Introduction to SAT

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SAT and SMT for Solving CSP's - Session 1

Seminar on Constraint Programming

29 March 2011

University of Bergen

Overview of the Session

- Propositional Logic
- DPLL procedure
- CDCL SAT solvers

Definition of Propositional Logic

- Syntax: Let us define formulas over a set of variables $\mathcal{P}$:
  - Every variable in $\mathcal{P}$ is a formula
  - If F is a formula, so is $\neg F$
  - If F and G are formulas, so are $(F \wedge G)$ and $(F \vee G)$
- Semantics:
  - An interpretation I over $\mathcal{P}$ is a function $I : \mathcal{P} \to \{0,1\}$
  - I satisfies F (written $I \models F$) if and only if $eval_I(F) = 1$
  - $eval_I : Formulas \to \{0,1\}$ is defined as follows:
    - $eval_I(p) = I(p)$
    - $eval_I(\neg F) = 1 - eval_I(F)$
    - $eval_I(\ (F \wedge G)\ ) = min\{eval_I(F), eval_I(G)\}$
    - $eval_I(\ (F \vee G)\ ) = max\{eval_I(F), eval_I(G)\}$
  - If $I \models F$ we say that I is a model of F

General Concepts in Logic

Let F and G be formulas. Then:

- F is satisfiable if it has at least one model
- F is unsatisfiable if it has no models
- F is a tautology if every interpretation is a model of F
- G is a logical consequence of F, denoted $F \models G$, if every model of F is a model of G
- F and G are logically equivalent, denoted $F \equiv G$, if F and G have the same models

SAT Problem. SAT Solver

The SAT problem consists in, given a formula F, return:

- YES if F is satisfiable
- NO if F is unsatisfiable

A program that solves the SAT problem is called a SAT solver

Detecting tautologies, logical consequences, ... reducible to SAT:

- F tautology iff $\neg F$ is unsatisfiable
- $F \models G$ iff $F \wedge \neg G$ is unsatisfiable
- $F \equiv G$ iff $(F \wedge \neg G) \vee (\neg F \wedge G)$ is unsatisfiable

Hence, having a SAT solver suffices to solve all these problems

Conjunctive Normal Form

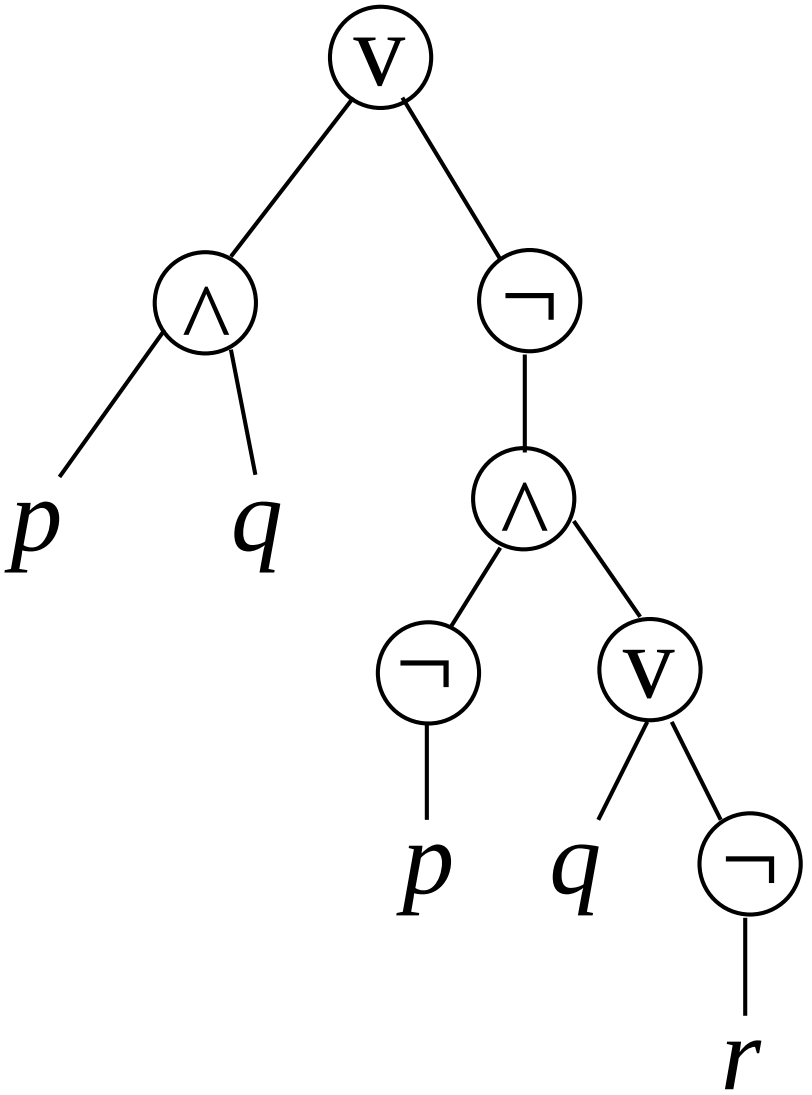
To build a SAT solver, it simplifies things to assume the input formula has a given format

- A literal is a prop. variable (p) or a negation of one ($\neg p$)
- A clause is a disjunction of zero or more literals ($l_1 \vee \ldots l_n$)
- The empty clause (zero lits.) is denoted $\square$ and is unsatisfiable
- A formula is in Conjunctive Normal Form (CNF) if it is a conjunction of zero or more clauses

For all our purposes, we will assume formulas are in CNF

Transformation to CNF via Tseitin

Let F be $(p \wedge q) \vee \neg(\ \neg p \wedge (q \vee \neg r)\ )$



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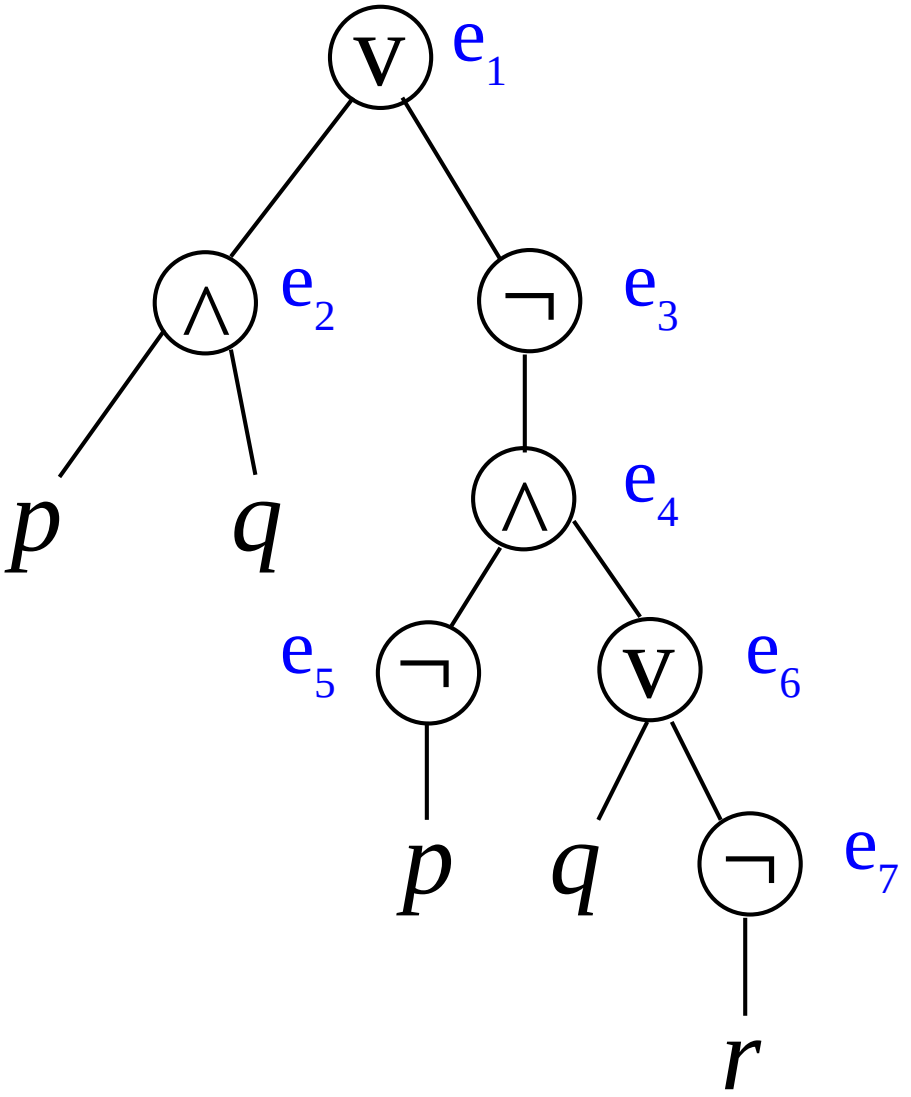

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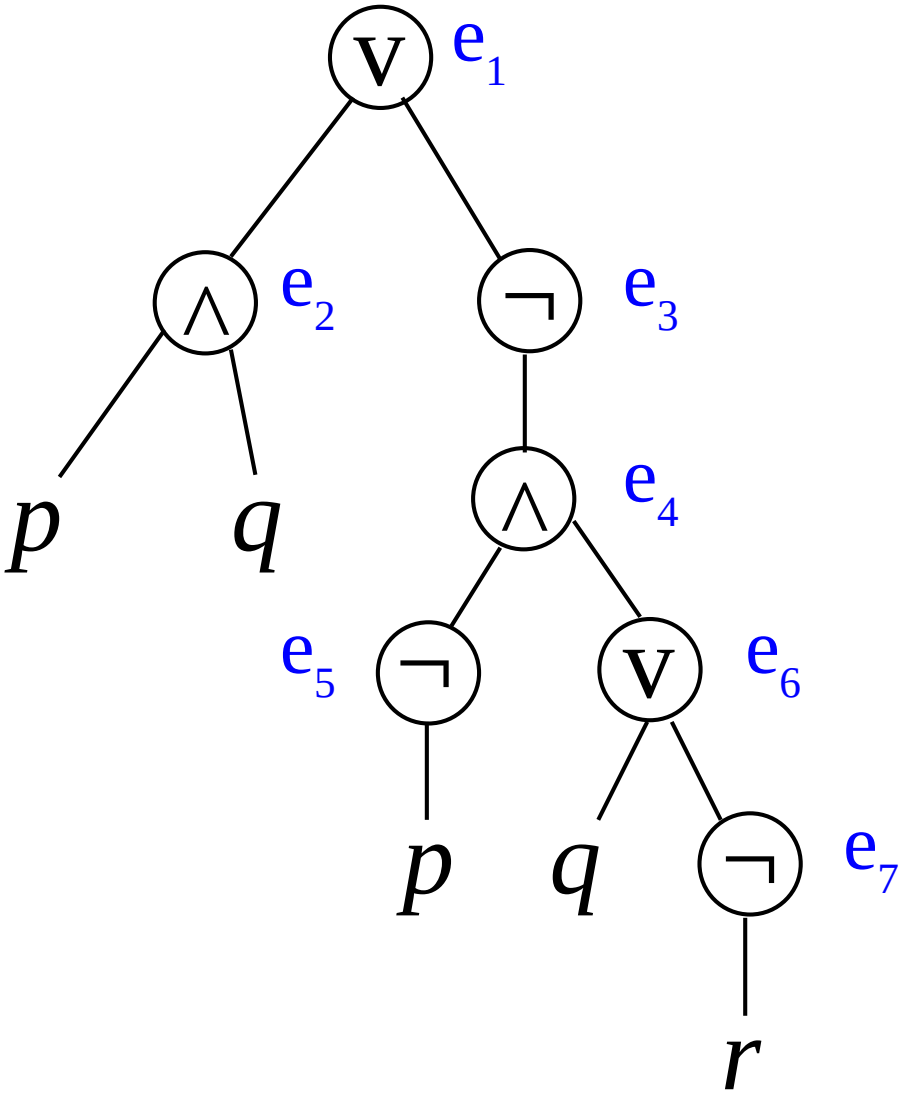

- e_1
- $e_1 \leftrightarrow e_2 \vee e_3$
- $e_2 \leftrightarrow p \wedge q$
- $e_3 \leftrightarrow \neg e_4$
- $e_4 \leftrightarrow e_5 \wedge e_6$
- $e_5 \leftrightarrow \neg p$
- $e_6 \leftrightarrow q \vee \neg e_7$
- $e_7 \leftrightarrow \neg r$

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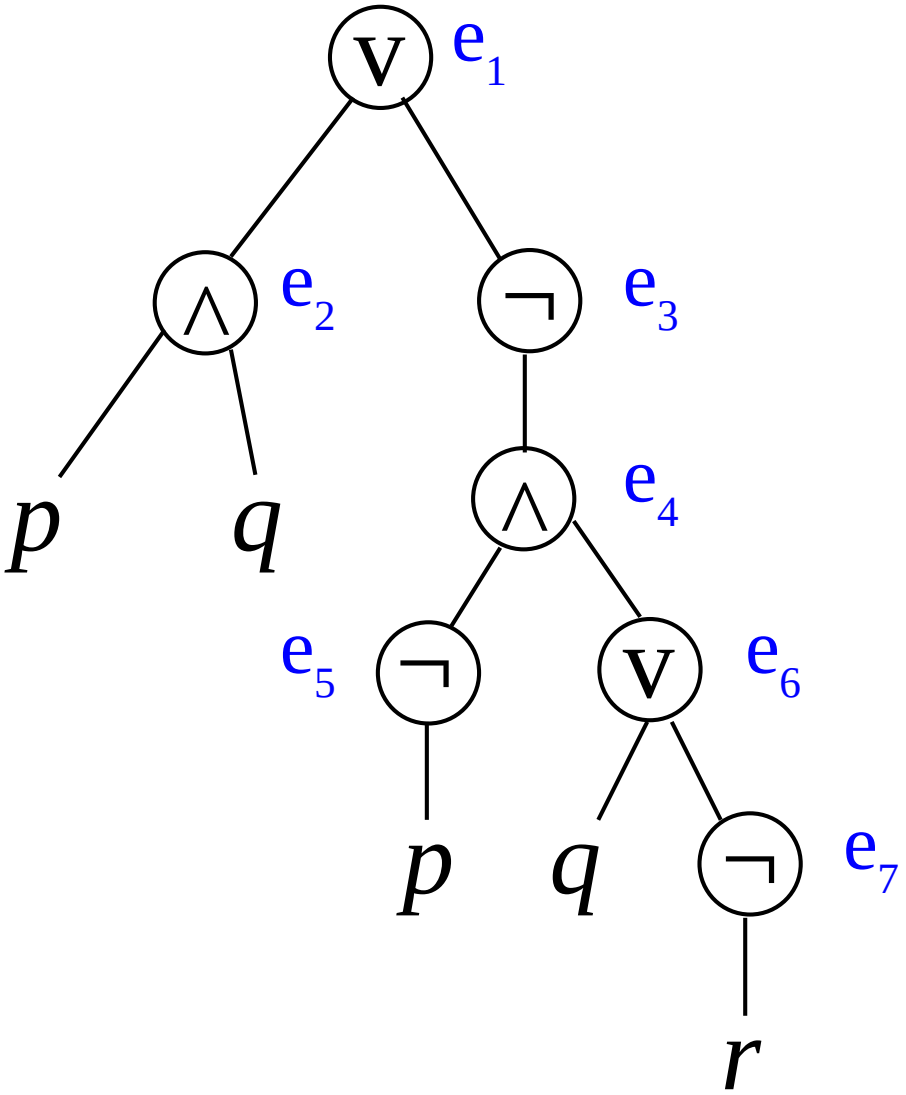

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- $e_1 \leftrightarrow e_2 \vee e_3$

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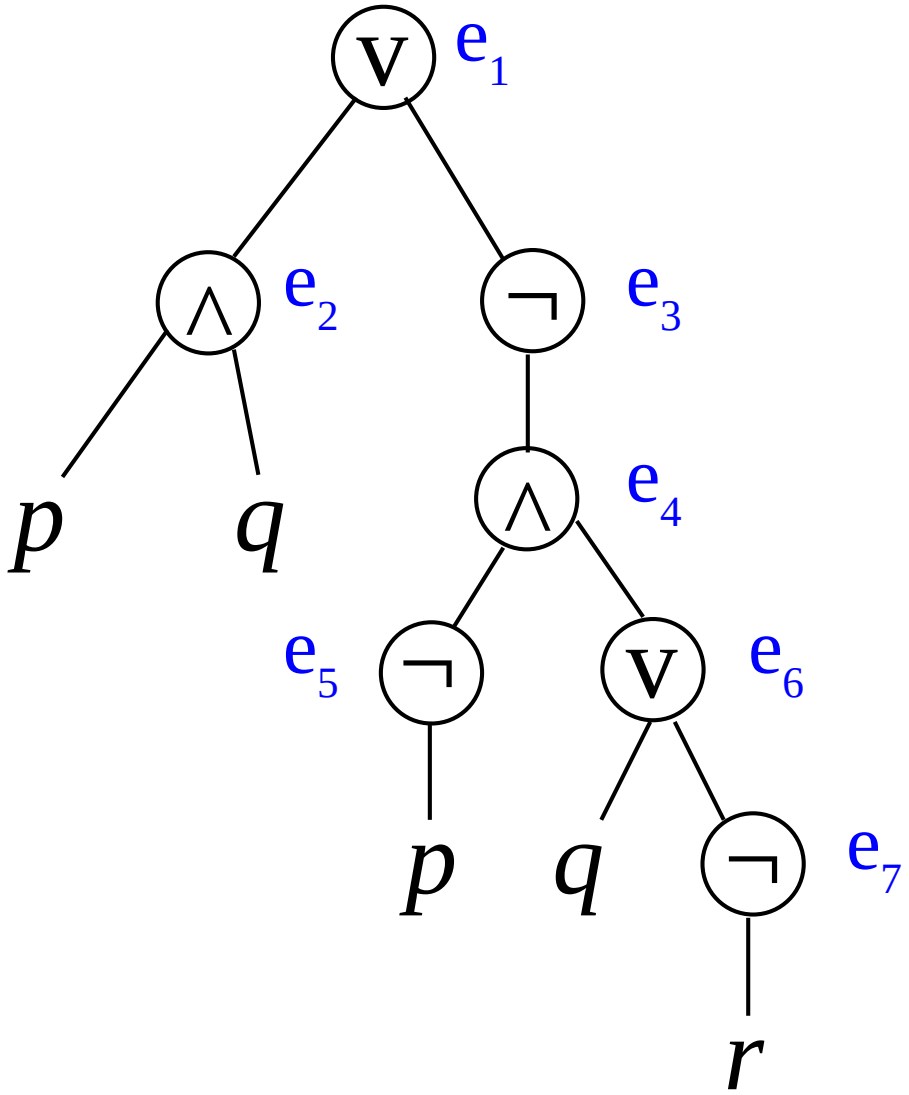

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- $e_2 \leftrightarrow p \wedge q$

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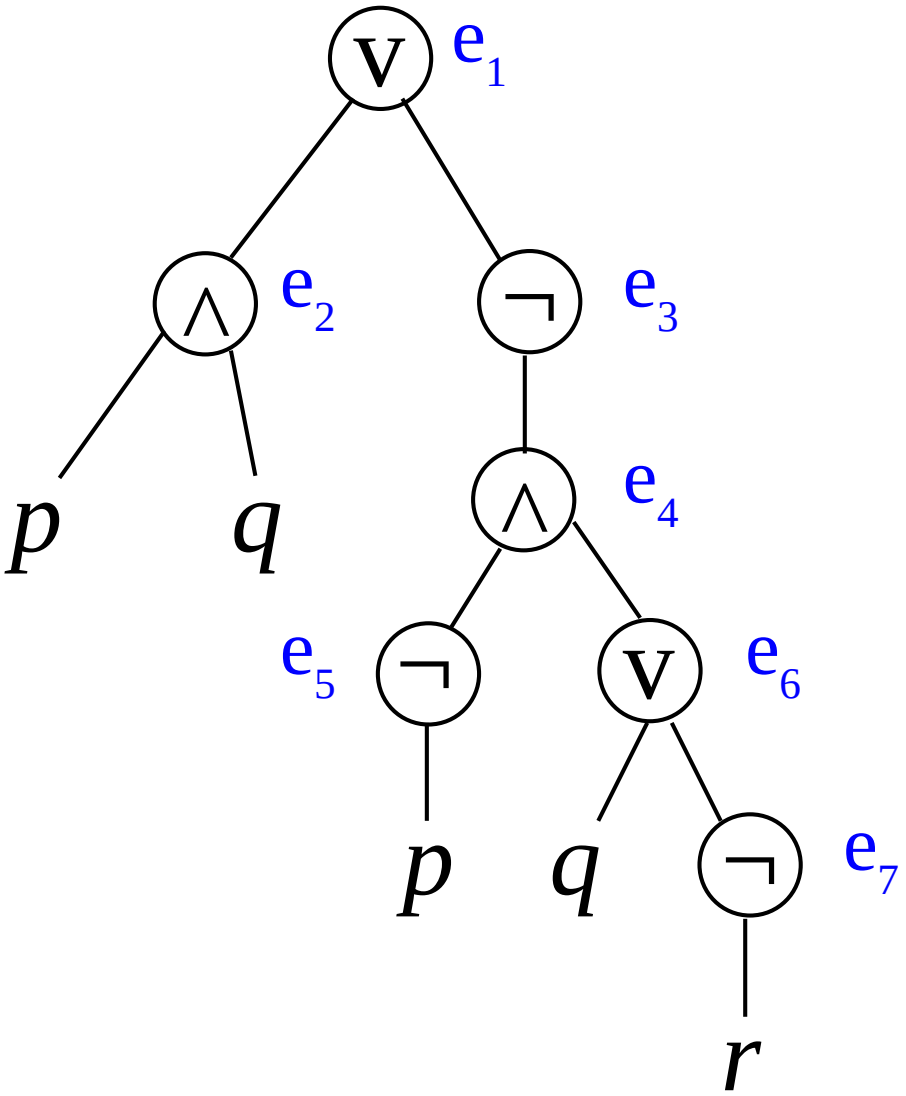

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- $e_7 \leftrightarrow \neg r$

Transformation to CNF via Tseitin (2)

- Tseitin does not produce an equivalent CNF
- Given F, the obtained CNF has 3 important properties:
  1. It is equisatisfiable to F
  2. Any model of CNF can be projected to the variables in F giving a model of F
  3. Any model of F can be extended to a model of the CNF
- Hence no model is lost nor added in the conversion
- Tseitin transformation works in linear time

Resolution

- The resolution rule is

$$\frac{p \vee C \quad \neg p \vee D}{C \vee D}$$

- $Res(S)$ = closure of set of clauses S under resolution =
  = clauses inferred in zero or more steps of resolution from S
- Properties:
  - Resolution is correct:
    $Res(S)$ only contains logical consequences
  - Resolution is refutationally complete:
    if S is unsatisfiable, then $\square \in Res(S)$
  - If S is a finite set of clauses, then $Res(S)$ is also finite
- So, given a set of clauses S, its satisfiability can be checked by:
  1. Computing $Res(S)$
  2. **If** $\square \in Res(S)$ **Then** UNSAT ; **Else** SAT

Overview of the Session

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Problem Solving with Propositional Logic

Example: Quasi-Group Completion (QGC)
Each row and column must contain $1, \ldots, n$

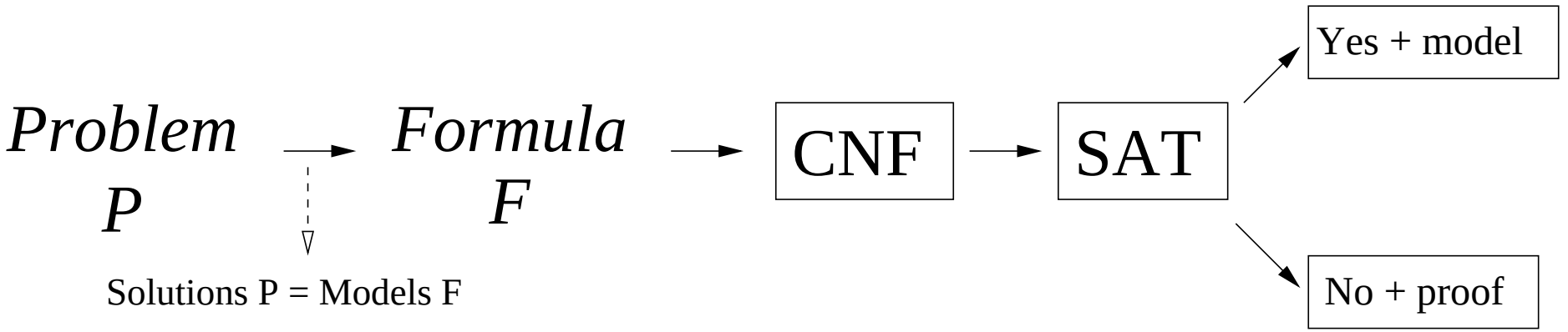
How to solve this with propositional logic?

1			
	3	4	
	4		2
4			3

- Consider variables p_{ijk} with the meaning:
  "square at row i column j has value k"
- Consider clauses expressing
  - at least one value k per row i and column j: $p_{ij1} \vee \ldots \vee p_{ijn}$
  - at most one value k per row i and column j: $\neg p_{ijk} \vee \neg p_{ijk'}$
  - same for exactly one row i per value k and column j
  - same for exactly one column j per value k and row i
  - some values are filled-in: e.g., p_{234}, p_{342}
- Models of CNF correspond to valid quasigroup completions
- This is known as the 3-D encoding into SAT

Problem Solving with Propositional Logic (2)

In general:



- This is the standard flow used for problem solving
- Transformation *P* to *F* (encoding) is problem-specific
- CNF conversion already reviewed
- Let us focus on how to design efficient SAT solvers

Designing an Efficient SAT Solver

- Specification of a SAT solver:

  INPUT: formula F in *CNF*

  OUTPUT:
  - If F is SAT: YES (+ model)
  - If F is UNSAT: NO (+ proof)

- Two possible methods:
  - resolution (already sketched)
  - DPLL (to be seen next)

- Due to efficiency, DPLL-based solvers are method of choice

Our Abstraction of DPLL

- DPLL stands for Davis–Putnam–Logemann–Loveland
- Given formula F in CNF, DPLL tries to build a model M for F
- Each step of the algorithm modifies M and/or F
- Interpretations M will be represented as sequences of literals:
  - Order in M does matter
  - No literal appears twice in M
  - No contradictory literals in M

  EXAMPLE: $p\bar{q}r$ is $M(p) = 1, M(q) = 0, M(r) = 1$
- Sequences might have decision literals, denoted l^{d}.
- We will introduce a transition system modelling DPLL
- States in transition system are pairs $M \parallel F$, where F is a CNF
- The rules in the transition system indicate which steps

  $$M \parallel F \Longrightarrow M' \parallel F'$$

  are allowed

Abstract DPLL - Rules

Extending the model:

UnitProp

$$M \parallel F, C \vee l \Longrightarrow M\, l \parallel F, C \vee l \quad \textbf{if} \begin{cases} M \models \neg C \\ l \text{ is undefined in } M \end{cases}$$

Decide

$$M \parallel F \Longrightarrow M\, l^{\mathsf{d}} \parallel F \quad \textbf{if} \begin{cases} l \text{ or } \neg l \text{ occurs in } F \\ l \text{ is undefined in } M \end{cases}$$

Abstract DPLL - Rules (2)

Repairing the model:

Fail

$$M \parallel F, C \Longrightarrow \mathit{fail} \ \textbf{if} \begin{cases} M \models \neg C \\ M \text{ contains no decisions} \end{cases}$$

Backtrack

$$M\, l^{\mathsf{d}}\, N \parallel F, C \Longrightarrow M \neg l \parallel F, C \ \textbf{if} \begin{cases} M\, l^{\mathsf{d}}\, N \models \neg C \\ N \text{ contains no decisions} \end{cases}$$

Abstract DPLL - Example 1

$$\emptyset \parallel \quad \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} \quad \Longrightarrow$$

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1^{d}\ 2\ 3^{d}\ 4 & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \quad (\text{Decide}) \\
1^{d}\ 2\ 3^{d}\ 4\ 5^{d} & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \quad (\text{UnitProp}) \\
1^{d}\ 2\ 3^{d}\ 4\ 5^{d}\ \overline{6} & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \quad (\text{Backtrack}) \\
1^{d}\ 2\ 3^{d}\ 4\ \overline{5} & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow
\end{array}
$$

Abstract DPLL - Example 1

$$\emptyset \;\|\; \bar{1} \vee 2,\ \bar{3} \vee 4,\ \bar{5} \vee \bar{6},\ 6 \vee \bar{5} \vee \bar{2} \;\Longrightarrow\; (\text{Decide})$$

$$1^{d} \;\|\; \bar{1} \vee 2,\ \bar{3} \vee 4,\ \bar{5} \vee \bar{6},\ 6 \vee \bar{5} \vee \bar{2} \;\Longrightarrow\; (\text{UnitProp})$$

$$1^{d}\ 2 \;\|\; \bar{1} \vee 2,\ \bar{3} \vee 4,\ \bar{5} \vee \bar{6},\ 6 \vee \bar{5} \vee \bar{2} \;\Longrightarrow\; (\text{Decide})$$

$$1^{d}\ 2\ 3^{d} \;\|\; \bar{1} \vee 2,\ \bar{3} \vee 4,\ \bar{5} \vee \bar{6},\ 6 \vee \bar{5} \vee \bar{2} \;\Longrightarrow\; (\text{UnitProp})$$

$$1^{d}\ 2\ 3^{d}\ 4 \;\|\; \bar{1} \vee 2,\ \bar{3} \vee 4,\ \bar{5} \vee \bar{6},\ 6 \vee \bar{5} \vee \bar{2} \;\Longrightarrow\; (\text{Decide})$$

$$1^{d}\ 2\ 3^{d}\ 4\ 5^{d} \;\|\; \bar{1} \vee 2,\ \bar{3} \vee 4,\ \bar{5} \vee \bar{6},\ 6 \vee \bar{5} \vee \bar{2} \;\Longrightarrow\; (\text{UnitProp})$$

$$1^{d}\ 2\ 3^{d}\ 4\ 5^{d}\ \bar{6} \;\|\; \bar{1} \vee 2,\ \bar{3} \vee 4,\ \bar{5} \vee \bar{6},\ 6 \vee \bar{5} \vee \bar{2} \;\Longrightarrow\; (\text{Backtrack})$$

$$1^{d}\ 2\ 3^{d}\ 4\ \bar{5} \;\|\; \bar{1} \vee 2,\ \bar{3} \vee 4,\ \bar{5} \vee \bar{6},\ 6 \vee \bar{5} \vee \bar{2} \;\Longrightarrow\; (\text{Decide})$$

Abstract DPLL - Example 1

$$
\begin{array}{rcll}
\emptyset & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \quad (\text{Decide}) \\
1^{d} & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \quad (\text{UnitProp}) \\
1^{d}\ 2 & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \quad (\text{Decide}) \\
1^{d}\ 2\ 3^{d} & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \quad (\text{UnitProp}) \\
1^{d}\ 2\ 3^{d}\ 4 & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \quad (\text{Decide}) \\
1^{d}\ 2\ 3^{d}\ 4\ 5^{d} & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \quad (\text{UnitProp}) \\
1^{d}\ 2\ 3^{d}\ 4\ 5^{d}\ \overline{6} & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \quad (\text{Backtrack}) \\
1^{d}\ 2\ 3^{d}\ 4\ \overline{5} & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \quad (\text{Decide}) \\
1^{d}\ 2\ 3^{d}\ 4\ \overline{5}\ 6^{d} & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} &
\end{array}
$$

Abstract DPLL - Example 2

$$\emptyset \;\|\; \overline{1} \vee 2 \vee 3,\; 1,\; \overline{2} \vee 3,\; \overline{2} \vee \overline{3},\; 2 \vee 3,\; 2 \vee \overline{3} \quad \Longrightarrow$$

Abstract DPLL - Example 2

$$\emptyset \;\|\; \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \quad \Longrightarrow \quad (\text{UnitProp})$$

Abstract DPLL - Example 2

$$\emptyset \;\|\; \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \quad \Longrightarrow \quad (\text{UnitProp})$$

$$1 \;\|\; \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \quad \Longrightarrow$$

Abstract DPLL - Example 2

$$\emptyset \;\|\; \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \quad \Longrightarrow \quad (\text{UnitProp})$$

$$1 \;\|\; \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \quad \Longrightarrow \quad (\text{Decide})$$

Abstract DPLL - Example 2

$$\emptyset \;\|\; \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \;\Longrightarrow\; (\text{UnitProp})$$

$$1 \;\|\; \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \;\Longrightarrow\; (\text{Decide})$$

$$1\ 2^{d} \;\|\; \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \;\Longrightarrow$$

Abstract DPLL - Example 2

$$\emptyset \;\|\; \bar{1} \vee 2 \vee 3,\ 1,\ \bar{2} \vee 3,\ \bar{2} \vee \bar{3},\ 2 \vee 3,\ 2 \vee \bar{3} \;\Longrightarrow\; (\text{UnitProp})$$

$$1 \;\|\; \bar{1} \vee 2 \vee 3,\ 1,\ \bar{2} \vee 3,\ \bar{2} \vee \bar{3},\ 2 \vee 3,\ 2 \vee \bar{3} \;\Longrightarrow\; (\text{Decide})$$

$$1\ 2^{d} \;\|\; \bar{1} \vee 2 \vee 3,\ 1,\ \bar{2} \vee 3,\ \bar{2} \vee \bar{3},\ 2 \vee 3,\ 2 \vee \bar{3} \;\Longrightarrow\; (\text{UnitProp})$$

Abstract DPLL - Example 2

$$\begin{array}{rcll}
\emptyset & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{Decide}) \\
1\ 2^{d} & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1\ 2^{d}\ 3 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow
\end{array}$$

Abstract DPLL - Example 2

$$\begin{array}{rcll}
\emptyset & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{Decide}) \\
1\ 2^{\mathrm{d}} & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1\ 2^{\mathrm{d}}\ 3 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{Backtrack})
\end{array}$$

Abstract DPLL - Example 2

$$\emptyset \;\|\; \overline{1}\vee 2\vee 3,\ 1,\ \overline{2}\vee 3,\ \overline{2}\vee\overline{3},\ 2\vee 3,\ 2\vee\overline{3} \;\Longrightarrow\; (\text{UnitProp})$$

$$1 \;\|\; \overline{1}\vee 2\vee 3,\ 1,\ \overline{2}\vee 3,\ \overline{2}\vee\overline{3},\ 2\vee 3,\ 2\vee\overline{3} \;\Longrightarrow\; (\text{Decide})$$

$$1\ 2^{d} \;\|\; \overline{1}\vee 2\vee 3,\ 1,\ \overline{2}\vee 3,\ \overline{2}\vee\overline{3},\ 2\vee 3,\ 2\vee\overline{3} \;\Longrightarrow\; (\text{UnitProp})$$

$$1\ 2^{d}\ 3 \;\|\; \overline{1}\vee 2\vee 3,\ 1,\ \overline{2}\vee 3,\ \overline{2}\vee\overline{3},\ 2\vee 3,\ 2\vee\overline{3} \;\Longrightarrow\; (\text{Backtrack})$$

$$1\,\overline{2} \;\|\; \overline{1}\vee 2\vee 3,\ 1,\ \overline{2}\vee 3,\ \overline{2}\vee\overline{3},\ 2\vee 3,\ 2\vee\overline{3} \;\Longrightarrow$$

Abstract DPLL - Example 2

$$\begin{array}{rcll}
\emptyset & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{Decide}) \\
1\ 2^{d} & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1\ 2^{d}\ 3 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{Backtrack}) \\
1\ \overline{2} & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp})
\end{array}$$

Abstract DPLL - Example 2

$$
\begin{array}{rcll}
\emptyset & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{Decide}) \\
1\ 2^{d} & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1\ 2^{d}\ 3 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{Backtrack}) \\
1\ \overline{2} & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1\ \overline{2}\ 3 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow
\end{array}
$$

Abstract DPLL - Example 2

$$
\begin{array}{rcll}
\emptyset & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{Decide}) \\
1\ 2^{d} & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1\ 2^{d}\ 3 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{Backtrack}) \\
1\ \overline{2} & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1\ \overline{2}\ 3 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{Fail})
\end{array}
$$

Abstract DPLL - Example 2

$$
\begin{array}{rcll}
\emptyset & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{Decide}) \\
1\ 2^{d} & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1\ 2^{d}\ 3 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{Backtrack}) \\
1\ \overline{2} & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{UnitProp}) \\
1\ \overline{2}\ 3 & \| & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow \quad (\text{Fail}) \\
\multicolumn{4}{c}{\textit{fail}}
\end{array}
$$

Abstract DPLL - Theoretical Results

- There are no infinite sequences of the form $\emptyset \parallel F \Longrightarrow \ldots$
- If $\emptyset \parallel F \Longrightarrow^* M \parallel F$ with final state $M \parallel F$, then
  - F is satisfiable
  - M is a model of F
- If $\emptyset \parallel F \Longrightarrow^* \mathit{fail}$ then F is unsatisfiable

Hence the transition system gives a decision procedure for SAT

Overview of the Session

- Propositional Logic
- DPLL procedure
- CDCL SAT solvers

CDCL SAT Solvers

- State-of-the-art SAT solvers implement DPLL procedure with the following improvements:
  - Conflict-analysis Driven Clause Learning (CDCL)
  - Lemma Removal
  - Activity-based Decision Heuristics
  - Restarts
  - Efficient Implementation of UnitProp

Motivating Example

$$\emptyset \Longrightarrow$$

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$

$p_{10} \vee \overline{p}_8 \vee p_1$

$p_{10} \vee p_3$

$\overline{p}_3 \vee p_{26}$

$p_{10} \vee \overline{p}_5$

$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$

$p_{21} \vee \overline{p}_6$

$p_{21} \vee \overline{p}_{17}$

$\overline{p}_{22} \vee \overline{p}_{13}$

$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

Motivating Example

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$

$p_{10} \vee \overline{p}_8 \vee p_1$

$p_{10} \vee p_3$

$\overline{p}_3 \vee p_{26}$

$p_{10} \vee \overline{p}_5$

$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$

$p_{21} \vee \overline{p}_6$

$p_{21} \vee \overline{p}_{17}$

$\overline{p}_{22} \vee \overline{p}_{13}$

$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

$\emptyset \Longrightarrow$

$p_{25} \Longrightarrow$

Motivating Example

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$

$p_{10} \vee \overline{p}_8 \vee p_1$

$p_{10} \vee p_3$

$\overline{p}_3 \vee p_{26}$

$p_{10} \vee \overline{p}_5$

$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$

$p_{21} \vee \overline{p}_6$

$p_{21} \vee \overline{p}_{17}$

$\overline{p}_{22} \vee \overline{p}_{13}$

$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

$\emptyset \Longrightarrow$

$p_{25} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d} \Longrightarrow$

Motivating Example

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$

$p_{10} \vee \overline{p}_8 \vee p_1$

$p_{10} \vee p_3$

$\overline{p}_3 \vee p_{26}$

$p_{10} \vee \overline{p}_5$

$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

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$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

$\emptyset \Longrightarrow$

$p_{25} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6 \Longrightarrow$

Motivating Example

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

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$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$

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$p_{21} \vee \overline{p}_{17}$

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$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

$\emptyset \Longrightarrow$

$p_{25} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6 \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17} \Longrightarrow$

Motivating Example

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

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$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

$\emptyset \Longrightarrow$

$p_{25} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6 \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d} \Longrightarrow$

Motivating Example

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$

$p_{10} \vee \overline{p}_8 \vee p_1$

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$\overline{p}_3 \vee p_{26}$

$p_{10} \vee \overline{p}_5$

$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$

$p_{21} \vee \overline{p}_6$

$p_{21} \vee \overline{p}_{17}$

$\overline{p}_{22} \vee \overline{p}_{13}$

$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

$\emptyset \Longrightarrow$

$p_{25} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6 \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13} \Longrightarrow$

Motivating Example

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$

$p_{10} \vee \overline{p}_8 \vee p_1$

$p_{10} \vee p_3$

$\overline{p}_3 \vee p_{26}$

$p_{10} \vee \overline{p}_5$

$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$

$p_{21} \vee \overline{p}_6$

$p_{21} \vee \overline{p}_{17}$

$\overline{p}_{22} \vee \overline{p}_{13}$

$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

$\emptyset \Longrightarrow$

$p_{25} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6 \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8 \Longrightarrow$

Motivating Example

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$

$p_{10} \vee \overline{p}_8 \vee p_1$

$p_{10} \vee p_3$

$\overline{p}_3 \vee p_{26}$

$p_{10} \vee \overline{p}_5$

$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$

$p_{21} \vee \overline{p}_6$

$p_{21} \vee \overline{p}_{17}$

$\overline{p}_{22} \vee \overline{p}_{13}$

$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

$\emptyset \Longrightarrow$

$p_{25} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6 \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8 \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d} \Longrightarrow$

Motivating Example

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$

$p_{10} \vee \overline{p}_8 \vee p_1$

$p_{10} \vee p_3$

$\overline{p}_3 \vee p_{26}$

$p_{10} \vee \overline{p}_5$

$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$

$p_{21} \vee \overline{p}_6$

$p_{21} \vee \overline{p}_{17}$

$\overline{p}_{22} \vee \overline{p}_{13}$

$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

$\emptyset \Longrightarrow$

$p_{25} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6 \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8 \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4 \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19} \Longrightarrow$

Motivating Example

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$

$p_{10} \vee \overline{p}_8 \vee p_1$

$p_{10} \vee p_3$

$\overline{p}_3 \vee p_{26}$

$p_{10} \vee \overline{p}_5$

$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$

$p_{21} \vee \overline{p}_6$

$p_{21} \vee \overline{p}_{17}$

$\overline{p}_{22} \vee \overline{p}_{13}$

$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

$\emptyset \Longrightarrow$

$p_{25} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6 \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8 \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d}\, p_{19} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d}\, p_{19}\, \overline{p}_{20}^{d} \Longrightarrow$

Motivating Example

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$

$p_{10} \vee \overline{p}_8 \vee p_1$

$p_{10} \vee p_3$

$\overline{p}_3 \vee p_{26}$

$p_{10} \vee \overline{p}_5$

$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$

$p_{21} \vee \overline{p}_6$

$p_{21} \vee \overline{p}_{17}$

$\overline{p}_{22} \vee \overline{p}_{13}$

$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

$\emptyset \Longrightarrow$

$p_{25} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6 \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8 \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d}\, p_{19} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d}\, p_{19}\, \overline{p}_{20}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d}\, p_{19}\, \overline{p}_{20}^{d}\, p_{23} \Longrightarrow$

Motivating Example

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$

$p_{10} \vee \overline{p}_8 \vee p_1$

$p_{10} \vee p_3$

$\overline{p}_3 \vee p_{26}$

$p_{10} \vee \overline{p}_5$

$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$

$p_{21} \vee \overline{p}_6$

$p_{21} \vee \overline{p}_{17}$

$\overline{p}_{22} \vee \overline{p}_{13}$

$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

$\emptyset \Longrightarrow$

$p_{25} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6 \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8 \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4 \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23} \Longrightarrow$

$\underbrace{p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}}_{M}\; p^d_{11} \Longrightarrow$

Motivating Example

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$

$p_{10} \vee \overline{p}_8 \vee p_1$

$p_{10} \vee p_3$

$\overline{p}_3 \vee p_{26}$

$p_{10} \vee \overline{p}_5$

$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$

$p_{21} \vee \overline{p}_6$

$p_{21} \vee \overline{p}_{17}$

$\overline{p}_{22} \vee \overline{p}_{13}$

$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

$\emptyset \Longrightarrow$

$p_{25} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6 \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8 \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d}\, p_{19} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d}\, p_{19}\, \overline{p}_{20}^{d} \Longrightarrow$

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d}\, p_{19}\, \overline{p}_{20}^{d}\, p_{23} \Longrightarrow$

$\underbrace{p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d}\, p_{19}\, \overline{p}_{20}^{d}\, p_{23}}_{M}\, p_{11}^{d} \Longrightarrow$

$M\, p_{11}^{d} \Longrightarrow$

Motivating Example

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$

$p_{10} \vee \overline{p}_8 \vee p_1$

$p_{10} \vee p_3$

$\overline{p}_3 \vee p_{26}$

$p_{10} \vee \overline{p}_5$

$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$

$p_{21} \vee \overline{p}_6$

$p_{21} \vee \overline{p}_{17}$

$\overline{p}_{22} \vee \overline{p}_{13}$

$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

$\emptyset \Longrightarrow$

$p_{25} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6 \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8 \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4 \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20} \Longrightarrow$

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23} \Longrightarrow$

$\underbrace{p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}}_{M}\; p^d_{11} \Longrightarrow$

$M\, p^d_{11} \Longrightarrow$

Before we continue, some notation:

Literal p_{25} belongs to decision level 0

Literals $\overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}$ belong to decision level 1

...

Motivating Example (2)

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$

$\overline{p}_{11} \vee p_{13} \vee p_{16}$

$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$

$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$

$p_{10} \vee \overline{p}_8 \vee p_1$

$p_{10} \vee p_3$

$\overline{p}_3 \vee p_{26}$

$p_{10} \vee \overline{p}_5$

$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$

$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$

$p_{21} \vee \overline{p}_6$

$p_{21} \vee \overline{p}_{17}$

$\overline{p}_{22} \vee \overline{p}_{13}$

$p_{13} \vee p_8$

$\overline{p}_4 \vee p_{19}$

$p_{20} \vee p_{23}$

$\overline{p}_{20} \vee p_{24}$

p_{25}

Remember M is

$p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d}\, p_{19}\, \overline{p}_{20}^{d}\, p_{23}$

Motivating Example (2)

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
$\overline{p}_{11} \vee p_{13} \vee p_{16}$
$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
$p_{10} \vee \overline{p}_8 \vee p_1$
$p_{10} \vee p_3$
$\overline{p}_3 \vee p_{26}$
$p_{10} \vee \overline{p}_5$
$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
$p_{21} \vee \overline{p}_6$
$p_{21} \vee \overline{p}_{17}$
$\overline{p}_{22} \vee \overline{p}_{13}$
$p_{13} \vee p_8$
$\overline{p}_4 \vee p_{19}$
$p_{20} \vee p_{23}$
$\overline{p}_{20} \vee p_{24}$
p_{25}

Remember M is

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

$M\, p^d_{11} \Longrightarrow$

Motivating Example (2)

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
$\overline{p}_{11} \vee p_{13} \vee p_{16}$
$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
$p_{10} \vee \overline{p}_8 \vee p_1$
$p_{10} \vee p_3$
$\overline{p}_3 \vee p_{26}$
$p_{10} \vee \overline{p}_5$
$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
$p_{21} \vee \overline{p}_6$
$p_{21} \vee \overline{p}_{17}$
$\overline{p}_{22} \vee \overline{p}_{13}$
$p_{13} \vee p_8$
$\overline{p}_4 \vee p_{19}$
$p_{20} \vee p_{23}$
$\overline{p}_{20} \vee p_{24}$
p_{25}

Remember M is

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

$M\, p^d_{11} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12} \Longrightarrow$

Motivating Example (2)

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
$\overline{p}_{11} \vee p_{13} \vee p_{16}$
$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
$p_{10} \vee \overline{p}_8 \vee p_1$
$p_{10} \vee p_3$
$\overline{p}_3 \vee p_{26}$
$p_{10} \vee \overline{p}_5$
$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
$p_{21} \vee \overline{p}_6$
$p_{21} \vee \overline{p}_{17}$
$\overline{p}_{22} \vee \overline{p}_{13}$
$p_{13} \vee p_8$
$\overline{p}_4 \vee p_{19}$
$p_{20} \vee p_{23}$
$\overline{p}_{20} \vee p_{24}$
p_{25}

Remember M is

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

$M\, p^d_{11} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16} \Longrightarrow$

Motivating Example (2)

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
$\overline{p}_{11} \vee p_{13} \vee p_{16}$
$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
$p_{10} \vee \overline{p}_8 \vee p_1$
$p_{10} \vee p_3$
$\overline{p}_3 \vee p_{26}$
$p_{10} \vee \overline{p}_5$
$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
$p_{21} \vee \overline{p}_6$
$p_{21} \vee \overline{p}_{17}$
$\overline{p}_{22} \vee \overline{p}_{13}$
$p_{13} \vee p_8$
$\overline{p}_4 \vee p_{19}$
$p_{20} \vee p_{23}$
$\overline{p}_{20} \vee p_{24}$
p_{25}

Remember M is

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

$M\, p^d_{11} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2 \Longrightarrow$

Motivating Example (2)

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
$\overline{p}_{11} \vee p_{13} \vee p_{16}$
$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
$p_{10} \vee \overline{p}_8 \vee p_1$
$p_{10} \vee p_3$
$\overline{p}_3 \vee p_{26}$
$p_{10} \vee \overline{p}_5$
$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
$p_{21} \vee \overline{p}_6$
$p_{21} \vee \overline{p}_{17}$
$\overline{p}_{22} \vee \overline{p}_{13}$
$p_{13} \vee p_8$
$\overline{p}_4 \vee p_{19}$
$p_{20} \vee p_{23}$
$\overline{p}_{20} \vee p_{24}$
p_{25}

Remember M is

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

$M\, p^d_{11} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10} \Longrightarrow$

Motivating Example (2)

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
$\overline{p}_{11} \vee p_{13} \vee p_{16}$
$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
$p_{10} \vee \overline{p}_8 \vee p_1$
$p_{10} \vee p_3$
$\overline{p}_3 \vee p_{26}$
$p_{10} \vee \overline{p}_5$
$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
$p_{21} \vee \overline{p}_6$
$p_{21} \vee \overline{p}_{17}$
$\overline{p}_{22} \vee \overline{p}_{13}$
$p_{13} \vee p_8$
$\overline{p}_4 \vee p_{19}$
$p_{20} \vee p_{23}$
$\overline{p}_{20} \vee p_{24}$
p_{25}

Remember M is

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

$M\, p^d_{11} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1 \Longrightarrow$

Motivating Example (2)

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
$\overline{p}_{11} \vee p_{13} \vee p_{16}$
$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
$p_{10} \vee \overline{p}_8 \vee p_1$
$p_{10} \vee p_3$
$\overline{p}_3 \vee p_{26}$
$p_{10} \vee \overline{p}_5$
$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
$p_{21} \vee \overline{p}_6$
$p_{21} \vee \overline{p}_{17}$
$\overline{p}_{22} \vee \overline{p}_{13}$
$p_{13} \vee p_8$
$\overline{p}_4 \vee p_{19}$
$p_{20} \vee p_{23}$
$\overline{p}_{20} \vee p_{24}$
p_{25}

Remember M is

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

$M\, p^d_{11} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3 \Longrightarrow$

Motivating Example (2)

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
$\overline{p}_{11} \vee p_{13} \vee p_{16}$
$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
$p_{10} \vee \overline{p}_8 \vee p_1$
$p_{10} \vee p_3$
$\overline{p}_3 \vee p_{26}$
$p_{10} \vee \overline{p}_5$
$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
$p_{21} \vee \overline{p}_6$
$p_{21} \vee \overline{p}_{17}$
$\overline{p}_{22} \vee \overline{p}_{13}$
$p_{13} \vee p_8$
$\overline{p}_4 \vee p_{19}$
$p_{20} \vee p_{23}$
$\overline{p}_{20} \vee p_{24}$
p_{25}

Remember M is

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

$M\, p^d_{11} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3\, p_{26} \Longrightarrow$

Motivating Example (2)

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
$\overline{p}_{11} \vee p_{13} \vee p_{16}$
$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
$p_{10} \vee \overline{p}_8 \vee p_1$
$p_{10} \vee p_3$
$\overline{p}_3 \vee p_{26}$
$p_{10} \vee \overline{p}_5$
$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
$p_{21} \vee \overline{p}_6$
$p_{21} \vee \overline{p}_{17}$
$\overline{p}_{22} \vee \overline{p}_{13}$
$p_{13} \vee p_8$
$\overline{p}_4 \vee p_{19}$
$p_{20} \vee p_{23}$
$\overline{p}_{20} \vee p_{24}$
p_{25}

Remember M is

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

$M\, p^d_{11} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3\, p_{26} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3\, p_{26}\, \overline{p}_5 \Longrightarrow$

Motivating Example (2)

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
$\overline{p}_{11} \vee p_{13} \vee p_{16}$
$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
$p_{10} \vee \overline{p}_8 \vee p_1$
$p_{10} \vee p_3$
$\overline{p}_3 \vee p_{26}$
$p_{10} \vee \overline{p}_5$
$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
$p_{21} \vee \overline{p}_6$
$p_{21} \vee \overline{p}_{17}$
$\overline{p}_{22} \vee \overline{p}_{13}$
$p_{13} \vee p_8$
$\overline{p}_4 \vee p_{19}$
$p_{20} \vee p_{23}$
$\overline{p}_{20} \vee p_{24}$
p_{25}

Remember M is

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

$M\, p^d_{11} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3\, p_{26} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3\, p_{26}\, \overline{p}_5 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3\, p_{26}\, \overline{p}_5\, p_{18} \Longrightarrow$

Motivating Example (2)

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
$\overline{p}_{11} \vee p_{13} \vee p_{16}$
$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
$p_{10} \vee \overline{p}_8 \vee p_1$
$p_{10} \vee p_3$
$\overline{p}_3 \vee p_{26}$
$p_{10} \vee \overline{p}_5$
$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
$p_{21} \vee \overline{p}_6$
$p_{21} \vee \overline{p}_{17}$
$\overline{p}_{22} \vee \overline{p}_{13}$
$p_{13} \vee p_8$
$\overline{p}_4 \vee p_{19}$
$p_{20} \vee p_{23}$
$\overline{p}_{20} \vee p_{24}$
p_{25}

Remember M is

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

$M\, p^d_{11} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3\, p_{26} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3\, p_{26}\, \overline{p}_5 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3\, p_{26}\, \overline{p}_5\, p_{18} \Longrightarrow$

Conflict!

Motivating Example (2)

$\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
$\overline{p}_{11} \vee p_{13} \vee p_{16}$
$p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
$\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
$p_{10} \vee \overline{p}_8 \vee p_1$
$p_{10} \vee p_3$
$\overline{p}_3 \vee p_{26}$
$p_{10} \vee \overline{p}_5$
$\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
$\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
$p_{21} \vee \overline{p}_6$
$p_{21} \vee \overline{p}_{17}$
$\overline{p}_{22} \vee \overline{p}_{13}$
$p_{13} \vee p_8$
$\overline{p}_4 \vee p_{19}$
$p_{20} \vee p_{23}$
$\overline{p}_{20} \vee p_{24}$
p_{25}

Remember M is

$p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

$M\, p^d_{11} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3\, p_{26} \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3\, p_{26}\, \overline{p}_5 \Longrightarrow$
$M\, p^d_{11}\, \overline{p}_{12}\, p_{16}\, \overline{p}_2\, \overline{p}_{10}\, p_1\, p_3\, p_{26}\, \overline{p}_5\, p_{18} \Longrightarrow$

Conflict!

- Let's try to find the causes of conflict
- First of all we compute, for each lit, the reason why it is true

Motivating Example (3)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
7. $\overline{p}_3 \vee p_{26}$
8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
11. $p_{21} \vee \overline{p}_6$
12. $p_{21} \vee \overline{p}_{17}$
13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
15. $\overline{p}_4 \vee p_{19}$
16. $p_{20} \vee p_{23}$
17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Motivating Example (3)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
7. $\overline{p}_3 \vee p_{26}$
8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
11. $p_{21} \vee \overline{p}_6$
12. $p_{21} \vee \overline{p}_{17}$
13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
15. $\overline{p}_4 \vee p_{19}$
16. $p_{20} \vee p_{23}$
17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Literal:	p_{11}^d	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

Motivating Example (3)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
7. $\overline{p}_3 \vee p_{26}$
8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
11. $p_{21} \vee \overline{p}_6$
12. $p_{21} \vee \overline{p}_{17}$
13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
15. $\overline{p}_4 \vee p_{19}$
16. $p_{20} \vee p_{23}$
17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Literal:	p_{11}^d	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

Let us take the conflicting clause $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$.

The reason why p_{18} is true is clause 9.

Resolution gives:

$$\frac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18} \qquad \overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17}}$$

Motivating Example (3)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
7. $\overline{p}_3 \vee p_{26}$
8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
11. $p_{21} \vee \overline{p}_6$
12. $p_{21} \vee \overline{p}_{17}$
13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
15. $\overline{p}_4 \vee p_{19}$
16. $p_{20} \vee p_{23}$
17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Literal:	p_{11}^d	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

Let us take the conflicting clause $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$.
The reason why p_{18} is true is clause 9.
Resolution gives:

$$\frac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18} \qquad \overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17}}$$

Last assigned false lit in resulting clause is p_5.
The reason why p_5 is false is clause 8.
Again, resolution:

$$\frac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17} \qquad p_{10} \vee \overline{p}_5}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}}$$

Motivating Example (3)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
7. $\overline{p}_3 \vee p_{26}$
8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
11. $p_{21} \vee \overline{p}_6$
12. $p_{21} \vee \overline{p}_{17}$
13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
15. $\overline{p}_4 \vee p_{19}$
16. $p_{20} \vee p_{23}$
17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Literal:	p_{11}^d	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

Let us take the conflicting clause $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$.
The reason why p_{18} is true is clause 9.
Resolution gives:

$$\frac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18} \qquad \overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17}}$$

Last assigned false lit in resulting clause is p_5.
The reason why p_5 is false is clause 8.
Again, resolution:

$$\frac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17} \qquad p_{10} \vee \overline{p}_5}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}}$$

The process is now iterated...

Motivating Example (4)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
7. $\overline{p}_3 \vee p_{26}$
8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
11. $p_{21} \vee \overline{p}_6$
12. $p_{21} \vee \overline{p}_{17}$
13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
15. $\overline{p}_4 \vee p_{19}$
16. $p_{20} \vee p_{23}$
17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Remember M is $p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

Literal:	p^d_{11}	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

Motivating Example (4)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
7. $\overline{p}_3 \vee p_{26}$
8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
11. $p_{21} \vee \overline{p}_6$
12. $p_{21} \vee \overline{p}_{17}$
13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
15. $\overline{p}_4 \vee p_{19}$
16. $p_{20} \vee p_{23}$
17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Remember M is $p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

Literal:	p^d_{11}	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

$$\dfrac{\dfrac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18} \qquad \overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17}} \qquad p_{10} \vee \overline{p}_5}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}}$$

Motivating Example (4)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
7. $\overline{p}_3 \vee p_{26}$
8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
11. $p_{21} \vee \overline{p}_6$
12. $p_{21} \vee \overline{p}_{17}$
13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
15. $\overline{p}_4 \vee p_{19}$
16. $p_{20} \vee p_{23}$
17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Remember M is $p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

Literal:	p^d_{11}	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

$$\dfrac{\dfrac{\dfrac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18} \qquad \overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17}} \qquad p_{10} \vee \overline{p}_5}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}} \qquad p_{10} \vee p_3}{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}}$$

Motivating Example (4)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
7. $\overline{p}_3 \vee p_{26}$
8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
11. $p_{21} \vee \overline{p}_6$
12. $p_{21} \vee \overline{p}_{17}$
13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
15. $\overline{p}_4 \vee p_{19}$
16. $p_{20} \vee p_{23}$
17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Remember M is $p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

Literal:	p^d_{11}	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18} \qquad \overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17}} \qquad p_{10} \vee \overline{p}_5}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}} \qquad p_{10} \vee p_3}{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}}$$

$$\dfrac{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10} \qquad p_{10} \vee \overline{p}_8 \vee p_1}{\overline{p}_{19} \vee p_{17} \vee p_{10} \vee \overline{p}_8}$$

Motivating Example (4)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
7. $\overline{p}_3 \vee p_{26}$
8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
11. $p_{21} \vee \overline{p}_6$
12. $p_{21} \vee \overline{p}_{17}$
13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
15. $\overline{p}_4 \vee p_{19}$
16. $p_{20} \vee p_{23}$
17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Remember M is $p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d}\, p_{19}\, \overline{p}_{20}^{d}\, p_{23}$

Literal:	p_{11}^{d}	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

$$\dfrac{\dfrac{\dfrac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18} \qquad \overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17}} \qquad p_{10} \vee \overline{p}_5}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}} \qquad p_{10} \vee p_3}{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}}$$

$$\dfrac{\dfrac{\dfrac{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10} \qquad p_{10} \vee \overline{p}_8 \vee p_1}{\overline{p}_{19} \vee p_{17} \vee p_{10} \vee \overline{p}_8} \qquad p_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee p_2 \vee \overline{p}_4 \vee p_{20}}}$$

Motivating Example (4)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
7. $\overline{p}_3 \vee p_{26}$
8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
11. $p_{21} \vee \overline{p}_6$
12. $p_{21} \vee \overline{p}_{17}$
13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
15. $\overline{p}_4 \vee p_{19}$
16. $p_{20} \vee p_{23}$
17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Remember M is $p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d}\, p_{19}\, \overline{p}_{20}^{d}\, p_{23}$

Literal:	p_{11}^{d}	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18} \qquad \overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17}} \qquad p_{10} \vee \overline{p}_5}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}} \qquad p_{10} \vee p_3}{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}}$$

$$\dfrac{\dfrac{\dfrac{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10} \qquad p_{10} \vee \overline{p}_8 \vee p_1}{\overline{p}_{19} \vee p_{17} \vee p_{10} \vee \overline{p}_8} \qquad p_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee p_2 \vee \overline{p}_4 \vee p_{20}} \qquad p_{12} \vee \overline{p}_{16} \vee \overline{p}_2}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{16}}$$

Motivating Example (4)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
7. $\overline{p}_3 \vee p_{26}$
8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
11. $p_{21} \vee \overline{p}_6$
12. $p_{21} \vee \overline{p}_{17}$
13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
15. $\overline{p}_4 \vee p_{19}$
16. $p_{20} \vee p_{23}$
17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Remember M is $p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

Literal:	p^d_{11}	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

$$\frac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18} \qquad \overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17} \qquad p_{10} \vee \overline{p}_5}$$

$$\frac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17} \qquad p_{10} \vee \overline{p}_5}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10} \qquad p_{10} \vee p_3}$$

$$\frac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10} \qquad p_{10} \vee p_3}{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}}$$

$$\frac{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10} \qquad p_{10} \vee \overline{p}_8 \vee p_1}{\overline{p}_{19} \vee p_{17} \vee p_{10} \vee \overline{p}_8 \qquad p_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}}$$

$$\frac{\overline{p}_{19} \vee p_{17} \vee p_{10} \vee \overline{p}_8 \qquad p_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee p_2 \vee \overline{p}_4 \vee p_{20} \qquad p_{12} \vee \overline{p}_{16} \vee \overline{p}_2}$$

$$\frac{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee p_2 \vee \overline{p}_4 \vee p_{20} \qquad p_{12} \vee \overline{p}_{16} \vee \overline{p}_2}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{16}}$$

$$\frac{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{16} \qquad \overline{p}_{11} \vee p_{13} \vee p_{16}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{11} \vee p_{13}}$$

Motivating Example (4)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
7. $\overline{p}_3 \vee p_{26}$
8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
11. $p_{21} \vee \overline{p}_6$
12. $p_{21} \vee \overline{p}_{17}$
13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
15. $\overline{p}_4 \vee p_{19}$
16. $p_{20} \vee p_{23}$
17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Remember M is $p_{25}\, \overline{p}_{21}^{d}\, \overline{p}_6\, \overline{p}_{17}\, p_{22}^{d}\, \overline{p}_{13}\, p_8\, p_4^{d}\, p_{19}\, \overline{p}_{20}^{d}\, p_{23}$

Literal:	p_{11}^{d}	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18} \qquad \overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17}} \qquad p_{10} \vee \overline{p}_5}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}} \qquad p_{10} \vee p_3}{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10} \qquad p_{10} \vee \overline{p}_8 \vee p_1}{\overline{p}_{19} \vee p_{17} \vee p_{10} \vee \overline{p}_8} \qquad p_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee p_2 \vee \overline{p}_4 \vee p_{20}} \qquad p_{12} \vee \overline{p}_{16} \vee \overline{p}_2}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{16}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{16} \qquad \overline{p}_{11} \vee p_{13} \vee p_{16}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{11} \vee p_{13}} \qquad \overline{p}_{11} \vee p_6 \vee \overline{p}_{12}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{11} \vee p_{13} \vee p_6}}$$

Motivating Example (4)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
7. $\overline{p}_3 \vee p_{26}$
8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
11. $p_{21} \vee \overline{p}_6$
12. $p_{21} \vee \overline{p}_{17}$
13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
15. $\overline{p}_4 \vee p_{19}$
16. $p_{20} \vee p_{23}$
17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Remember M is $p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

Literal:	p^d_{11}	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

$$\frac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18} \qquad \overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17} \qquad p_{10} \vee \overline{p}_5}$$

$$\frac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17} \qquad p_{10} \vee \overline{p}_5}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10} \qquad p_{10} \vee p_3}$$

$$\frac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10} \qquad p_{10} \vee p_3}{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}}$$

$$\frac{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10} \qquad p_{10} \vee \overline{p}_8 \vee p_1}{\overline{p}_{19} \vee p_{17} \vee p_{10} \vee \overline{p}_8 \qquad p_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}}$$

$$\frac{\overline{p}_{19} \vee p_{17} \vee p_{10} \vee \overline{p}_8 \qquad p_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee p_2 \vee \overline{p}_4 \vee p_{20} \qquad p_{12} \vee \overline{p}_{16} \vee \overline{p}_2}$$

$$\frac{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee p_2 \vee \overline{p}_4 \vee p_{20} \qquad p_{12} \vee \overline{p}_{16} \vee \overline{p}_2}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{16}}$$

$$\frac{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{16} \qquad \overline{p}_{11} \vee p_{13} \vee p_{16}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{11} \vee p_{13} \qquad \overline{p}_{11} \vee p_6 \vee \overline{p}_{12}}$$

$$\frac{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{11} \vee p_{13} \qquad \overline{p}_{11} \vee p_6 \vee \overline{p}_{12}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{11} \vee p_{13} \vee p_6}$$

Note that process now can't continue

Motivating Example (4)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
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6. $p_{10} \vee p_3$
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8. $p_{10} \vee \overline{p}_5$
9. $\overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}$
10. $\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18}$
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17. $\overline{p}_{20} \vee p_{24}$
18. p_{25}

Remember M is $p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

Literal:	p^d_{11}	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

$$\dfrac{\dfrac{\dfrac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18} \qquad \overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17}} \qquad p_{10} \vee \overline{p}_5}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}} \qquad p_{10} \vee p_3}{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}}$$

$$\dfrac{\dfrac{\dfrac{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10} \qquad p_{10} \vee \overline{p}_8 \vee p_1}{\overline{p}_{19} \vee p_{17} \vee p_{10} \vee \overline{p}_8} \qquad p_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee p_2 \vee \overline{p}_4 \vee p_{20}} \qquad p_{12} \vee \overline{p}_{16} \vee \overline{p}_2}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{16}}$$

$$\dfrac{\dfrac{\dfrac{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{16} \qquad \overline{p}_{11} \vee p_{13} \vee p_{16}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{11} \vee p_{13}} \qquad \overline{p}_{11} \vee p_6 \vee \overline{p}_{12}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{11} \vee p_{13} \vee p_6}}$$

Note also that all obtained clauses are false in assignment.

Motivating Example (4)

1. $\overline{p}_{11} \vee p_6 \vee \overline{p}_{12}$
2. $\overline{p}_{11} \vee p_{13} \vee p_{16}$
3. $p_{12} \vee \overline{p}_{16} \vee \overline{p}_2$
4. $\overline{p}_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}$
5. $p_{10} \vee \overline{p}_8 \vee p_1$
6. $p_{10} \vee p_3$
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13. $\overline{p}_{22} \vee \overline{p}_{13}$
14. $p_{13} \vee p_8$
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Remember M is $p_{25}\, \overline{p}^d_{21}\, \overline{p}_6\, \overline{p}_{17}\, p^d_{22}\, \overline{p}_{13}\, p_8\, p^d_4\, p_{19}\, \overline{p}^d_{20}\, p_{23}$

Literal:	p^d_{11}	$\overline{p}_{12}$	p_{16}	$\overline{p}_2$	$\overline{p}_{10}$	p_1	p_3	p_{26}	$\overline{p}_5$	p_{18}
Reason:	$\emptyset$	1	2	3	4	5	6	7	8	9

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_{18} \qquad \overline{p}_1 \vee \overline{p}_3 \vee p_5 \vee p_{17} \vee p_{18}}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_5 \vee p_{17}} \qquad p_{10} \vee \overline{p}_5}{\overline{p}_3 \vee \overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}} \qquad p_{10} \vee p_3}{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10}}$$

$$\dfrac{\dfrac{\dfrac{\overline{p}_{19} \vee \overline{p}_1 \vee p_{17} \vee p_{10} \qquad p_{10} \vee \overline{p}_8 \vee p_1}{\overline{p}_{19} \vee p_{17} \vee p_{10} \vee \overline{p}_8} \qquad p_2 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{10}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee p_2 \vee \overline{p}_4 \vee p_{20}} \qquad p_{12} \vee \overline{p}_{16} \vee \overline{p}_2}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{16}}$$

$$\dfrac{\dfrac{\dfrac{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{16} \qquad \overline{p}_{11} \vee p_{13} \vee p_{16}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee p_{12} \vee \overline{p}_{11} \vee p_{13}} \qquad \overline{p}_{11} \vee p_6 \vee \overline{p}_{12}}{\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{11} \vee p_{13} \vee p_6}$$

Now, in blue lits false at the current decision level (5)

Motivating Example (5)

- Three clauses with only one literal assigned at the last DL (5):
  - $\overline{p}_{19} \vee p_{17} \vee p_{10} \vee \overline{p}_8$ (max DL of others:3)
  - $\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee p_2 \vee \overline{p}_4 \vee p_{20}$ (max DL of others:4)
  - $\overline{p}_{19} \vee p_{17} \vee \overline{p}_8 \vee \overline{p}_4 \vee p_{20} \vee \overline{p}_{11} \vee p_{13} \vee p_6$ (max DL of others:4)
- If we had had those clauses:
  - At DL. 3 we could've propagated p_{10}
  - At DL. 4 we could've propagated p_2
  - At DL. 4 we could've propagated $\overline{p}_{11}$
- In practice procedure stops as soon as such clause is found, as:
  - It is the cheapest one to find
  - It can propagate literals at a lower DL

Backjump Rule

This example motivates us to introduce the rule:

Backjump

$$M\, l^{\mathsf{d}}\, N \parallel F \Longrightarrow M\, l' \parallel F \quad \textbf{if} \begin{cases} \text{for some clause } C \vee l' : \\ \quad F \models C \vee l' \text{ and } M \models \neg C \\ \quad l' \text{ is undefined in } M \\ \quad l' \text{ or } \neg l' \text{ occurs in } F \end{cases}$$

The only thing we need is a backjump clause $C \vee l'$ such that:

1. It is a logical consequence of the rest of the clauses
2. All its literals are false at some previous decision level d, except one which was undefined at d

Conflict Analysis

- The procedure shown in the example is called conflict analysis
- Why the obtained clause is a logical consequence of the input?
  - Because resolution is correct

Conflict Analysis (2)

- The procedure shown in the example is called conflict analysis
- Why always a false clause with only one lit set at the last decision level (dl) is obtained?
  - Conflicting clause has at least two lits false at dl (provided UnitProp applied before any decision)
  - Each non-decision lit l false at dl can be resolved away. l is replaced by lits $l_1, \ldots, l_n$ such that:
    1. All of them are false
    2. All of them have been added to the assignment before l (hence their decision level is $\leq dl$)
    3. At least one was set at dl (again, provided ...)
  - By 3, obtained clauses contain at least one lit false at dl
  - Procedure terminates because of 2. In the worst case, with last decision lit being the only set to false at dl

Lemma Learning

- Every time a conflict is found, conflict analysis is started
- Backjump clause is added to the clause database:

  Learn

  $$M \parallel F \Longrightarrow M \parallel F, C \textbf{ if } \begin{cases} \text{all atoms of } C \text{ occur in } F \\ F \models C \end{cases}$$

- Backjump clauses are usually known as lemmas
- Learning them helps to prevent future similar conflicts

Lemma Removal

- Effects of adding lemmas:
  - \+ Reduces the search space
  - \- Space traversal slower since UnitProp becomes expensive
- Hence we cannot keep all generated lemmas. We need:

  Forget

  $M \parallel F, C \Longrightarrow M \parallel F$ **if** $F \models C$

- Which lemmas to keep and which ones to forget?
  - Each lemma has a number called activity
  - Activity incremented when lemma is used in conflict analysis
  - From time to time, lemmas with low activity are removed
  - Mixed policies: short lemmas, recent lemmas kept, ...

Decision Heuristic

- SAT instances have thousands of variables
- We can't keep enough lemmas to store info about all vars
- Most SAT instances have clusters of variables:
  sets of variables that are semantically linked

GOAL: force the SAT solver to work on one cluster at a time

- Each var/lit has an associated activity
- Each time it appears in a conflict analysis, activity incremented
- Recent activity should be given more importance:
  all activities are divided by a constant factor from time to time
- Decide chooses unassigned lit with highest activity
- Note that heuristic does not depend on clauses: CHEAP!

Restarts

- Sometimes SAT solver gets trapped in parts of the search space
- Restarts are introduced to overcome this problem:

  Restart

  $$M \parallel F \Longrightarrow \emptyset \parallel F$$

- Unrestricted application of Restart leads to non-termination
- Restart is applied with increasing periodicity
  (inner-outer geometric sequence, Luby sequence)

Efficient UnitProp: Occur Lists

- Most time of the SAT solver ($\approx 80\%$) is spent on UnitProp (also called BCP, Boolean Constraint Propagation)
- Critical to have efficient BCP!
- BCP only has to detect unit or conflicting clauses (there is no need to detect that all clauses are true)
- Occur lists data structure

  Instead of traversing the whole clause set again and again:

  - For each literal, store the clauses where it appears
  - Every time a new lit l is added to the assignment, only clauses containing $\bar{l}$ need to be visited

Efficient UnitProp: 2-watched literals scheme

- How to improve on occur lists?
- Clauses with 2 non-false lits can't be unit or conflicting
- For each clause we will try to watch two non-false literals
- Enough to visit a clause when a watched literal becomes false
- Advantages
  - Each clause is visited far less often
  - Upon backtrack, nothing has to be done
  - Inactive literals tend to be watched,
    hence further reducing the number of clauses to be visited
  - Very effective for long clauses (e.g. lemmas)
  - For binary clauses specialized data structures are used

Overall CDCL Algorithm

```
while(true){

    while (propagate_gives_conflict()){
        if (decision_level==0) return UNSAT;
        else analyze_conflict();
    }

    restart_if_applicable();
    remove_lemmas_if_applicable();

    if (!decide()) returns SAT; // All vars assigned
}
```

Why Are SAT Solvers Really Good?

Three key ingredients that only work if used TOGETHER:

1. Learn at each conflict the backjump clause as a lemma:
   - makes UnitProp more powerful
   - prevents future similar conflicts

2. Decide on variable with most occurrences in recent conflicts:
   - so-called activity-based heuristics
   - idea: work off clusters of related variables + first fail pp.

3. Forget from time to time low-activity lemmas:
   - crucial to keep UnitProp fast and afford memory usage
   - idea: lemmas from worked off clusters no longer needed!

Bibliography - Some further reading

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