Automatic Generation of Polynomial Invariants for System Verification

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Plan of the Talk

- Introduction
  - Need for program verification
  - Invariants and abstract interpretation
  - Polynomial invariants

Plan of the Talk

- Introduction
- Generation of Invariant Polynomial Equalities
(with D. Kapur: ISSAC'04, SAS'04)
    - Related work
    - Abstract domain of ideals
    - Particular case: loops without nesting

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- Introduction
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
  - Imperative programs
    (with D. Kapur: ICTAC'04)
  - Petri nets
    (with R. Clarisó, J. Cortadella: ATPN'05)
  - Hybrid systems
    (with A. Tiwari: HSCC'05)

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- Introduction
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
- Generation of Invariant Polynomial Inequalities (with R. Bagnara, E. Zaffanella: SAS'05)
  - Abstract domain of polynomial cones

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  - safety
  - security
  - ...

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Failure of the Ariane 5 launcher in 1996

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Failure of the Ariane 5 launcher in 1996

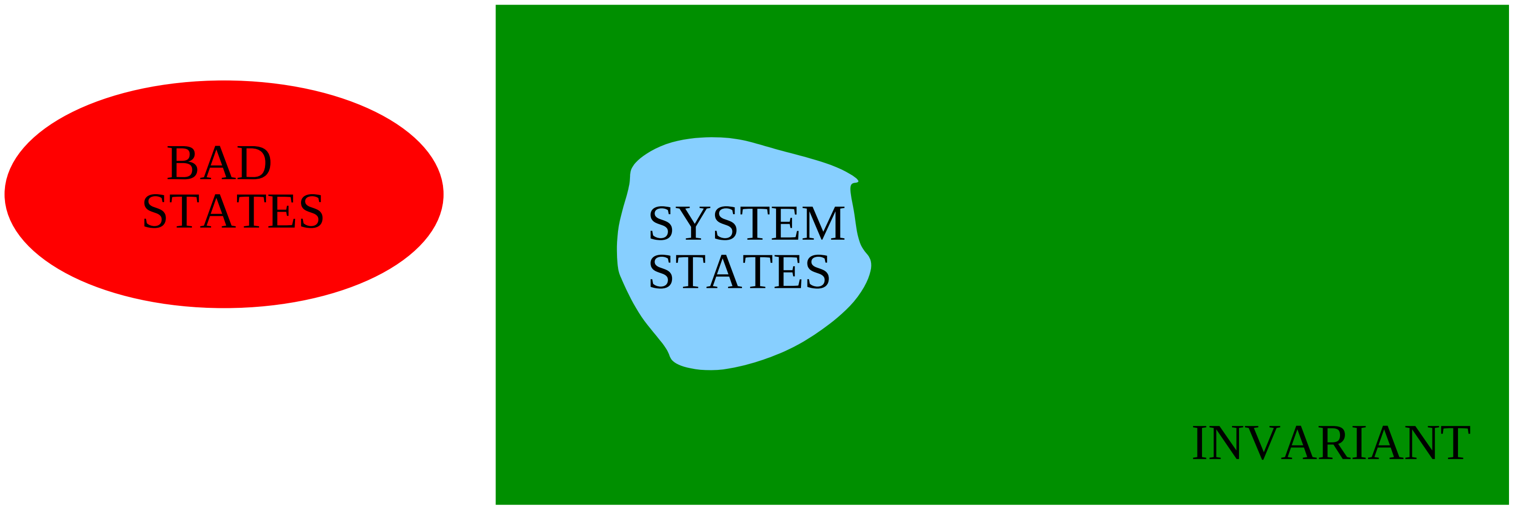
- Fundamental finding errors asap.
- Invariants are crucial for program verification!

Invariants in Verification

CORRECTNESS OF THE SYSTEM:

$$\text{SYSTEM STATES} \;\cap\; \text{BAD STATES} \;=\; \emptyset$$

Invariants in Verification



CORRECTNESS OF THE SYSTEM:

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SUFFICIENT CONDITION:

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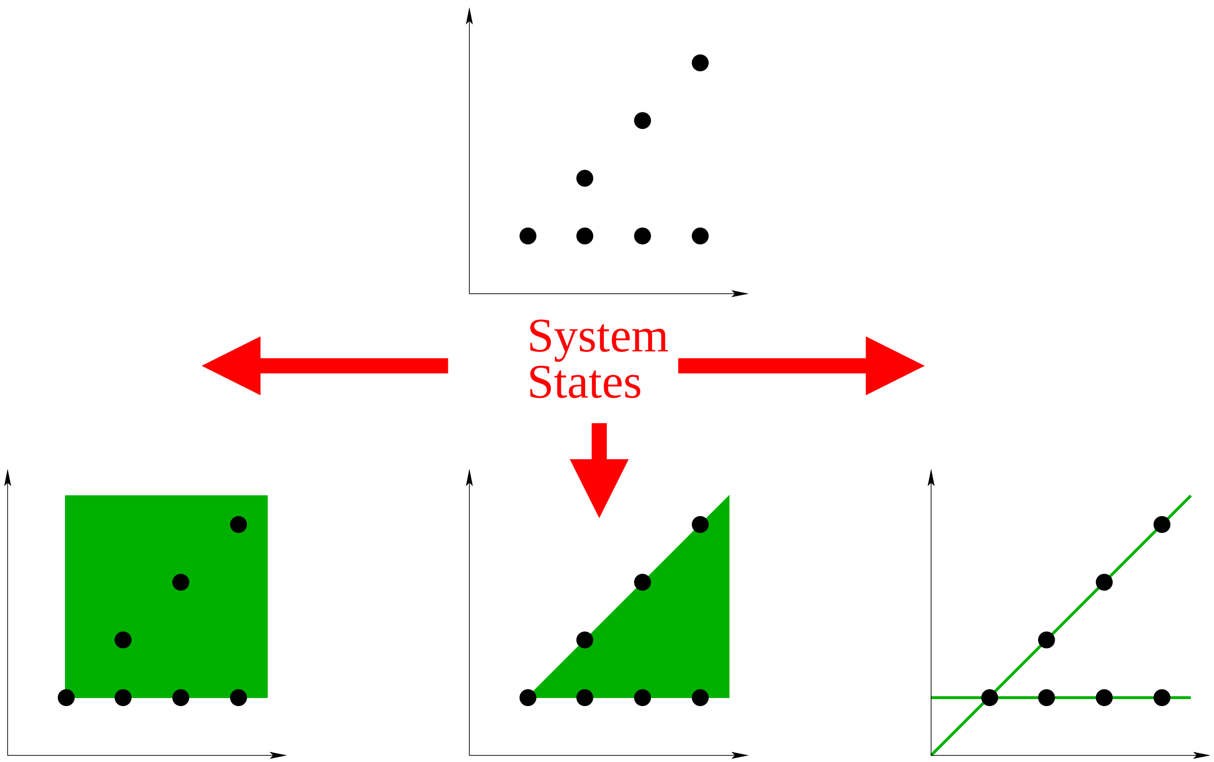
$$x + 2y - 3z \leq 3$$

- polynomial equalities and inequalities

$$x = y^2 \qquad (a+1)^2 > b^2 \geq a^2$$

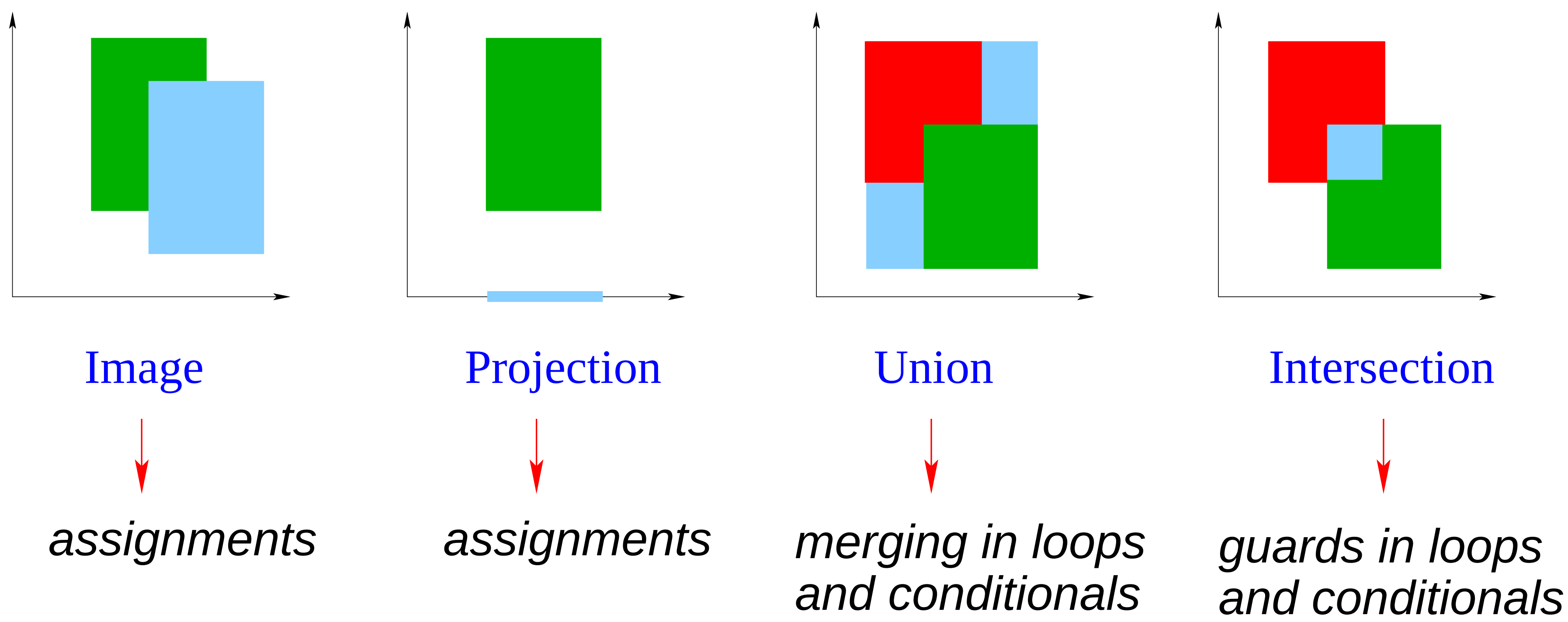
Abstract Interpretation: Overapproximation

Sets of variable values overapproximated by abstract values



Abstract Interpretation: Operations

- Invariants computed by symbolic execution of the system with abstract values
- This requires abstracting concrete operations on states:



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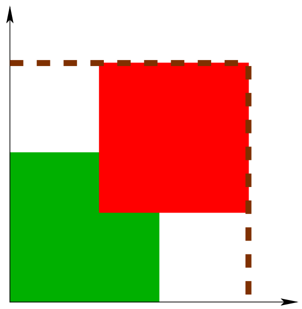

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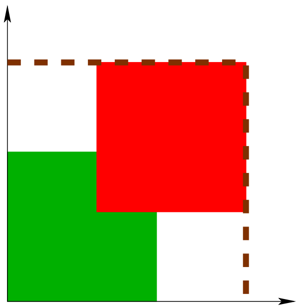

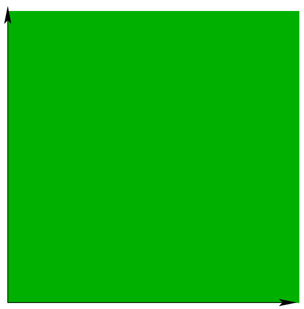

Why Care about Polynomial Invariants?

- Linear invariants used to verify many classes of systems:
  - Imperative programs
  - Logic programs
  - Hybrid systems
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- Linear invariants used to verify many classes of systems:
  - Imperative programs
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- But some applications require polynomial invariants:

The abstract interpreter ASTRÉE employs
polynomial invariants to verify absence of run-time errors
in flight control software

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Related Work (1)

- Iterative fixpoint approaches
  - Forward propagation
    - Rodríguez-Carbonell & Kapur 2004
    - Colón 2004
  - Backward propagation
    - Müller-Olm & Seidl 2004
- Constraint-based approaches
  - Sankaranarayanan & Sipma & Manna 2004

Related Work (2)

Work	Restrictions	Conds $=$	Conds $\neq$	Complete
MOS, POPL'04	bounded deg	no	no	yes
SSM, POPL'04	fi xed form	yes	no	no
MOS, IPL'04	fi xed form	no	yes	yes
COL, SAS'04	bounded deg	yes	no	no
RCK, SAS'04	bounded deg	yes	yes	yes*
RCK, ISSAC'04	no restriction	no	no	yes

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- Implementation successfully applied to many programs
- Ideals of polynomials represented by special finite bases of generators: Gröbner bases
- Many tools available manipulating ideals, Gröbner bases, e.g. Macaulay 2, Maple

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- An ideal is a set of polynomials I such that
  - $0 \in I$
  - If $p, q \in I$, then $p + q \in I$
  - If $p \in I$ and q any polynomial, $pq \in I$

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  - If p evaluates to 0 on S, then pq evaluates to 0 on S

$$\forall \omega \in S, \quad p(\omega) = 0 \Longrightarrow p(\omega) \cdot q(\omega) = 0$$

Ideals of Polynomials (3)

- E.g. multiples of a polynomial p, $\langle p \rangle$
  - $0 = 0 \cdot p \in \langle p \rangle$
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- Hilbert's basis theorem: all ideals are finitely generated $\longrightarrow$ there is finite representation for ideals

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  - quotient: $I : J = \{p \mid \forall q \in J, p \cdot q \in I\}$
  - intersection: $I \cap J$
- All operations implemented using Gröbner bases
- These are used in abstraction of concrete semantics

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- Parametric widening $I \nabla_d J$
- Based on taking polynomials of $I \cap J$ of degree $\leq d$
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Example

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$\quad a := a + 2b + 1; b := b + 1;$

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$F_0(I) = \langle 0 \rangle$

$F_1(I) = (\langle a \rangle + \langle I_0(a \leftarrow a') \rangle) \cap \mathbb{C}[a, b, c]$

$F_2(I) = (\langle b \rangle + \langle I_1(b \leftarrow b') \rangle) \cap \mathbb{C}[a, b, c]$

$F_3(I) = I_3 \nabla_2 (I_2 \cap I_6)$

$F_4(I) = \langle I_3 \rangle : \langle b - c \rangle$

$F_5(I) = I_4(a \leftarrow a - 2b - 1)$

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$F_7(I) = \mathrm{I}(\mathrm{V}(I_3 + \langle b - c \rangle))$

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In 6 steps found loop invariant:

$a = b^2$

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$$
\begin{array}{l}
a := 0 \; ; \\
b := 0 \; ; \\
\textbf{while} \;\; b \neq c \;\; \textbf{do} \\
\qquad a := a + 2b + 1 \; ; \\
\qquad b := b + 1; \\
\textbf{end while}
\end{array}
$$

Overview of the Method

- $(a_n, b_n, c_n) \equiv$ program state after n loop iterations

$$\left\{\begin{array}{lcl} a_{n+1} & = & a_n + 2b_n + 1 \\ b_{n+1} & = & b_n + 1 \end{array}\right. , \quad \left\{\begin{array}{lcl} a_0 & = & 0 \\ b_0 & = & 0 \end{array}\right.$$

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- Quantifi er elimination: $b = n \Longrightarrow a = b^2$ is loop invariant
- Gröbner bases can be used to eliminate loop counters

Our Handling of Conditional Statements (1)

$x := R;$

$y := 0;$

$r := R^2 - N;$

while $?$ **do**

  if $?$ **then**

    $r := r + 2x + 1;$

    $x := x + 1;$

  else

    $r := r - 2y - 1;$

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2. Compute common invariants for both loops

- Finding common invariants $\equiv$
  Finding intersection of invariant ideals

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- Finding common invariants $\equiv$
  Finding intersection of invariant ideals
- But this is not sound!

Our Handling of Conditional Statements (3)

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Program	**Algorithm**
$\bar{x} := \bar{\alpha};$	$I' := \langle 1 \rangle; I := \langle x_1 - \alpha_1, \cdots, x_m - \alpha_m \rangle;$
while $?$ **do**	**while** $I' \neq I$ **do**
$\bar{x} := f(\bar{x});$	$I' := I;$
or	$I := \bigcap_{n=0}^{\infty} [\, I(\bar{x} \leftarrow f^{-n}(\bar{x}))$
$\bar{x} := g(\bar{x});$	$\bigcap I(\bar{x} \leftarrow g^{-n}(\bar{x}))\,];$
end while	**end while**

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- Correct and complete:
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- Implemented in Maple:

  1. Solving recurrences

  2. Eliminating variables
  3. Intersecting ideals

  (2–3: Gröbner bases)

Example

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Invariant polynomial equality:

$$x^2 - y^2 = r + N$$

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Imperative Programs

Pre: $\{\ N \geq 1\}$

$x := R;\ y := 0;\ r := R^2 - N;$

Inv: $\{\ N \geq 1\ \wedge\ x^2 - y^2 = r + N\ \}$

while $r \neq 0$ **do**

if $r < 0$ **then**

$r := r + 2x + 1;$

$x := x + 1;$

else

$r := r - 2y - 1;$

$y := y + 1;$

end if

end while

Post: $\{x \neq y \wedge N \bmod (x - y) = 0\}$

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- $N \geq 1 \implies$
  $R^2 - 0^2 = (R^2 - N) + N$
- $x^2 - y^2 = r + N \; \wedge \; r < 0 \implies$
  $(x+1)^2 - y^2 = (r + 2x + 1) + N$

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- $N \geq 1 \implies$
  $R^2 - 0^2 = (R^2 - N) + N$
- $x^2 - y^2 = r + N \ \wedge\ r < 0 \implies$
  $(x+1)^2 - y^2 = (r+2x+1) + N$
- $x^2 - y^2 = r + N \ \wedge\ r > 0 \implies$
  $x^2 - (y+1)^2 = (r-2y-1) + N$

Imperative Programs

Pre: $\{ N \geq 1\}$

$x := R;\ y := 0;\ r := R^2 - N;$

Inv: $\{ N \geq 1 \ \wedge\ x^2 - y^2 = r + N \ \}$

while $r \neq 0$ **do**

  if $r < 0$ **then**

    $r := r + 2x + 1;$

    $x := x + 1;$

  else

    $r := r - 2y - 1;$

    $y := y + 1;$

  end if

end while

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Petri Nets: Introduction

- Petri nets: mathematical model for studying systems
  - concurrency
  - parallelism
  - non-determinism

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- Petri nets: mathematical model for studying systems
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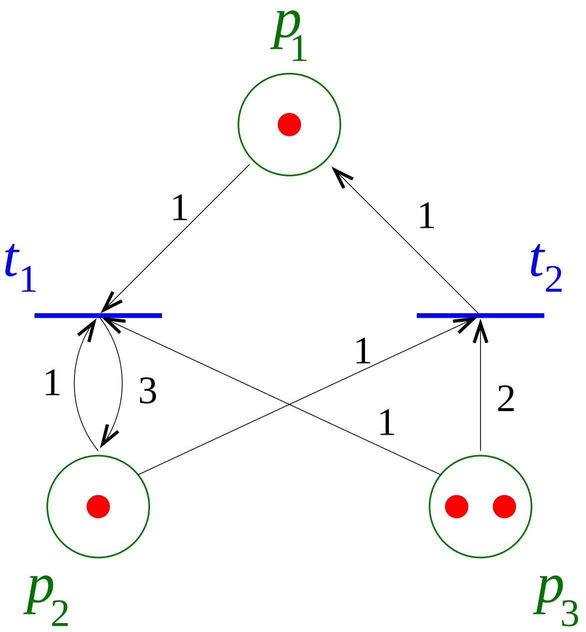
- Applications:
  - Manufacturing and Task Planning
  - Communication Networks
  - Hardware Design

Definitions

- A Petri net is a bipartite directed graph where:
  - Nodes partitioned into places (◯) and transitions (|)
  - Arcs are labelled with weights
- A marking maps a number of tokens to each place



$p_1 \longrightarrow 1$

$p_2 \longrightarrow 1$

$p_3 \longrightarrow 2$

Dynamics (1)

- Dynamics of a Petri net described by
  - initial marking
  - firing of transitions

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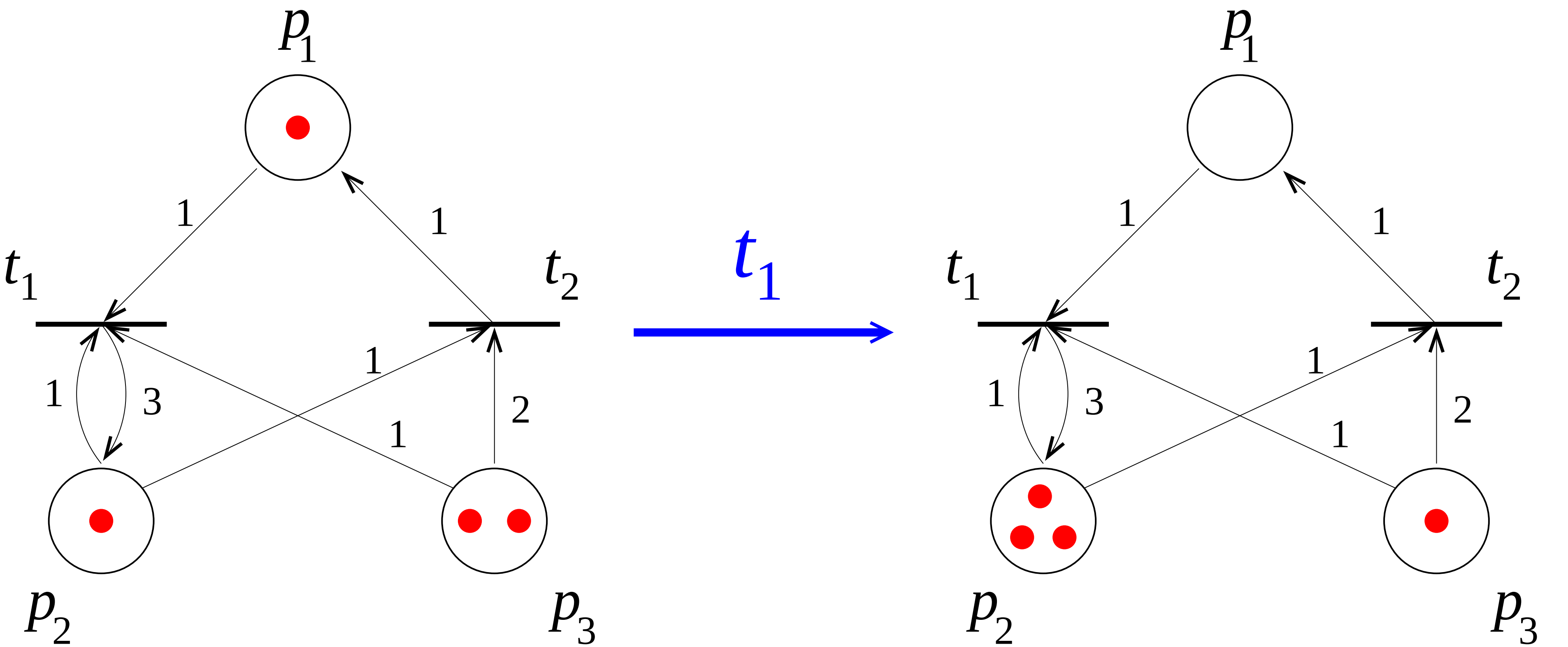
- A transition is enabled if there are $\geq$ tokens in each input place than indicated in the arcs
- When a transition is enabled, it can fire:
  the number of tokens indicated in the arcs is

  1. removed from input places
  2. added to output places

Dynamics (2)



Dynamics (3)

- Enabling of transitions may also depend on inhibitor arcs
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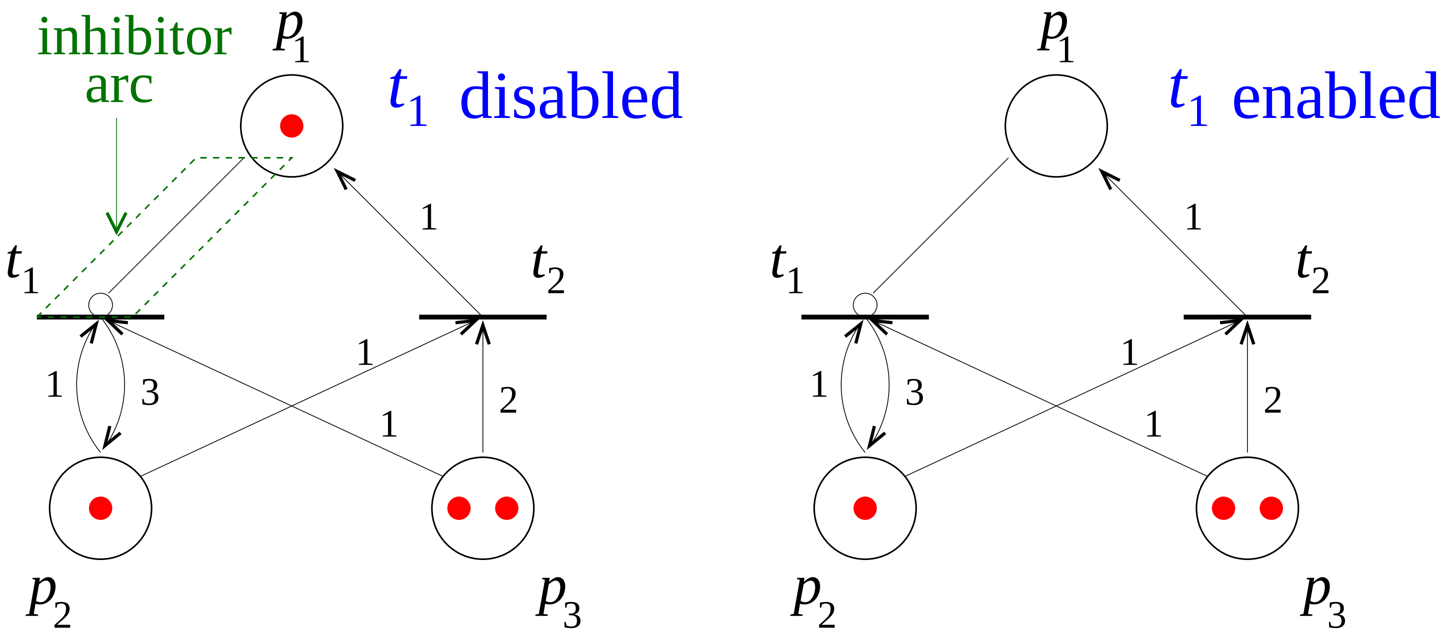

Dynamics (4)

- Deadlocks are markings for which all transitions are disabled

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- Deadlocks are markings for which all transitions are disabled
- Given a Petri net with an initial marking:
  - Invariant properties of reachable states ?
  - Any deadlocks ?

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- Enabling of a transition with inhibitor place p_i: $x_i = 0$
- Firing of a transition
  - with input place p_i and label c_i: $x_i := x_i - c_i;$
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Generating Polynomial Invariants (1)

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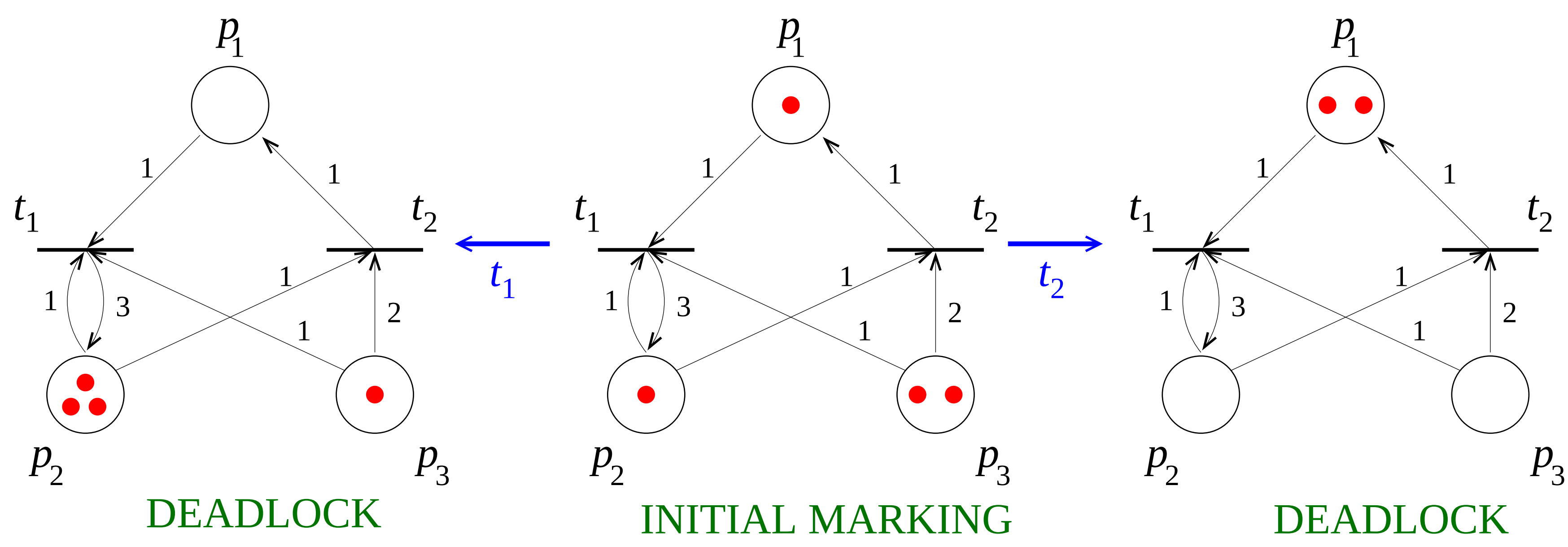
- Example:



Generating Polynomial Invariants (2)

- Polynomial invariants obtained:

$$Inv = \begin{cases} 5x_1 + 3x_2 + x_3 - 10 & = 0 \\ 5x_3^2 + 2x_2 - 11x_3 & = 0 \\ x_2x_3 + 2x_3^2 - 5x_3 & = 0 \\ 5x_2^2 - 17x_2 + 6x_3 & = 0 \end{cases}$$

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$$Inv \Leftrightarrow (x_1, x_2, x_3) \in \{(0,3,1),(1,1,2),(2,0,0)\}$$

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- In general overapproximation of reach set is obtained

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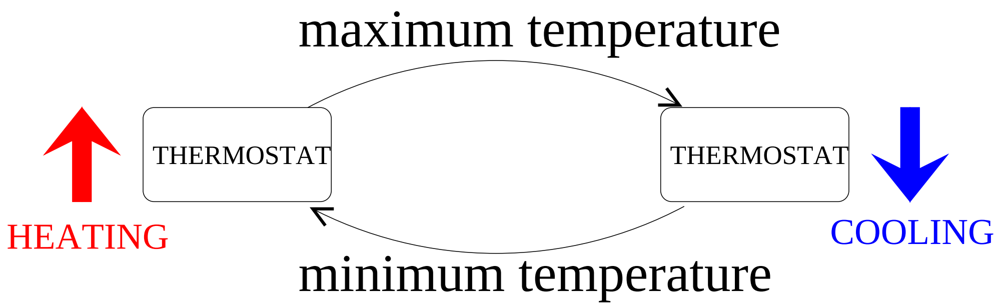
- Examples:
  - A thermostat that heats/cools depending on the temperature in the room



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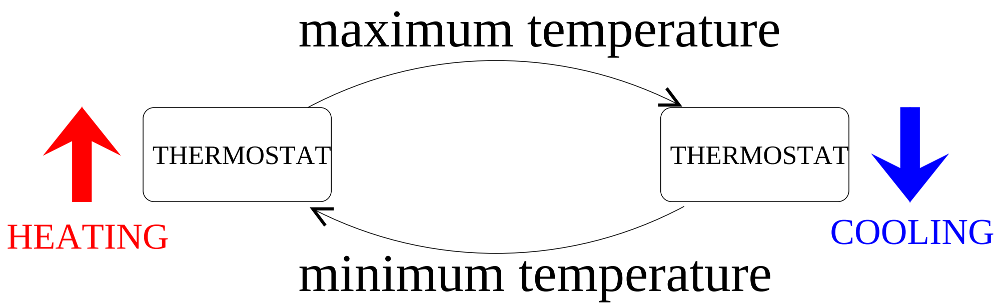

  - A robot controller that changes the direction of movement if the robot is too close to a wall.

Definition

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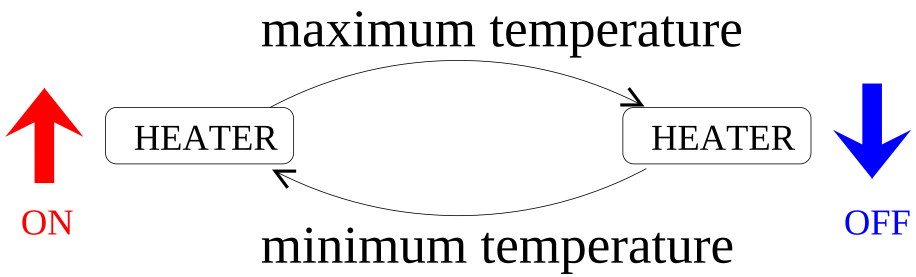

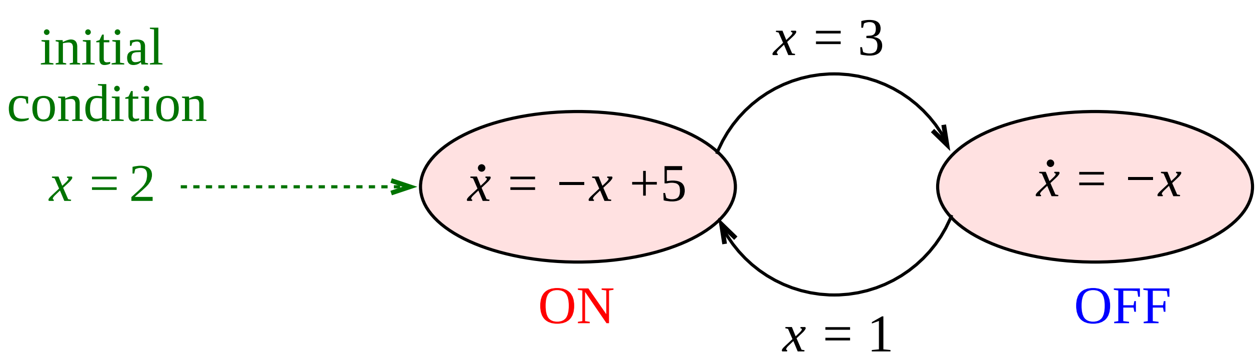

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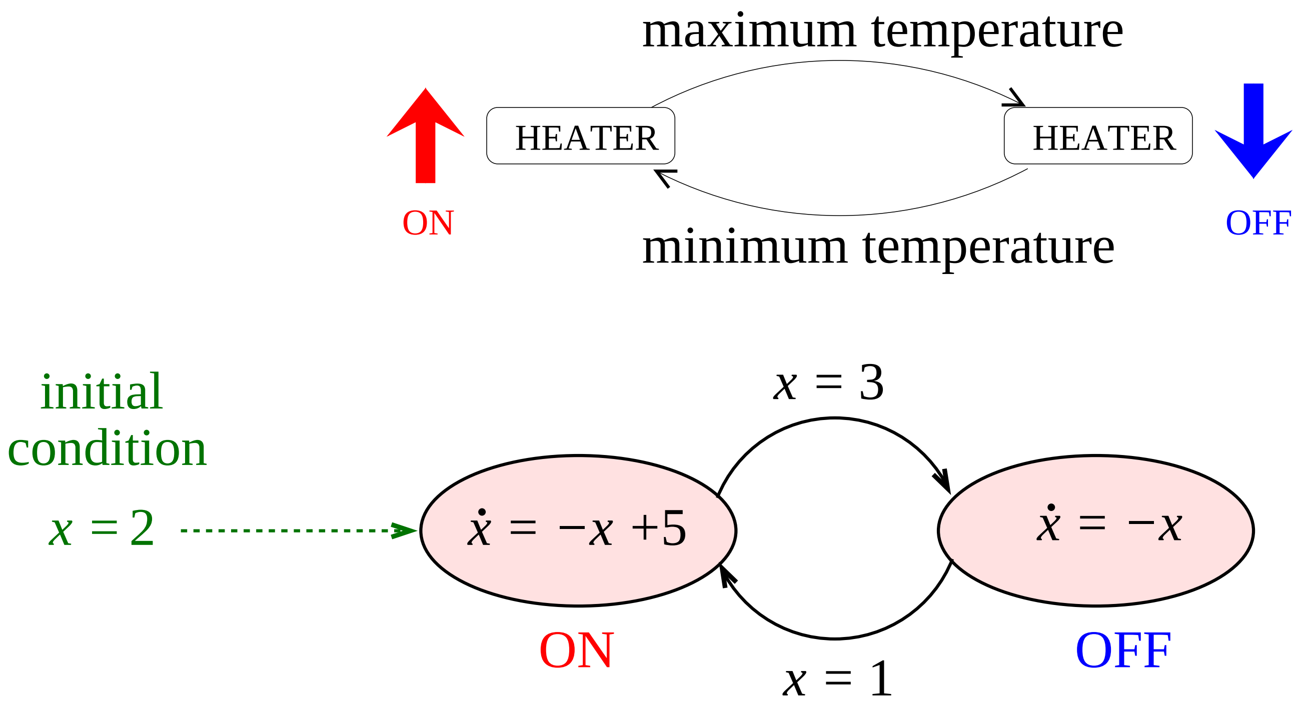

- We restrict to linear differential equations at locations

Dynamics (1)

- A computation is a sequence of states (discrete location, valuation of variables)

$$(l_0, x_0), (l_1, x_1), (l_2, x_2), \ldots$$

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     - Continuous evolution: there is a trajectory going from x_i to x_{i+1} along the flow determined by the differential equation $\dot{x} = Ax + B$ at location $l_i = l_{i+1}$

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  - We know how to deal with discrete systems
  - How to handle continuous evolution?
- Problem:
  computing polynomial invariants of linear systems of differential equations

Form of the Solution

Solution to $\dot{x} = Ax + B$ can be expressed as polynomials in t, $e^{\pm at}$, $\cos(bt)$, $\sin(bt)$, where $\lambda = a + bi$ are eigenvalues of matrix A.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{v_x} \\ \dot{v_y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix}$$

$$\begin{cases} x = x^* + 2\sin(t/2)\, v_x^* + (2\cos(t/2) - 2)\, v_y^* \\ y = y^* + (-2\cos(t/2) + 2)\, v_x^* + 2\sin(t/2)\, v_y^* \\ v_x = \cos(t/2)\, v_x^* - \sin(t/2)\, v_y^* \\ v_y = \sin(t/2)\, v_x^* + \cos(t/2)\, v_y^* \end{cases}$$

Elimination of Time

Idea: eliminate terms depending on t from solution:

- transform solution into polynomials using new variables
- eliminate by means of Gröbner bases using auxiliary equations

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SOLUTION

$$\begin{cases} x &= x^* + 2zv_x^* + (2w-2)v_y^* \\ y &= y^* + (-2w+2)\,v_x^* + 2zv_y^* \\ v_x &= wv_x^* - zv_y^* \\ v_y &= zv_x^* + wv_y^* \end{cases}$$

INITIAL CONDITIONS

$$\begin{cases} v_x^* &= 2 \\ v_y^* &= -2 \end{cases}$$

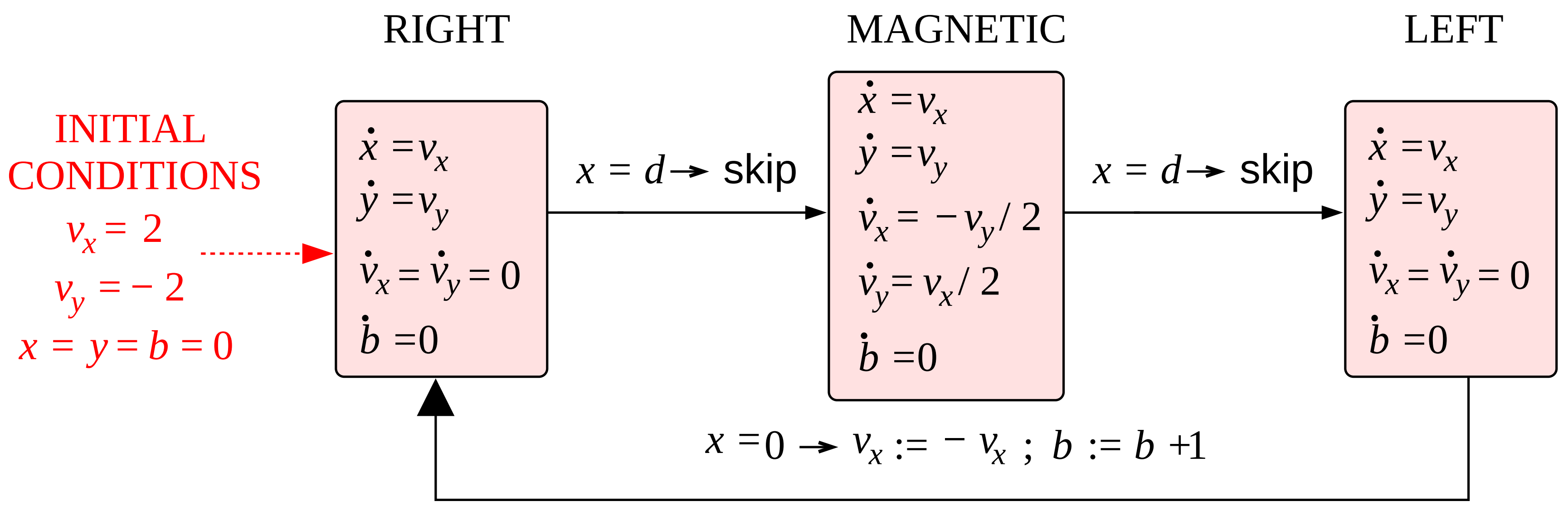
AUXILIARY EQUATIONS

$$\{\ w^2 + z^2 = 1$$

$$\Downarrow$$

$v_x^2 + v_y^2 = 8$ (conservation of energy)

Example



RIGHT $\rightarrow$ $v_y = -2 \,\wedge\, v_x = 2 \,\wedge\, 2db - 8b + y + x = 0$

MAGNETIC $\rightarrow$ $x - 2v_y - d = 4 \wedge v_x^2 + v_y^2 = 8 \wedge 2v_x + y + 2db - 8b + d = 4$

LEFT $\rightarrow$ $v_y = -2 \,\wedge\, v_x = -2 \,\wedge\, 2db - 8b + y - x = 8$

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Drawing a Parallel from Equalities

Linear equalities
[Karr'76]

↓

Polynomial equalities
[Colon'04]

Drawing a Parallel from Equalities

Linear equalities	Linear inequalities
[Karr'76]	[Cousot & Halbwachs'78]
↓	↓
Polynomial equalities	Polynomial inequalities
[Colon'04]	[Bagnara & Rodríguez-Carbonell & Zaffanella'05]

From Linear to Polynomial Equalities

$a := 0$;
$b := 0$;
$c := 1$;

while $?$ **do**

$a := a + 1$;
$b := b + c$;
$c := c + 2$;

end while

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$\{ \ c = 2a + 1 \ \}$
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Loop invariant
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Introduce new variable s standing for a^2

Extend program with new variable s

$$a := 0 \quad \rightarrow \quad s := 0$$

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Loop invariant
$\{\ b = a^2 \wedge c = 2a + 1\ \}$
is more precise

From Linear to Polynomial Inequalities

$\{ \text{ Pre}: \; b \geq 0 \; \}$

$a := 0 \; ;$

while $(a+1)^2 \leq b$ **do**

$\quad a := a + 1 \; ;$

end while

$\{ \text{ Post}: \; (a+1)^2 > b \wedge b \geq a^2 \; \}$

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Linear analysis cannot deal with the quadratic condition

$$(a+1)^2 \leq b$$

From Linear to Polynomial Inequalities

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Loop invariant $\{ \ a \geq 0 \wedge b \geq 0 \ \}$

not precise enough

From Linear to Polynomial Inequalities

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Loop invariant

$\{\ b \geq a^2 \wedge \cdots\ \}$

enough to prove partial correctness

Linearization of Polynomial Constraints

- Abstract values = sets of constraints
- Given a degree bound d, all terms x^α with $\deg(x^\alpha) \leq d$ are considered as different and independent variables

Vector Spaces $\leftrightarrow$ Polynomial Cones

polynomial $= 0$

- $\forall$ polynomial p, $p \sim p = 0$
- Vector space = set of polynomials closed under

$$\frac{}{0 = 0}$$

$$\frac{p = 0 \quad q = 0 \quad \lambda, \mu \in \mathbb{R}}{\lambda p + \mu q = 0}$$

Vector Spaces $\leftrightarrow$ Polynomial Cones

polynomial $= 0$	polynomial ≥ 0
• $\forall$ polynomial p, $p \sim p = 0$	• $\forall$ polynomial p, $p \sim p \geq 0$
• Vector space = set of polynomials closed under	• Polynomial cone = set of polynomials closed under
$\dfrac{}{0 = 0}$	$\dfrac{}{1 \geq 0}$
$\dfrac{p = 0 \quad q = 0 \quad \lambda, \mu \in \mathbb{R}}{\lambda p + \mu q = 0}$	$\dfrac{p \geq 0 \quad q \geq 0 \quad \lambda, \mu \in \mathbb{R}_+}{\lambda p + \mu q \geq 0}$

Explicitly Adding Other Inference Rules

polynomial $= 0$

$$\frac{p = 0 \quad \deg(pq) \leq d}{pq = 0}$$

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Future Work

- Extend the techniques to interprocedural analyses

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- But I am now working on something different:
  Satisfiability Modulo Theories (SMT)
  See `http://www.barcelogic.org`

Thank you!