \ \wedge \ \tau_i \implies R > R'$
3. $I \ \wedge \ \tau_j \implies R \geq R'$ for all j

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1. (p_1, w_1) where p_1 represents the bound condition (1)
2. (p_2, w_2) where p_2 represents the strict-decreasing condition (2)
3. (p_3, w_3) where p_3 represents the non-increasing condition (3)

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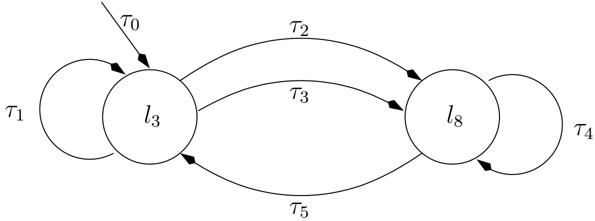
Once the problem is encoded in Max-SMT(NA):

- The Max-SMT solver looks for the best solution getting a ranking function if possible
- Otherwise, the weights can guide the search to get invariants and quasi-ranking functions that satisfy as many conditions as possible

Example



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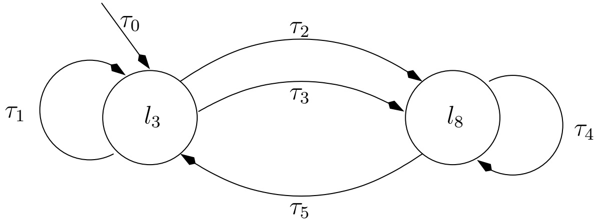

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Solver finds invariant $b \geq 1$ at l_8 and ranking function b for τ_1

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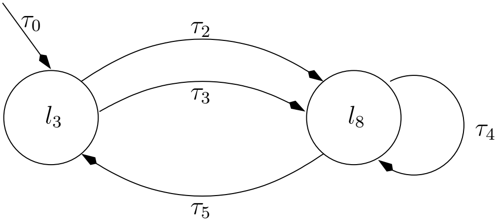

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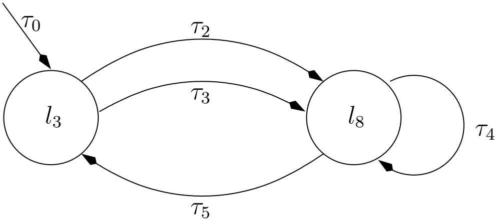

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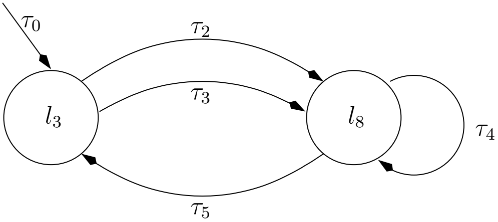
$$

Nothing else can be done, but ...

Example



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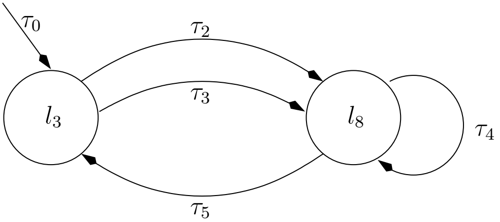

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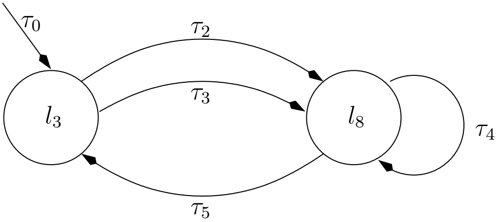
We can split τ_5 in three subcases and

Example



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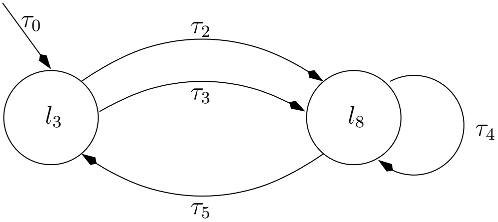
We can split τ_5 in three subcases and remove 5.2 by strict decreasingness

Example



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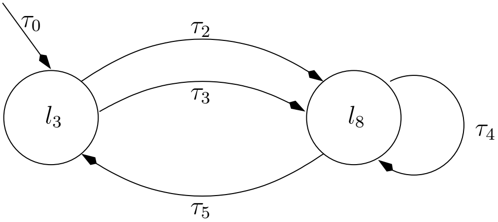
We can split τ_5 in three subcases and remove 5.1 by unfeasibility

Example



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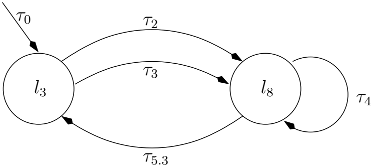
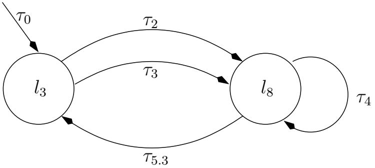
$\tau_{5.3}$: $b \geq z,\quad b \geq 0,\quad b = b',\quad a' = tmp,\quad b' = z,\quad tmp' = tmp,\quad z' = z$

Example



$$
\begin{array}{llllllll}
\tau_0: & & & & a' = ?, & b' = ?, & tmp' = ?, & z' = ? \\
\tau_2: & b \geq 1, & a \geq 0, & a < b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_3: & b \geq 1, & a \geq 0, & a > b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_4: & b < z, & & & a' = a, & b' = b, & tmp' = tmp, & z' = z - b \\
\tau_{5.3}: & b \geq z, & b \geq 0, & b = b', & a' = tmp, & b' = z, & tmp' = tmp, & z' = z
\end{array}
$$

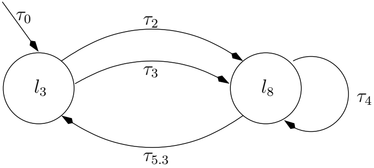
Example



$$
\begin{array}{llllllll}
\tau_0: & & & & a' = ?, & b' = ?, & tmp' = ?, & z' = ? \\
\tau_2: & b \geq 1, & a \geq 0, & a < b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_3: & b \geq 1, & a \geq 0, & a > b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_4: & b < z, & & & a' = a, & b' = b, & tmp' = tmp, & z' = z - b \\
\tau_{5.3}: & b \geq z, & b \geq 0, & b = b', & a' = tmp, & b' = z, & tmp' = tmp, & z' = z
\end{array}
$$

Now, we cannot find a ranking function but get the invariant $a \geq z$ at l_8.

Example



$$
\begin{array}{lllllllll}
\tau_0 : & & & & a' = ?, & b' = ?, & tmp' = ?, & z' = ? \\
\tau_2 : & b \geq 1, & a \geq 0, & a < b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_3 : & b \geq 1, & a \geq 0, & a > b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
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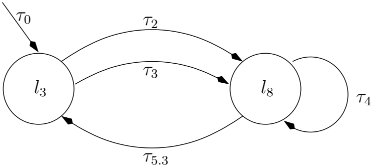
\tau_{5.3} : & b \geq z, & b \geq 0, & b = b', & a' = tmp, & b' = z, & tmp' = tmp, & z' = z
\end{array}
$$

Now, we cannot find a ranking function but get the invariant $a \geq z$ at l_8.
Next, again, we only generate the invariant $tmp = b$ at l_8.

Example

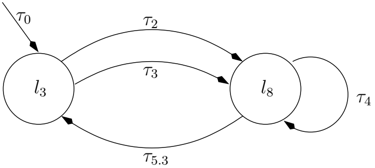


τ_0 :				$a' = ?$,	$b' = ?$,	$tmp' = ?$,	$z' = ?$
τ_2 :	$b \geq 1$,	$a \geq 0$,	$a < b$,	$a' = a$,	$b' = b$,	$tmp' = b$,	$z' = a$
τ_3 :	$b \geq 1$,	$a \geq 0$,	$a > b$,	$a' = a$,	$b' = b$,	$tmp' = b$,	$z' = a$
τ_4 :	$b < z$,			$a' = a$,	$b' = b$,	$tmp' = tmp$,	$z' = z - b$
$\tau_{5.3}$:	$b \geq z$,	$b \geq 0$,	$b = b'$,	$a' = tmp$,	$b' = z$,	$tmp' = tmp$,	$z' = z$

With the invariant $a \geq 0$ at l_8 we have that function $a + b$ fulfills for $\tau_{5.3}$:

p_1 (bounded) and p_3 (non-increasing) but not p_2 (strict-decreasing)

Example



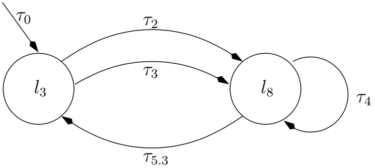
$$
\begin{array}{llllllll}
\tau_0 : & & & & a' = ?, & b' = ?, & tmp' = ?, & z' = ? \\
\tau_2 : & b \geq 1, & a \geq 0, & a < b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_3 : & b \geq 1, & a \geq 0, & a > b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_4 : & b < z, & & & a' = a, & b' = b, & tmp' = tmp, & z' = z - b \\
\tau_{5.3} : & b \geq z, & b \geq 0, & b = b', & a' = tmp, & b' = z, & tmp' = tmp, & z' = z
\end{array}
$$

With the invariant $a \geq 0$ at l_8 we have that function $a + b$ fulfills for $\tau_{5.3}$:

p_1 (bounded) and p_3 (non-increasing) but not p_2 (strict-decreasing)

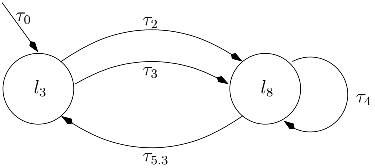
The Max-SMT solver generates $a + b$

Example



$$
\begin{array}{lllllllll}
\tau_0 : & & & & a' = ?, & b' = ?, & tmp' = ?, & z' = ? \\
\tau_2 : & b \geq 1, & a \geq 0, & a < b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_3 : & b \geq 1, & a \geq 0, & a > b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_4 : & b < z, & & & a' = a, & b' = b, & tmp' = tmp, & z' = z - b \\
\tau_{5.3} : & b \geq z, & b \geq 0, & b = b', & a' = tmp, & b' = z, & tmp' = tmp, & z' = z
\end{array}
$$

Example

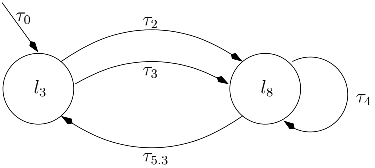


$$
\begin{array}{llllllll}
\tau_0: & & & & a' = ?, & b' = ?, & tmp' = ?, & z' = ? \\
\tau_2: & b \geq 1, & a \geq 0, & a < b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_3: & b \geq 1, & a \geq 0, & a > b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_4: & b < z, & & & a' = a, & b' = b, & tmp' = tmp, & z' = z - b \\
\tau_{5.3}: & b \geq z, & b \geq 0, & b = b', & a' = tmp, & b' = z, & tmp' = tmp, & z' = z
\end{array}
$$

With ranking function $a + b$ we can split $\tau_{5.3}$ into

$$\tau_{5.4} : \tau_{5.3} \wedge a + b > a' + b' \qquad \tau_{5.5} : \tau_{5.3} \wedge a + b = a' + b'$$

Example



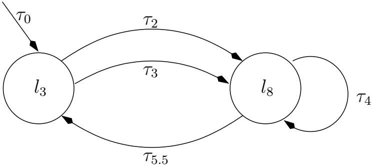
$$
\begin{array}{llllllll}
\tau_0: & & & & a' = ?, & b' = ?, & tmp' = ?, & z' = ? \\
\tau_2: & b \geq 1, & a \geq 0, & a < b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_3: & b \geq 1, & a \geq 0, & a > b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_4: & b < z, & & & a' = a, & b' = b, & tmp' = tmp, & z' = z - b \\
\tau_{5.3}: & b \geq z, & b \geq 0, & b = b', & a' = tmp, & b' = z, & tmp' = tmp, & z' = z
\end{array}
$$

With ranking function $a + b$ we can split $\tau_{5.3}$ into

$$\tau_{5.4} : \tau_{5.3} \ \wedge \ a + b > a' + b' \qquad\qquad \tau_{5.5} : \tau_{5.3} \ \wedge \ a + b = a' + b'$$

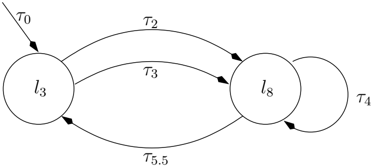
Then $\tau_{5.4}$ can be removed and $\tau_{5.5}$ simplified: $\tau_{5.5} : \tau_{5.3} \ \wedge \ a = a'$

Example



$$
\begin{array}{llllllll}
\tau_0: & & & & a' = ?, & b' = ?, & tmp' = ?, & z' = ? \\
\tau_2: & b \geq 1, & a \geq 0, & a < b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_3: & b \geq 1, & a \geq 0, & a > b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_4: & b < z, & & & a' = a, & b' = b, & tmp' = tmp, & z' = z - b \\
\tau_{5.3}: & b \geq z, & b \geq 0, & b = b', & a' = tmp, & b' = z, & tmp' = tmp, & z' = z
\end{array}
$$

Example



$$
\begin{array}{llllllll}
\tau_0: & & & & a' = ?, & b' = ?, & tmp' = ?, & z' = ? \\
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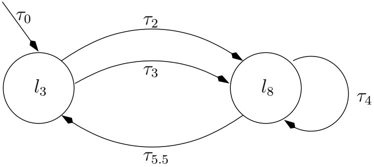
\tau_3: & b \geq 1, & a \geq 0, & a > b, & a' = a, & b' = b, & tmp' = b, & z' = a \\
\tau_4: & b < z, & & & a' = a, & b' = b, & tmp' = tmp, & z' = z - b \\
\tau_{5.5}: & b \geq z, & b \geq 0, & b = b', & a' = tmp, & b' = z, & tmp' = tmp, & z' = z \\
 & a' = a & & & & & &
\end{array}
$$

Using the information of the transitions we can infer that $a = b$ after $\tau_{5.5}$.

Example



$$
\begin{array}{lllllllll}
\tau_0: & & & & a' = ?, & b' = ?, & tmp' = ?, & z' = ? \\
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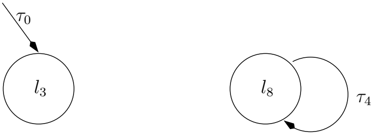
\tau_4: & b < z, & & & a' = a, & b' = b, & tmp' = tmp, & z' = z - b \\
\tau_{5.5}: & b \geq z, & b \geq 0, & b = b', & a' = tmp, & b' = z, & tmp' = tmp, & z' = z \\
 & a' = a & & & & & &
\end{array}
$$

Using the information of the transitions we can infer that $a = b$ after $\tau_{5.5}$. Then the connections between $\tau_{5.5}$ and τ_2 or τ_3 are unfeasible.

Example

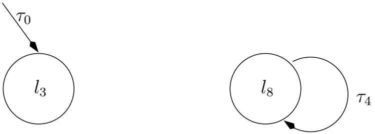


τ_0 : $a' = ?$, $b' = ?$, $tmp' = ?$, $z' = ?$

τ_4 : $b < z$, $a' = a$, $b' = b$, $tmp' = tmp$, $z' = z - b$

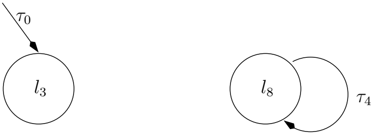
Using the information of the transitions we can infer that $a = b$ after $\tau_{5.5}$. Then the connections between $\tau_{5.5}$ and τ_2 or τ_3 are unfeasible.

Example



$$
\begin{array}{llllll}
\tau_0 : & & a' = ?, & b' = ?, & tmp' = ?, & z' = ? \\
\tau_4 : & b < z, & a' = a, & b' = b, & tmp' = tmp, & z' = z - b
\end{array}
$$

Example



$$
\begin{array}{llllll}
\tau_0 : & & a' = ?, & b' = ?, & tmp' = ?, & z' = ? \\
\tau_4 : & b < z, & a' = a, & b' = b, & tmp' = tmp, & z' = z - b
\end{array}
$$

Solver generates ranking function $z - b$ for τ_4

Example

$$\tau_0 : \qquad a' = ?, \quad b' = ?, \quad tmp' = ?, \quad z' = ?$$

We are DONE!

Using Max-SMT to improve termination analysis

Advantages of the method:

- Using Max-SMT we can characterize different ways of progress depending on whether p_1, p_2 or p_3 are fulfilled.
- Using different weights we can encode which conditions are more important than others.

Implementation and experiments

- We have implemented these techniques
- The prototype reads C code
- Possible answers:
  - YES
  - NO (few cases)
  - Unknown

Implementation and experiments

- Experiments:
  - Benchmarks used in the Termination Competition for Java programs. 111 instances of iterative programs and 41 instances of recursive programs where termination follows from scalar information.
- Results are very promising:
  - Our first implementation is already competitive compared with tools for Java programs that have been developed since many years ago.

  Results from the TermComp full-run December 2011:

	Iterative			Recursive		
	YES	NO	MAYBE	YES	NO	MAYBE
AProVE	77	0	36	32	0	9
Costa	64	0	49	28	0	13
Julia	72	21	20	35	0	6
Max-SMT	76	22	18	32	0	9

Implementation and experiments

- Experiments:
  - Programs made by students (can be ugly code). Obtained from an on-line learning environment (Jutge.org). 7924 instances coming from 12 different programming problems.
- Results are very promising:
  - These programs can be considered challenging.
    Most often they are not the most elegant solution but a working one with many more conditional statements than necessary.

	YES	NO	MAYBE
Max-SMT	6139	15	1770

Implementation and experiments

- Experiments:
  - Benchmarks taken from [Cook et al., CAV'13] coming from Windows device drivers, the Apache web server, the PostgreSQL server, integer approximations of numerical programs from a book on numerical recipes, integer approximations of benchmarks from LLBMC, ...
    260 instances known to be terminating.

- Results are very promising:

	YES
Cooperating-T2	245
Terminator	177
T2	189
ARMC	138
AproVE	197
AproVE+Interproc	185
KITTeL	196
Max-SMT	197

Conclusions

- Approach to SMT(NA) that directly extends to Max-SMT(NA)
- Approach to termination analysis relying on Max-SMT
- Our prototype is already a competitive tool

Future work

There is a very long list...

- Improve invariant generation techniques.
  (e.g., by combining with abstract interpretation)

- Improve termination of recursive functions.

- Termination in presence of other data types (arrays, etc.)

- Improve the NA solver combining Barcelogic solver with other methods that are much better proving unsatisfiability
  (like [Jovanovic and De Moura, IJCAR'12])

Thank you!