

Smoothed Analysis of the Expected Number of Maximal Points in Two Dimensions

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Abstract

The *Maximal* points in a set S are those that aren't *dominated* by any other point in S . Such points arise in multiple application settings in which they are called by a variety of different names, e.g., maxima, Pareto optimums, skylines. Because of their ubiquity, there is a large literature on the *expected* number of maxima in a set S of n points chosen IID from some distribution. Most such results assume that the underlying distribution is uniform over some spatial region and strongly use this uniformity in their analysis.

This work was initially motivated by the question of how this expected number changes if the input distribution is perturbed by random noise. More specifically, let $\mathbf{B}_p$ denote the uniform distribution from the 2-d unit L_p ball, $\delta\mathbf{B}_q$ denote the 2-d L_q -ball, of radius δ and $\mathbf{B}_p + \delta\mathbf{B}_q$ be the convolution of the two distributions, i.e., a point $v \in \mathbf{B}_p$ is reported with an error chosen from $\delta\mathbf{B}_q$. The question is how the expected number of maxima change as a function of δ . Although the original motivation is for small δ the problem is well defined for any δ and our analysis treats the general case.

More specifically, we study, as a function of n, δ , the expected number of maximal points when the n points in S are chosen IID from distributions of the type $\mathbf{B}_p + \delta\mathbf{B}_q$ where $p, q \in \{1, 2, \infty\}$ for $\delta > 0$ and also of the type $\mathbf{B}_\infty + \delta\mathbf{B}_q$ where $q \in [1, \infty)$ for $\delta > 0$.

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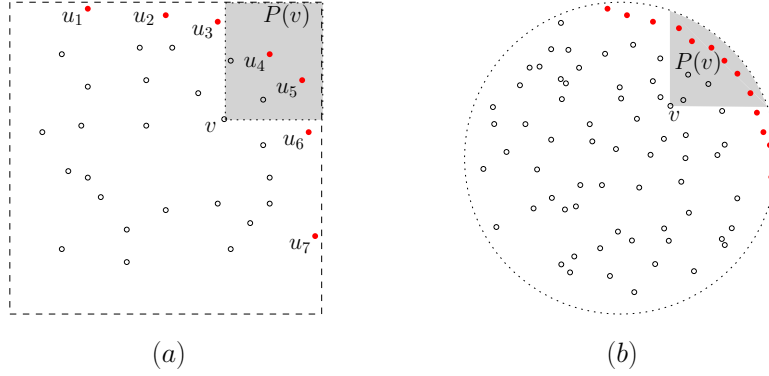


Figure 1: The diagram shows $\text{MAX}(S_n)$ for two point sets S_n . In both (a) and (b) the circles – both empty and filled – denote the points in S_n and the filled circles are $\text{MAX}(S_n)$. If the points are considered as being drawn from region D $P(v)$ denotes the region in D that dominates v . In (a) D is the dotted square and in (b) D is the dotted circle.

1 Introduction

Let S be a set of 2-dimensional points. The *largest* points in S are its *maximal points* of S and are a well-studied object. More formally¹

Definition 1 For $u \in \mathbb{R}^2$ let $u.x$ ($u.y$) denote the x (y) coordinate of u . For $u, v \in \mathbb{R}^2$, u is dominated by v if $u \neq v$, $u.x \leq v.x$ and $u.y \leq v.y$. If $S \subset \mathbb{R}^2$ then

$$\text{MAX}(S) = \{u \in S : u \text{ is not dominated by any point in } S \setminus \{u\}\}.$$

$\text{MAX}(S)$ are the maximal points of S .

The problems of finding and estimating the number of maximal points of a set in $\mathbb{R}^2$, appear very often in many fields under different denominations, *maximal vectors*, *skylines*, *Pareto frontier/points* and others, see, e.g., [18, 5, 17, 12, 14], and for a more exhaustive history of the problems and further references, Sections 1 and 2 in [7].

¹We restrict our definition to $S \subset \mathbb{R}^2$ because that is what this paper addresses; the concept of maxima generalize naturally to $S \subset \mathbb{R}^d$ for $d > 2$ and have been well-studied there as well. We discuss this in more detail in the Conclusions and Extensions section.

Recall that the L_p metric for points u, v in the d -dimensional space is defined by

$$\|u - v\|_p = \begin{cases} \left(\sum_{i=1}^d |u_i - v_i|^p \right)^{1/p} & \text{for real } p \geq 0, \\ \max_i (|u_i - v_i|) & \text{if } p = \infty. \end{cases}$$

Let S_n denote a set of n points chosen Independently Identically Distributed (IID) from some 2-D distribution $\mathbf{D}$ and $M_n = |\text{MAX}(S_n)|$ be the random variable counting the number of maximal points in S_n . Because maxima are so ubiquitous, understanding the expected number of maxima has been important in many areas and many properties of M_n have been studied.

More specifically, if $\mathbf{D}$ is the uniform distribution drawn from an L_p ball with $p \geq 1$ then, it is well known [12, 2, 13, 6] that

- If $p = \infty$, then $\mathbf{E}[M_n] = H_n \sim \ln n$.

The same result holds if the points are drawn from some distribution $\mathbf{D} = (\mathbf{X}, \mathbf{Y})$ where $\mathbf{X}$ and $\mathbf{Y}$ are ANY two 1-dimensional distributions that are independent of each other.

- If $p > 1$, then

$$\lim_{n \rightarrow \infty} \frac{\mathbf{E}[M_n]}{\sqrt{n}} = C_p$$

where C_p is a constant dependent only upon p .

- Similar results to the above, i.e., that $\mathbf{E}[M_n] = O(\sqrt{n})$, derived using similar techniques, are known if $\mathbf{D}$ is a *uniform* distribution from ANY convex region [11].

It is also known [15, 16] that if the n points are chosen IID from a 2-D Gaussian distribution then $\mathbf{E}[M_n] \sim \ln n$. There are also generalizations of these results (both the $\mathbf{B}_p$ ones and the Gaussian one) to higher dimensions. See [13] for a table containing most known results.

Surprisingly, given the importance of the problem, not much else is known. The motivation for this work is to extend the family of distributions for which $\mathbf{E}[M_n]$ can be derived.

Consider a point u that is originally generated from some uniform distribution over a unit L_p ball but, has some δ error in the L_q metric when measured or reported. The actual reported point can be equivalently considered as being chosen from a new distribution which we denote by $\mathbf{B}_p + \delta \mathbf{B}_q$

(the next section provides formal definitions). Note that the support of this distribution is the Minkowski sum of the two balls.

As an example, Figure 2 shows the support of $\mathbf{B}_\infty + \delta\mathbf{B}_1$. In the diagram, the shaded inner square is the unit ∞ -ball. A point u_1 chosen from that square is then perturbed by the addition of another point u_2 , drawn uniformly from the B_1 ball with radius δ . The support of this convoluted distribution is the interior of the dotted region in the figure.

Note that the distribution is NOT uniform in this support. Towards the centre the density is uniform but it decreases approaching the boundary of the support where it becomes zero. Note too that the rate of decrease differs in different parts of the support. It is this non-uniformity that will cause complications in calculating $\mathbf{E}[M_n]$.

Although the problem described above was for small δ it is well defined for all $\delta > 0$ which is what we analyze in this paper. More specifically, the motivation for the present work is twofold:

- Explain how $\mathbf{E}[M_n]$ changes when the distribution is perturbed and
- Increase the families of distributions for which $\mathbf{E}[M_n]$ is understood.

The idea of analyzing how quantities change under perturbations is *smoothed analysis* [20, 21]. In the classic setting, smoothed analysis of the number of maxima would mean analyzing how, given a *fixed* set S_n , $\mathbf{E}[M_n]$ would change under small perturbations (as a function of the original set S_n). This was the approach in [9, 8] (see also similar work for convex hulls in [10]). This paper differs in that it is the *Distribution* that is being smoothed (or convoluted) and not the point set. This paper also differs from recent work [22, 1] on the *most-likely* skyline and convex hull problems in that those papers assume that each point has a given probability distribution and they are attempting to find the subset of points that has the highest probability of being the skyline (or convex hull).

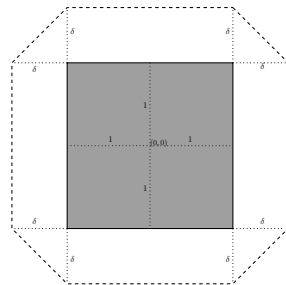


Figure 2: $\mathbf{B}_\infty + \delta\mathbf{B}_1$.

2 Definitions and Results

Definition 2 $p \in [1, \infty)$ or $p \geq 1$ will denote that p is a real number ≥ 1 . $p \in [1, \infty]$ will denote that $p \in [1, \infty)$ OR $p = \infty$.

Definition 3 Let $\mathbf{D}$ be a distribution over $\mathbb{R}^2$.

- If $\delta \geq 0$, the distribution $\delta\mathbf{D}$ is generated by choosing a point u using $\mathbf{D}$ and then returning the point $(\delta \cdot u.x, \delta \cdot u.y)$.
- Let $\mathbf{D}_1, \mathbf{D}_2$ be two distributions over $\mathbb{R}^2$. $\mathbf{D}_1 + \mathbf{D}_2$ is the convolution of $\mathbf{D}_1, \mathbf{D}_2$. It is generated by choosing a point u_1 from $\mathbf{D}_1$ and a point u_2 from $\mathbf{D}_2$ and returning $(u_1.x + u_2.x, u_1.y + u_2.y)$.
- A set of $S_n = \{u_1, \dots, u_n\}$ is Chosen from $\mathbf{D}$ if the u_i are IID with each u being generated using distribution $\mathbf{D}$.

Definition 4 Let $D \subseteq \mathbb{R}^2$ be a set and $\delta \geq 0$.

Then $\delta D = \{(\delta \cdot u.x, \delta \cdot u.y) : u \in D\}$.

The Minkowski sum of sets D_1 and D_2 is

$$D_1 + D_2 = \{(u_1.x + u_2.x, u_1.y + u_2.y) : u_1 \in D_1, u_2 \in D_2\}.$$

If $u \in \mathbb{R}^2$, let $u + D$ will denote the set $\{u\} + D$.

Definition 5 Let $u \in \mathbb{R}^2$, $r > 0$ and $p \in [1, \infty]$,

- The L_p ball of radius r around u is

$$B_p(u, r) = \{(x, y) : |x - u.x|^p + |y - u.y|^p \leq r^p\}.$$

- The L_∞ ball of radius r around u is

$$B_\infty(u, r) = \{(x, y) : \max(|x - u.x|, |y - u.y|) \leq r\}.$$

Let $B_p = B_p((0, 0), 1)$ and $B_\infty = B_\infty((0, 0), 1)$ denote the respective unit balls and $a_p = \text{Area}(B_p)$, $a_\infty = \text{Area}(B_\infty) = 2$ denote their respective areas.

- For all $p \in [1, \infty]$, $\mathbf{B}_p$ will denote the uniform distribution that selects a point uniformly from B_p . This distribution has support B_p with uniform density $1/a_p$ within B_p .
- $\mathbf{B}_p + \delta\mathbf{B}_q$ will be the convolution of distributions $\mathbf{B}_p$ and $\delta\mathbf{B}_q$. This distribution's support is the Minkowski sum $B_p + \delta B_q$. Note that the density of $\mathbf{B}_p + \delta\mathbf{B}_q$ is NOT uniform in $B_p + \delta B_q$.

The main result of this paper is

Theorem 1 Fix p, q so that either $p, q \in \{1, 2, \infty\}$ or $p = \infty$ and $q \geq 1$. Let S_n be n points chosen from the distribution $\mathbf{B}_p + \delta \mathbf{B}_q$ and $M_n = |\text{MAX}(S_n)|$. Let $\delta \geq 0$ be a function of n . Then $\mathbf{E}[M_n]$ behaves as below.

	(a)	(b)	(c)	(d)	(e)	(f)
	D =	$0 \leq \delta$				$\delta = 1$
(i)	$\mathbf{B}_\infty + \delta \mathbf{B}_\infty$	$\Theta(\ln n)$				$\Theta(\ln n)$
		$\delta \leq \frac{1}{\sqrt{n}}$	$\frac{1}{\sqrt{n}} \leq \delta \leq 1$	$1 \leq \delta \leq \sqrt{n}$	$\sqrt{n} \leq \delta$	
(ii)	$\mathbf{B}_1 + \delta \mathbf{B}_1$	$\Theta(\sqrt{n})$	$\Theta\left(\frac{n^{1/3}}{\delta^{1/3}}\right)$	$\Theta\left(\delta^{1/3} n^{1/3}\right)$	$\Theta(\sqrt{n})$	$\Theta\left(n^{1/3}\right)$
(iii)	$\mathbf{B}_2 + \delta \mathbf{B}_2$	$\Theta(\sqrt{n})$	$\Theta\left(\frac{n^{2/7}}{\delta^{3/7}}\right)$	$\Theta\left(\delta^{3/7} n^{2/7}\right)$	$\Theta(\sqrt{n})$	$\Theta\left(n^{2/7}\right)$
		$\delta \leq \frac{1}{\sqrt{n}}$	$\frac{1}{\sqrt{n}} \leq \delta \leq \sqrt{n}$		$\sqrt{n} \leq \delta$	
(iv)	$\mathbf{B}_\infty + \delta \mathbf{B}_q$	$\Theta(\ln n)$	$\Theta\left(\ln n + \sqrt{\delta} n^{1/4}\right)$		$\Theta(\sqrt{n})$	$\Theta\left(n^{1/4}\right)$
		$\delta \leq \frac{1}{\sqrt{n}}$	$\frac{1}{\sqrt{n}} \leq \delta \leq n^{1/26}$	$n^{1/26} \leq \delta \leq \sqrt{n}$	$\sqrt{n} \leq \delta$	
(v)	$\mathbf{B}_1 + \delta \mathbf{B}_2$	$\Theta(\sqrt{n})$	$\Theta\left(\frac{n^{2/7}}{\delta^{3/7}}\right)$	$\Theta\left(\sqrt{\delta} n^{1/4}\right)$	$\Theta(\sqrt{n})$	$\Theta\left(n^{2/7}\right)$

Observations: In $\mathbf{B}_p + \delta \mathbf{B}_q$

- When $p = q = \infty$, M_n has exactly the same distribution as if S_n were chosen from $\mathbf{B}_\infty$ so this is an uninteresting case.
- When δ is small enough ($\leq 1/\sqrt{n}$), $\mathbf{E}[M_n]$ behaves almost as if S_n were chosen from $\mathbf{B}_p$ and when δ is large enough ($\geq \sqrt{n}$) it behaves almost as if S_n were chosen from $\mathbf{B}_q$.
- Later Lemma 9 will show that M_n has the same distribution for S_n chosen from both $\mathbf{B}_p + \delta \mathbf{B}_q$ and $\mathbf{B}_q + \frac{1}{\delta} \mathbf{B}_p$. Thus row (iv) gives the behavior for $\mathbf{B}_q + \delta \mathbf{B}_\infty$ for any $q \geq 1$ and row (v) the behavior for $\mathbf{B}_2 + \delta \mathbf{B}_1$.
- When $p = q \in \{1, 2\}$ the behavior starts at $\Theta(\sqrt{n})$, smoothly decreases until reaching $\delta = 1$ and then increases again until reaching $\Theta(\sqrt{n})$. The behavior in the middle is different for 1 and 2. In both cases there is symmetry between δ and $1/\delta$ (from Lemma 9).
- When $p = 1, q = 2$ there is no symmetry. Behavior starts at $\Theta(\sqrt{n})$, decreases to $\Theta(n^{7/26})$ at $\delta = n^{1/26}$ and then increases again at a different rate to $\Theta(\sqrt{n})$.
- When $p = \infty$, the behavior is asymptotically equivalent for all $q \in [1, \infty)$, not just $q = 1, 2$. The only difference is in the value of the constant hidden by

the Θ . The behavior starts at $\Theta(\ln n)$, stays there for a short while and then smoothly increases to $\Theta(\sqrt{n})$.

3 Basic Lemmas

The following collection of Lemmas comprise the basic toolkit used to derive Theorem 1. They are only stated here, with complete proofs being provided in Section 5.

Definition 6 Let $\mathbf{D}$ be a distribution over $\mathbb{R}^2$, $x \in \mathbb{R}^2$ and $A \subset \mathbb{R}^2$ a measurable region.

- $f_{\mathbf{D}}(x)$ will denote the density function of $\mathbf{D}$.
- $\mu_{\mathbf{D}}(A) = \int_A f_{\mathbf{D}}(x)dx$ will denote the measure of A .

If $\mathbf{D}$ is understood we often simply write $f(x)$ and $\mu(A)$.

Definition 7 Let $D \subseteq \mathbb{R}^2$, $v \in D$ and $A \subseteq D$.

- $P(v) = \{u \in D : u \text{ dominates } v\} \cup \{v\}$.
- $P(A) = \bigcup_{v \in A} P(v)$.
- A is dominant in D or a dominant region in D , if $P(A) = A$.

Note that, by definition, $\forall v \in D$, $P(v)$ is a dominant region in D .

Lemma 1 Let v and S_n be chosen from $\mathbf{D}$ and $A \subseteq D$. Then

$$\begin{aligned} (a) \quad \Pr(v \in A) &= \mu(A). \\ (b) \quad \mathbf{E}[|A \cap S_n|] &= n\mu(A). \\ (c) \quad \Pr(A \cap S_n = \emptyset) &= (1 - \mu(A))^n. \end{aligned}$$

The following observation will be used to prove most of our lower bounds.

Lemma 2 (Lower Bound) Let S_n be chosen from $\mathbf{D}$. Further let $A_1, A_2, \dots, A_m$ be a collection of pairwise disjoint dominant regions in D with $\mu(A_i) = \Omega(1/n)$ for all i . Then

$$\mathbf{E}[M_n] \geq \mathbf{E}\left[\left|\text{MAX}\left(S_n \cap \bigcup_{i=1}^m A_i\right)\right|\right] = \Omega(m).$$

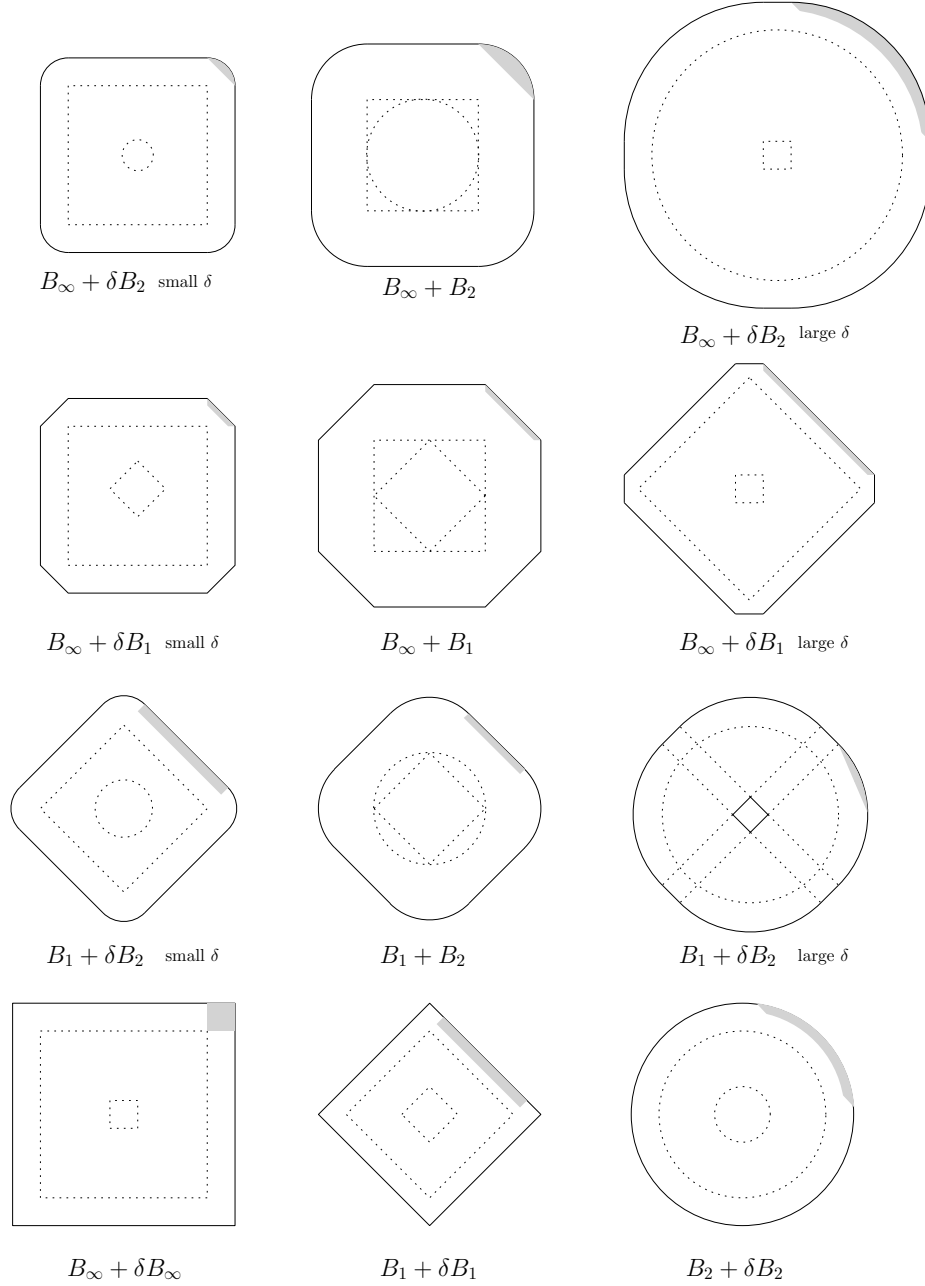


Figure 3: Illustrations of the supports of some of the different distributions in the form $\mathbf{B}_p + \delta \mathbf{B}_q$ examined in Theorem 1. The dotted lines denote the B_p and δB_q balls centered at 0. Note that in all cases the density is uniform near the center of the support but then decreases to 0 as the boundary is approached. The gray areas denote, approximately, where the maxima of S_n are concentrated.

Definition 8 Let $D = B_p + \delta B_q$. For $v \in D$ define

$$P'(v) = B_q(v, \delta) \cap B_p = (v + \delta B_q) \cap B_p,$$

the preimage of point v in B_p .

Lemma 3 Fix $p, q \in [1, \infty]$. Let $\mathbf{D} = \mathbf{B}_p + \delta \mathbf{B}_q$ and let v be a point chosen from $\mathbf{D}$. Let $A \subseteq \mathbb{R}^2$. Then

$$\begin{aligned} f(v) &= \frac{1}{a_p a_q} \frac{\text{Area}(\{u \in B_p : v - u \in \delta B_q\})}{\delta^2} \\ &= \frac{1}{a_p a_q} \frac{\text{Area}(P'v)}{\delta^2} \end{aligned} \quad (1)$$

$$\mu(A) = \frac{1}{a_p a_q} \int_{u \in B_p} \frac{\text{Area}((u + \delta B_q) \cap A)}{\delta^2} du. \quad (2)$$

Lemma 4 Fix $p, q \in [1, \infty]$. Let $\mathbf{D} = \mathbf{B}_p + \delta \mathbf{B}_q$ and $\kappa > 0$ be any constant.

- (a) If $v \in D \Rightarrow f(v) = O(1)$.
- (b) If $v \in B_p \Rightarrow f(v) = \begin{cases} \Theta(1) & \text{if } \delta \leq \kappa \\ \Theta\left(\frac{1}{\delta^2}\right) & \text{if } \delta > \kappa \end{cases}$
- (c) If $A \subseteq D \Rightarrow \mu(A) = O(\text{Area}(A))$.
- (d) If $A \subseteq B_p \Rightarrow \mu(A) = \begin{cases} \Theta(\text{Area}(A)) & \text{if } \delta \leq \kappa \\ \Theta\left(\frac{\text{Area}(A)}{\delta^2}\right) & \text{if } \delta > \kappa \end{cases}$

The constants implicit in the $O()$ in (a) and (c) are only dependent upon p, q while the constants implicit in the $\Theta()$ in (b) and (d) are only dependent upon p, q, κ .

Definition 9 (Asymptotic and Continuous Dominance)

Let $\mathbf{D}$ be any distribution with a continuous density function $f(u)$. Let A, B be two disjoint regions in the support D .

We say that A asymptotically dominates B in measure if

1. A, B are parameterized by $t \in [0, T]$, i.e.,

- $A(T) = A; B(T) = B$.
- (Monotonicity in t) $\forall t_1 < t_2, A(t_1) \subseteq A(t_2)$ and $B(t_1) \subseteq B(t_2)$.

2. $\forall t, \mu(B(t)) = O(\mu(A(t)))$.

If A asymptotically dominates B in measure and, in addition,

$$\forall t \in [0, T], \quad \text{if } v \in B \setminus B(t) \text{ and } u \in A(t) \quad \Rightarrow \quad u \text{ dominates } v.$$

we further say that A continuously dominates B .

Lemma 5 (Mirror) Let $\mathbf{D}$ be any distribution with a continuous density function $f(u)$ and S_n a set of points chosen from $\mathbf{D}$. Let $A, B \subset D$ such that A asymptotically dominates B in measure and define the random variable

$$t' = \begin{cases} \min\{t : A(t) \cap S_n \neq \emptyset\}, & \text{if } A \cap S_n \neq \emptyset, \\ T & \text{if } A \cap S_n = \emptyset. \end{cases} \quad \text{and} \quad X = |S_n \cup B(t')|.$$

Then

$$\mathbf{E}[X(t')] = O(1).$$

Lemma 6 (Sweep) Let $\mathbf{D}$ be any distribution with a continuous density function $f(u)$ and S_n a set of points chosen from $\mathbf{D}$. Let $A, B \subset D$ such that A continuously dominates B . Then,

$$\mathbf{E}[|S_n \cap \text{MAX}(S_n)|] = O(1).$$

Corollary 7 Fix $p, q \in [1, \infty]$ and choose S_n from $\mathbf{D} = \mathbf{B}_p + \delta \mathbf{B}_q$. Let Q_1 be the upper-right quadrant of the plane and O_1 the first octant, i.e.,

$$Q_1 = \{u \in \mathbb{R}^2 : 0 \leq u.x, 0 \leq u.y\} \quad \text{and} \quad O_1 = \{u \in \mathbb{R}^2 : 0 \leq u.y \leq u.x\}$$

Then

$$\mathbf{E}[M_n] = \mathbf{E}[|\text{MAX}(S_n)|] = \mathbf{E}[|Q_1 \cap \text{MAX}(S_n)|] + O(1) \quad (3)$$

$$= \Theta\left(\mathbf{E}[|O_1 \cap \text{MAX}(S_n)|]\right). \quad (4)$$

Proof: Set

$$A = D \cap \{u \in \mathbb{R}^2 : u.y \geq 0\}, \quad B = D \cap \{u \in \mathbb{R}^2 : u.y < 0\}.$$

For $t \in [0, 2]$ set

$$A(t) = \{u \in A : u.x \geq 1 - t\}, \quad B(t) = \{u \in B : u.x \geq 1 - t\}.$$

Condition (10 pf Definition 9 trivially hold. Condition (2) holds because, by symmetry around the x -axis $\mu(B(t)) = \mu(A(t))$. The final continuous

domination condition holds because every point in $B \setminus B(t)$ is below and to the left of every point in $A(t)$. Thus the expected number of maximal points in S_n below the x -axis is $O(1)$. Note that this is *independent* of n . Similarly, the expected number of maximal points to the left of the y -axis is $O(1)$. This proves Eq. 3

To prove Eq. 4 define the second octant to be

$$O_2 = \{u \in \mathbb{R}^2 : 0 \leq u.x \leq u.y\}.$$

By the symmetry between the x and y coordinates in the distribution,

$$\mathbf{E}[|O_1 \cap \text{MAX}(S_n)|] = \mathbf{E}[|O_2 \cap \text{MAX}(S_n)|].$$

Futhermore, since O_1 and O_2 partition Q_1 ,

$$\begin{aligned} \mathbf{E}[|Q_1 \cap \text{MAX}(S_n)|] &= \mathbf{E}[|O_1 \cap \text{MAX}(S_n)|] + \mathbf{E}[|O_2 \cap \text{MAX}(S_n)|] \\ &= 2\mathbf{E}[|O_1 \cap \text{MAX}(S_n)|]. \end{aligned}$$

Thus

$$\mathbf{E}[M_n] = \mathbf{E}[Q_1 \cap |\text{MAX}(S_n)|] + O(1) = \Theta(\mathbf{E}[|O_1 \cap \text{MAX}(S_n)|]).$$

□

The Sweep lemma will not usually be used in isolation but by partitioning a region into multiple strips and showing that each strip contains $O(1)$ maxima. The following formulation is used many times

Lemma 8 (Strip) *Let $\mathbf{D}$ be any distribution with a continuous density function $f(u)$ and S_n a set of points chosen from $\mathbf{D}$. Let $\bar{B}_1, \dots, \bar{B}_m, A_1, \dots, A_m$ be regions in D where, for all i ,*

- $\bar{B}_i = B_i \cup B'_i$ where
- $\mu(B'_i) = O(1/n)$ and
- A_i continuously dominates B_i .

Then

$$\mathbf{E}\left[\left|\text{MAX}(S_n) \cap \left(\bigcup_{i=1}^m \bar{B}_i\right)\right|\right] = O(m).$$

Proof: From Lemma 6, $\forall i, \mathbf{E} [|\text{MAX}(S_n) \cap B_i|] = O(1)$. Thus

$$\begin{aligned} \mathbf{E} [|\text{MAX}(S_n) \cap \bar{B}_i|] &\leq \mathbf{E} [|\text{MAX}(S_n) \cap B_i|] + \mathbf{E} [|\text{MAX}(S_n) \cap B'_i|] \\ &\leq O(1) + \mathbf{E} [|\text{MAX}(S_n) \cap B'_i|] = O(1) + n\mu(B'_i) = O(1). \end{aligned}$$

So

$$\mathbf{E} \left[\left| \text{MAX}(S_n) \cap \left(\bigcup_{i=1}^m \bar{B}_i \right) \right| \right] \leq \sum_{i=1}^m \mathbf{E} [|\text{MAX}(S_n) \cap \bar{B}_i|] = O(m).$$

□

The fact that for $\delta > 0$, u dominates v if and only if δu dominates δv implies

Lemma 9 (Scaling) Fix $p, q \in [1, \infty]$, $\mathbf{D} = \mathbf{B}_p + \delta \mathbf{B}_q$ and $\mathbf{D}' = \mathbf{B}_q + \frac{1}{\delta} \mathbf{B}_p$. Let S_n be n points chosen from $\mathbf{D}$ and S'_n n points chosen from $\mathbf{D}'$. Then $\text{MAX}(S_n)$ and $\text{MAX}(S'_n)$ have exactly the same distribution. In particular

$$\mathbf{E} [|\text{MAX}(S_n)|] = \mathbf{E} [|\text{MAX}(S'_n)|].$$

Lemma 10 (Rectangle Behavior) Let $\mathbf{D}$ be any distribution with a continuous density function $f(u)$. Rectangle $R = [\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \subseteq D$ is coordinate-independent rectangle if

$$\forall u_1, u_2 \in R, \quad \text{if } u_1.x = u_2.x \text{ then } f(u_1) = f(u_2).$$

Let S_n be chosen from $\mathbf{D}$ and $R \subseteq D$ be a coordinate-independent rectangle. Then

$$\mathbf{E} [|\text{MAX}(S_n \cap R)|] = \mathbf{E} [\ln X] = \begin{cases} \Theta(\ln n) & \text{if } \mu(R) = \Omega(n^{-\epsilon}) \text{ for any } \epsilon > 0, \\ O(\ln n) & \text{In all cases.} \end{cases}$$

where X is a $B(n, n\mu(R))$ Binomial Random Variable.

Proof: (of Lemma 10).

The distribution of $S_n \cap R$ is equivalent to the one generated by

- (a) Choosing random variable X from a binomial distribution $B(n, \mu(R))$.
- (b) For $v \in R$, setting $\bar{f}(v) = \frac{f(v)}{\mu(R)}$ to be the probability density function for choosing a point v from $\mathbf{D}$ conditioned on knowing that $v \in R$.

(c) Choosing X points (in R) from the distribution defined by $\bar{f}(v)$.

In particular, point (b) implies that if $u_1.x = u_2.x$ then $\bar{f}(u_1) = \bar{f}(u_2)$. As stated in the introduction, the number of maxima for m points chosen from such a distribution behaves exactly as if m points are chosen from $\mathbf{B}_\infty$. Thus, if m points are chosen using $\bar{f}(u)$, the expected number of maxima among them will be $\Theta(\ln m)$. Thus

$$\mathbf{E}[\text{MAX}(S_n \cap R)] = \sum_{m=0}^n \Pr(X = m) \Theta(\ln m) = \mathbf{E}[\ln X].$$

Part (1) immediately follows. $\square$

Lemma 11 (Limiting Behavior) *Let $p \in [1, \infty]$, $q \in [1, \infty)$, $\delta = O(1/\sqrt{n})$ and S_n chosen from $\mathbf{D} = \mathbf{B}_p + \delta \mathbf{B}_q$. Then*

$$\mathbf{E}[M_n] = \begin{cases} \Theta(\ln n) & \text{if } p = \infty, \\ \Theta(\sqrt{n}) & \text{if } p \neq \infty. \end{cases}$$

Note that if u chosen from $\mathbf{B}_p$, $u.x$ and $u.y$ are independent random variables. Thus, for any $\delta > 0$ if v is chosen from $\mathbf{D} = \mathbf{B}_\infty + \delta \mathbf{B}_\infty$ $v.x$ and $v.y$ are independent random variables. As noted in the introduction, this means that if S_n is chosen from $\mathbf{D}$, $\mathbf{E}[M_n]$ is exactly the same as if S_n was just chosen from $\mathbf{B}_\infty$, i.e., $\mathbf{E}[M_n] = \Theta(\ln n)$.

Now note that Lemma 11 combined with Lemma 9 immediately imply the limiting behavior in columns (b) and (e) of the table in Theorem 1. Note too that for rows (ii) and (iii), column (d) follows directly from applying Lemma 9 to column (c).

Thus, proving Theorem 1 reduces to proving cells (ii) c, (iii) c, (iv) c,d and (v) c,d. In the next sections we sketch how to derive these results with full proofs relegated to the appendix.

4 The General Approach

4.1 A Simple Example: $\mathbf{B}_1$.

Before sketching our results it is instructive to see how the Lemmas in the previous section can be used to re-derive that fact that, if $\mathbf{D} = \mathbf{B}_1$ then $\mathbf{E}[M_n] = \Theta(\sqrt{n})$. This is illustrated in Figure 4.

Even though the behavior for $\mathbf{D} = \mathbf{B}_1$ is already well known we provide this to illustrate the generic steps for deriving $\mathbf{E}[M_n]$. These are exactly

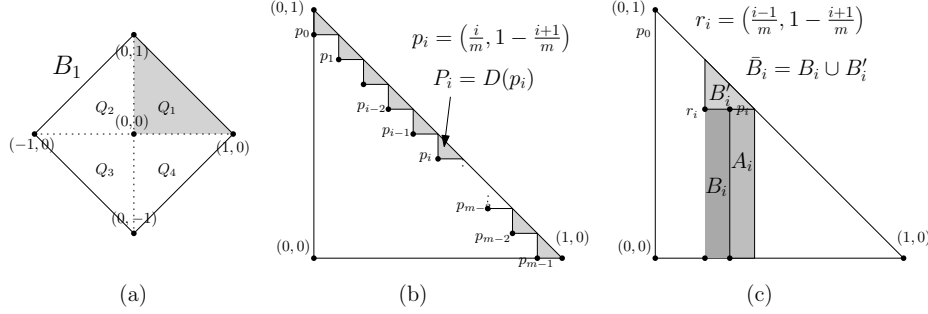


Figure 4: Illustration of proof $\mathbf{E}[M_n] = \Theta(\sqrt{n})$ when S_n is chosen from $\mathbf{B}_1$. All but $O(1)$ maxima will be in quadrant Q_1 ; (b) and (c) are restricted to Q_1 . (b) illustrates the lower bound and (c) the upper bound.

the same steps that are needed when $\mathbf{D} = \mathbf{B}_p + \delta \mathbf{B}_q$ and permits identifying where the complications can arise in those more general cases.

Set $m = \lfloor \sqrt{n} \rfloor$ and let p_i, r_i be the points defined in the figure with $P_i = P(p_i)$ and $B'_i = P(r_i)$. Also set

$$B_i = \left\{ (x, y) : \frac{i-1}{m} \leq x \leq \frac{i}{m}, 0 \leq y \leq 1 - \frac{i+1}{m} \right\}, \quad A_i = \left(\frac{1}{m}, 0 \right) + B_i$$

and $\bar{B}_i = B_i \cup B'_i$. Finally, for $0 \leq t \leq (1+i)/m$ set $B_i(t) = B_i \cap \{(x, y) : y \leq (1+i)/m - t\}$ and $A_i(t) = (\frac{1}{m}, 0) + B_i(t)$. The steps in the derivation are.

1. *Restricting to first Quadrant:* Corollary 7 implies that it is only necessary to analyze $\mathbf{E}[|Q_1 \cap \text{MAX}(S_n)|]$.
2. *Calculating Density and Measure:* Because $\mathbf{D}$ has a uniform density, $\mu(A) = \Theta(\text{Area}(A))$ for all regions $A \subseteq D$.
3. *Lower Bound:* The P_i are a collection of m pairwise disjoint dominant regions with $\mu(P_i) = \Theta(\text{Area}(P_i)) = \Theta(m^{-2}) = \Theta(1/n)$. Thus, from Lemma 2, $\mathbf{E}[M_n] = \Omega(m) = \Omega(\sqrt{n})$.
4. *Upper bound:* Note that $Q_1 \cap D = \left(\bigcup_{i=1}^{m-1} \bar{B}_i \right) \cup B'_m$ so

$$\mathbf{E}[|\text{MAX}(S_n) \cap Q_1|] = \mathbf{E} \left[\left| \text{MAX}(S_n) \cap \left(\bigcup_{i=1}^m \bar{B}_i \right) \right| \right] + \mathbf{E}[|\text{MAX}(S_n) \cap B'_m|].$$

Furthermore, $\forall i, \mu(B'_i) = \Theta(\text{Area}(B'_i)) = \Theta(1/n)$. In particular,

$$\mathbf{E} [| \text{MAX}(S_n) \cap B'_m |] \leq \mathbf{E} [| S_n \cap B'_m |] = O(n\mu(B'_m)) = O(1).$$

The crucial observation is that, $\forall i$, A_i continuously dominates B_i as defined in Lemma 6. Thus, plugging into Lemma 8 (Strip) yields

$$\mathbf{E} [| \text{MAX}(S_n) \cap Q_1 |] \leq O(m-1) + \mathbf{E} [| \text{MAX}(S_n) \cap B'_m |] = O(m) = O(\sqrt{n}).$$

Combining (3) and (4) gives the final result

$$|Q_1 \cap \text{MAX}(S_n)| \leq \sum_i \Theta(1) + \sum_i \Theta(1) = \Theta(m) = \Theta(\sqrt{n}).$$

4.2 The General Approach

The proof of Theorem 1 will require case-by-case analyses of $\mathbf{D} = \mathbf{B}_p + \delta\mathbf{B}_q$ for different pairs p, q . The analysis for each pair will follow exactly the same 5 steps as the analysis of $\mathbf{B}_1$ above.

Step 1: Restricting the analysis to quadrant Q_1 .

This will be the same for every case and will usually not be explicitly mentioned.

Step 2: Deriving the measure.

This will often be quite cumbersome. While Lemma 3 provides an integral formula this, in many cases, is unusable. The density varies quite widely near the border of the support D , which is where most of the maxima are located. A substantial amount of work is involved in finding usable functional representations for the densities/measures in different parts of the support. The results will be stated at the start of the section with the proof deferred to the end of the section.

Step 3: Deriving the lower bound.

This is usually a simple application of Lemma 2, given the results of Step 2.

Step 4: Deriving the upper bound,

This is the hardest step. It is usually derived using the sweep lemma with the difficulties arising from how to specify the regions to be swept. This strongly depends upon *how* the measure is represented .

5 Proofs of Basic Lemmas

Proof: of Lemma 2.

First note that, from Lemma 1, $\Pr(S_n \cap A_i = \emptyset) = (1 - \mu(A_i))^n$. Thus $\mu(A_i) = \Omega(1/n)$ implies

$$\Pr(S_n \cap A_i \neq \emptyset) = 1 - \Pr(S_n \cap A_i = \emptyset) = \Omega(1).$$

If region A is dominant then points in A can only be dominated by other points in A so $A \cap \text{MAX}(S_n) = \text{MAX}(S_n \cap A)$. Since each A_i is dominant, this implies

$$\mathbf{E} [|\text{MAX}(S_n) \cap A_i|] \geq \Pr(S_n \cap A_i \neq \emptyset) = \Omega(1).$$

Finally, since the A_i are pairwise disjoint,

$$\begin{aligned} \mathbf{E} [|\text{MAX}(S_n)|] &\geq \mathbf{E} \left[\left| \text{MAX}(S_n) \cap \left(\bigcup_i A_i \right) \right| \right] \\ &= \mathbf{E} \left[\left| \bigcup_i (\text{MAX}(S_n) \cap A_i) \right| \right] \\ &= \sum_{i=1}^m \mathbf{E} [|\text{MAX}(S_n) \cap A_i|] \geq \sum_{i=1}^m \Omega(1) = \Omega(m). \end{aligned}$$

□

Proof: of Lemma 3:

Note that for $u \in B + p$, $f_{\mathbf{B}_p}(u) = \frac{1}{a_p}$ and for $u' \in \delta B_q$, $f_{\delta \mathbf{B}_q}(u') = \frac{1}{a_q \delta^2}$.

To see Eq. 2 note that

$$\begin{aligned} \mu(A) &= \int_{u \in \mathbf{B}_p} \left(\int_{\substack{w \in \delta B_q \\ u+w \in A}} f_{\delta \mathbf{B}_q}(w) dw \right) f_{\mathbf{B}_p}(u) du \\ &= \frac{1}{a_p a_q \delta^2} \int_{u \in B_p} \left(\int_{\substack{w \in \delta B_q \\ u+w \in A}} dw \right) du \\ &= \frac{1}{a_p a_q} \int_{u \in B_p} \frac{\text{Area}((u + \delta B_q) \cap A)}{\delta^2} du. \end{aligned}$$

For Eq. 1 first note that

$$\begin{aligned}
\mu(A) &= \frac{1}{a_p a_q \delta^2} \int_{u \in B_p} \left(\int_{\substack{w \in \delta B_q \\ u+w \in A}} dw \right) du \\
&= \frac{1}{a_p a_q \delta^2} \int_{u \in B_p} \left(\int_{\substack{v \in u + \delta B_q \\ v \in A}} dv \right) du \\
&= \frac{1}{a_p a_q \delta^2} \int_{v \in A} \left(\int_{\substack{u \in v - \delta B_q \\ u \in \delta B_p}} du \right) dv \\
&= \frac{1}{a_p a_q} \int_{v \in A} \frac{\text{Area} \{u \in B_p : v - u \in \delta B_q\}}{\delta^2} dv
\end{aligned} \tag{5}$$

where (5) comes from the change of variables $v = u + w$. Differentiating around v yields Eq. 1. $\square$

Proof: of Lemma 4:

The proof for (a) follows easily from the fact that, for all u ,

$$\text{Area}(B_p \cap (u + \delta B_q)) \leq \text{Area}(u + \delta B_q) = \text{Area}(\delta B_q) = a_q \delta^2,$$

so from Eq. 1, $f(v) = O(1)$. Furthermore, if $u \in B_p$ then

$$\text{Area}(B_p \cap (u + \delta B_q)) \geq c \text{Area}(u + \delta B_q) = c a_q \delta^2$$

where c is only dependent upon p, q, κ . Thus, again from Eq. 1, $f(v) = \Theta(1)$, proving (b). The proofs for (c) and (d) follow from plugging these inequalities into Eq. 2. $\square$

Proof: of Mirror Lemma (Lemma 5):

Without loss of generality smoothly rescale t so that $\mu(A(t)) = t$, and thus $T = \mu(A)$.

The informal intuition of the Lemma is that since the “first” point in A appears when the sweep line is t' , $\mu(A(t')) \sim \frac{1}{n}$. Since B is asymptotically dominated in measure by A , $\mu(B(t')) = O(1/n)$ and thus $\mathbf{E}[X(t')] = n\mu(B(t')) = O(1)$.

Note that by the continuity of the measure we know that $\Pr(|S_n \cap A(t')| = 1) = 1$. That is, we may assume that $|D \setminus A(t')| = n - 1$.

Now assume that t' is known. *Conditioned on known t'* , the remaining $n - 1$ points in S_n are chosen from $D \setminus A(t')$ with the associated conditional

distribution. More specifically, if u is any one of those $n - 1$ points.

$$\Pr(u \in B(t') \mid t') = \frac{\mu(B(t'))}{\mu(D \setminus A(t'))} = \frac{\mu(B(t'))}{1 - \mu(A(t'))}.$$

Thus, conditioning on t' , and applying Lemma 1 (b)

$$\mathbf{E}[X(t') \mid t'] = (n - 1) \frac{\mu(B(t'))}{1 - \mu(A(t'))}$$

and therefore

$$\mathbf{E}[X(t')] = \mathbf{E}[\mathbf{E}[X(t') \mid t']] = \mathbf{E}\left[(n - 1) \frac{\mu(B(t'))}{1 - \mu(A(t'))}\right].$$

From the definition of t' and Lemma 1 (c), $\mu(A(t')) > 1/2$ with exponentially low probability. Therefore, recalling that $\mu(A(t)) = t$,

$$\begin{aligned} \mathbf{E}[X(t')] &= (n - 1) \mathbf{E}[O(\mu(B(t')))] \\ &= (n - 1) \mathbf{E}[O(\mu(A(t')))] = (n - 1) O(\mathbf{E}[t']). \end{aligned}$$

Another application of Lemma 1 (c) shows

$$\begin{aligned} \mathbf{E}[t'] &= \int_{\alpha=0}^T \Pr(t' \geq \alpha) d\alpha = \int_{\alpha=0}^T (1 - \mu(A(\alpha)))^{n-1} d\alpha \\ &= \int_{\alpha=0}^T e^{-(n-1)\mu(A(\alpha))} (1 + O((n-1)\mu^2(A(\alpha)))) d\alpha \\ &= \int_{\alpha=0}^T e^{-(n-1)\alpha} (1 + O((n-1)\alpha^2)) d\alpha = O\left(\frac{1}{n-1}\right). \end{aligned}$$

Thus $\mathbf{E}[X(t')] = O(1)$. $\square$

Proof: of Sweep Lemma (Lemma 6):

From the setup in Lemma 5, for all t , all points in $B \setminus B(t)$ are dominated by all points in $A(t)$. By the definition of t' , $A(t') \cap S_n$ contains (exactly) one point. Thus no point in $(B \setminus B(t')) \cap S_n$ can be maximal, i.e.,

$$\text{MAX}(S_n) \cap (B \setminus B(t')) = \emptyset.$$

The proof follows from

$$\mathbf{E}[|\text{MAX}(S_n) \cap B|] = \mathbf{E}[|\text{MAX}(S_n) \cap B(t')|] \leq \mathbf{E}[|S_n \cap B(t')|] = O(1).$$

$\square$

Proof: of Lemma 9:

Let $S_n = \{u_1, \dots, u_n\}$ be chosen from $\mathbf{D}$. Recall that the process of choosing point u from $\mathbf{D}$ is to choose w from $\mathbf{B}_p$, v from $\mathbf{B}_q$ and return $u = w + \delta v$. Choosing a point u' from $\mathbf{D}'$ is the same except that it returns $p' = v + \frac{1}{\delta}w = \frac{1}{\delta}u_i$. Thus the distribution of choosing $S_n = \{u_1, \dots, u_n\}$ from $\mathbf{D}$ is exactly the same as choosing $S_n = \{\frac{1}{\delta}u_1, \dots, \frac{1}{\delta}u_n\}$ from $\mathbf{D}'$.

Finally, note that dominance is invariant under multiplication by a scalar, i.e., p_i dominates p_j if and only if $\frac{1}{\delta}u_i$ dominates $\frac{1}{\delta}u_j$. Thus $|\text{MAX}(S_n)|$ and $|\text{MAX}(S'_n)|$ have the same distribution and $\mathbf{E}[|\text{MAX}(S_n)|] = \mathbf{E}[|\text{MAX}(S'_n)|]$. $\square$

The proof of Lemma 11 will need an observation that will be reused multiple times in the analysis of $\mathbf{B}_\infty + \delta\mathbf{B}_q$ and is therefore stated first, in its own lemma.

Lemma 12 Recall $P(v)$ from Definition 7. Fix $q \in [1, \infty)$ and set $\mathbf{D} = \mathbf{B}_\infty + \delta\mathbf{B}_q$. Let $\Delta \in (2\delta, 1/2)$. Define regions (Fig. 5(a))

$$\begin{aligned} T(\Delta) &= P((z, 0)) \cap P((0, z)), \\ R_1(\Delta) &= P((z, 0)) \setminus T(\Delta), \\ R_2(\Delta) &= P((0, z)) \setminus T(\Delta). \end{aligned}$$

(a) Then

$$\mathbf{E}[|\text{MAX}(S_n \cap R_1(\Delta))|] = O(\ln n). \quad (6)$$

(b) Furthermore, if $\mu(R_1(\Delta)) = \Omega(1/\sqrt{n})$, then²

$$\mathbf{E}\left[|\text{MAX}(S_n \cap R_1(\Delta))| \mid S_n \cap T(\Delta) = \emptyset\right] = \Omega(\ln n). \quad (7)$$

(c) Parts (a) and (b) remain correct if $R_1(\Delta)$ is replaced by $R_2(\Delta)$.

Proof: Suppose $u, v \in R_1(\Delta)$ satisfy $u.x = v.x$. Note that $v + \delta B_q$ is just $u + \delta B_q$ shifted vertically $v, x - u.x$. Then, because $\Delta \in (2\delta, 1/2)$,

$$\text{Area}((u + \delta B_q) \cap B_\infty) = \text{Area}((v + \delta B_q) \cap B_\infty).$$

(In Fig. 5(a) this is illustrated by noting that the two white cross-hatched areas are vertically shifted versions of each other.) Thus, from Lemma 3, $f(u) = f(v)$.

² $R \mid B$ denotes random variable R conditioned upon event B .

Plugging into Lemma 10 with $R = R_1(\Delta)$ immediately implies

$$\mathbf{E} [|\text{MAX}(S_n \cap R_1(\Delta))|] \leq \mathbf{E} [|\text{MAX}(S_n) \cap R_1(\Delta)|] = O(\ln n),$$

proving (a).

To prove (b) assume that the only information known is that $S_n \cap T(\Delta) = \emptyset$. Conditioned on this event, the new density for $v \in (B_\infty + \delta B_q) \setminus T(\Delta)$ is $\hat{f}(v) = \frac{f(v)}{1 - \mu(T(\Delta))} > f(v)$. In particular, this implies that $\hat{\mu}(R_1(\Delta)) = \Omega(1/\sqrt{n})$. It also implies that if $u, v \in R$ and $u.x = v.x$ then $\bar{f}(u) = \bar{f}(v)$.

We now again apply Lemma 10 with $R = R_1(\Delta)$ (this assumes that S_n is chosen from $\bar{\mathbf{D}}$ defined as the distribution on $B_\infty + \delta B_q \setminus T(\Delta)$ with density $\bar{f}(u)$) to see that

$$\mathbf{E} \left[|\text{MAX}(S_n) \cap R_1(\Delta)| \mid S_n \cap T(\Delta) = \emptyset \right] = \Theta(\ln n).$$

The proof of part (b) follows from the fact that the only points that can dominate points in $R_1(\Delta)$ are points in $T(\Delta)$ so

$$\mathbf{E} \left[|\text{MAX}(S_n \cap R_1(\Delta))| \mid S_n \cap T(\Delta) = \emptyset \right] = \mathbf{E} \left[|\text{MAX}(S_n) \cap R_1(\Delta)| \mid S_n \cap T(\Delta) = \emptyset \right]$$

By symmetry, the proof of (c) is exactly the same as the proofs of (a) and (b). $\square$

Proof: of Lemma 11: See Fig. 5.

The proofs of the cases (a) $p \neq \infty$ and (b) $p = \infty$ are done separately.

(a) $p = \infty$: Corollary 7 states that $\mathbf{E} [M_n] = \mathbf{E} [|\text{MAX}(S_n) \cap Q_1|] + O(1)$. We therefore only need to analyze $\mathbf{E} [|\text{MAX}(S_n) \cap Q_1|]$.

Define $T(\Delta)$, $R_1(\Delta)$ and $R_2(\Delta)$ as in Lemma 12. Also set

$$U(\Delta) = (B_\infty + \delta B_q) \setminus (T(\Delta) \cup R_1(\Delta) \cup R_2(\Delta)).$$

By decomposition and symmetry

$$\begin{aligned} \mathbf{E} [|\text{MAX}(S_n) \cap Q_1|] &= \mathbf{E} [|\text{MAX}(S_n) \cap R_1(\Delta)|] + \mathbf{E} [|\text{MAX}(S_n) \cap R_2(\Delta)|] \\ &\quad + \mathbf{E} [|\text{MAX}(S_n) \cap T(\Delta)|] + \mathbf{E} [|\text{MAX}(S_n) \cap U(\Delta)|] \quad (8) \\ &= 2\mathbf{E} [|\text{MAX}(S_n) \cap R_1(\Delta)|] \\ &\quad + \mathbf{E} [|\text{MAX}(S_n) \cap T(\Delta)|] + \mathbf{E} [|\text{MAX}(S_n) \cap U(\Delta)|]. \end{aligned}$$

Upper Bound:

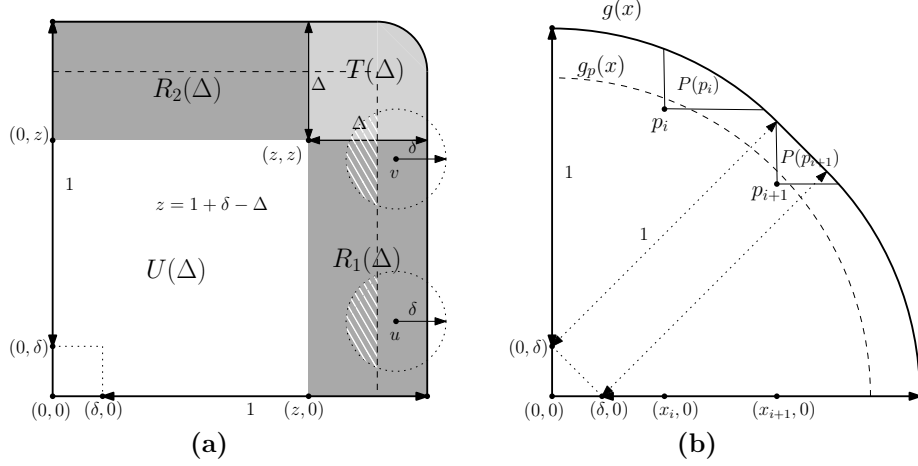


Figure 5: Illustration of the proofs of Lemmas 11 and 12. Note that only the support in the upper-right quadrant is shown. (a) illustrates the proof of Lemma 12 which is used in the proof of Lemma 11, when $p = \infty$. (b) illustrates the upper bound proof in Lemma 11 when $p \neq \infty$. Note that for clarity, diagrams are not drawn to scale. In (a), $\delta = O(1/\sqrt{n})$ and thus the part of $B_p + \delta B_q$ outside of B_p is extremely thin.

Let $\Delta = c\sqrt{\frac{\ln n}{n}} + \delta$ for constant c to be chosen later. From Lemma 4(d) $\mu(T(\Delta)) = \Theta\left(\frac{\ln n}{n}\right)$, where the constants implicit in the Θ grow to ∞ as c increases.

This immediately implies that

$$\mathbf{E}[|\text{MAX}(S_n) \cap T(\Delta)|] \leq \mathbf{E}[|S_n \cap T(\Delta)|] = n\mu(T(\Delta)) = O(\ln n). \quad (9)$$

From Lemma 1 (c), for any $c' > 0$ we can choose c large enough so that

$$\Pr(S_n \cap T(\Delta) = \emptyset) = (1 - \mu(T(\Delta)))^n = O(n^{-c'}).$$

Let $c' = 1$.

Next note that since any point in $T(\Delta)$ dominates ALL points in $U(\Delta)$, $S_n \cap T(\Delta) \neq \emptyset$ implies $|\text{MAX}(S_n) \cap U(\Delta)| = 0$. Since $\text{MAX}(S_n) \leq n$,

$$\mathbf{E}[|\text{MAX}(S_n) \cap U(\Delta)|] \leq n \Pr(S_n \cap T(\Delta) = \emptyset) = O(1). \quad (10)$$

Finally, note that from Lemma 12(a)

$$|\text{MAX}(S_n) \cap R_1(\Delta)| \leq |\text{MAX}(S_n \cap R_1(\Delta))| = O(\ln n). \quad (11)$$

Plugging Eqs. 9, 10 and 11 into Eq. 8 proves

$$\mathbf{E}[M_n] = O(\ln n).$$

Lower Bound:

Let $\Delta = \frac{1}{\sqrt{n}} + 2\delta$. From Lemma 4(c), $\mu(T(\Delta)) = O(1/n)$. From Lemma 4(c) (d), $\mu(R_1(\Delta)) = \Theta(1/\sqrt{n})$.

Thus, from Lemma 1, for large enough n ,

$$\Pr(S_n \cap T(\Delta) = \emptyset) = (1 - \mu(T(\Delta)))^n = \Omega(1).$$

Next observe that the only points outside of $R_1(\Delta)$ that can dominate a point in $R_1(\Delta)$ are points in $T(\Delta)$. Thus, if $S_n \cap T(\Delta) = \emptyset$ then

$$\text{MAX}(S_n) \cap R_1(\Delta) = \text{MAX}(S_n \cap R_1(\Delta)).$$

This combined with Lemma 12(b) yields

$$\begin{aligned} \mathbf{E}[M_n] &\geq \mathbf{E}[|\text{MAX}(S_n) \cap R_1(\Delta)|] \\ &\geq \mathbf{E}\left[|\text{MAX}(S_n) \cap R_1(\Delta)| \mid S_n \cap T(\Delta) = \emptyset\right] \cdot \Pr(S_n \cap T(\Delta) = \emptyset) \\ &= \mathbf{E}\left[|\text{MAX}(S_n \cap R_1(\Delta))| \mid S_n \cap T(\Delta) = \emptyset\right] \cdot \Pr(S_n \cap T(\Delta) = \emptyset) \\ &= \Omega(\ln n) \cdot \Omega(1) = \Omega(\ln n). \end{aligned}$$

(b) $p \neq \infty$:

Upper Bound:

Define $\bar{B}_p = B_p((0, 0), 1 - \delta)$ and $\bar{\mathbf{B}}_p$ as the uniform distribution over $\bar{B}_p$.

Let $S'_m = \{v'_1, v'_2, \dots, v'_n\}$ be a set m points chosen from $\bar{\mathbf{B}}_p$. Since scaling a L_p ball this way does not change the distribution of the number of maxima $\mathbf{E}[\text{MAX}(S'_m)] = \Theta(\sqrt{m})$.

Now let S_n be n points chosen from $\mathbf{B}_p + \delta\mathbf{B}_q$. Then

$$\begin{aligned} \mathbf{E}[|\text{MAX}(S_n)|] &= \mathbf{E}[|\text{MAX}(S_n) \cap \bar{B}_p|] + \mathbf{E}[|\text{MAX}(S_n) \cap ((B_p + \delta B_q) \setminus \bar{B}_p)|] \\ &\leq \mathbf{E}[|\text{MAX}(S_n \cap B_p)|] + \mathbf{E}[|S_n \cap ((B_p + \delta B_q) \setminus \bar{B}_p)|] \\ &\leq \mathbf{E}[|\text{MAX}(S_n \cap \bar{B}_p)|] + n\mu((B_p + \delta B_q) \setminus \bar{B}_p) \\ &= \mathbf{E}[|\text{MAX}(S_n \cap \bar{B}_p)|] + O(\sqrt{n}), \end{aligned}$$

where the last line uses the fact that

$$\begin{aligned}\mu((B_p + \delta B_q) \setminus B_p) &= O(\text{Area}((B_p + \delta B_q) \setminus B_p)) + O(\text{Area}(B_p \setminus \bar{B}_p)) \\ &= O(\delta) + O(\delta) = O\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

From the triangle inequality, if $v \in \bar{B}_p$ then $v + \delta B_q \subset B_p$. Thus $\{u \in B_p : v - u \in \delta B_q\} = B_q(v, \delta)$ and

$$f(v) = \frac{1}{a_p a_q} \frac{\text{Area}(\{u \in B_p : v - u \in \delta B_q\})}{\delta^2} = \frac{1}{a_p a_q} \frac{\text{Area}(B_q(v, \delta))}{\delta^2} = \frac{1}{a_p}.$$

Now note that $S_n \cap \bar{B}_p$ has an equivalent distribution to

1. Choosing random variable X , from a binomial distribution $B(n, \mu(\bar{B}_p))$.
2. For $v \in \bar{B}_p$, setting $\bar{f}(v) = \frac{f(v)}{\mu(\bar{B}_p)}$. $\bar{f}(v)$ is the conditional probability density function for choosing a point v from $\mathbf{B}_p + \delta \mathbf{B}_q$ conditioned on knowing that $v \in B_p$.
3. Choosing X points (in $\bar{B}_p$) from the distribution defined by $\bar{f}(p)$.

But since $f(v)$ is constant for $v \in \bar{B}_p$, $\bar{f}(v)$ is constant as well. This means that $S_n \cap \bar{B}_p$ has the same distribution as X points chosen uniformly from the ball $\bar{B}_p$. Conditioning on X this implies

$$\mathbf{E}[|MAX(S_n \cap \bar{B}_p)| \mid X] = \Theta(\sqrt{X}).$$

But $X \leq n$ so

$$\mathbf{E}[|MAX(S_n \cap B_p)|] = O(\sqrt{n})$$

and thus

$$\mathbf{E}[|MAX(S_n)|] \leq \mathbf{E}[|MAX(S_n \cap B_p)|] + O(\sqrt{n}) = O(\sqrt{n}).$$

Lower Bound See Fig. 5(a).

Let $g_p(x) = (1 - x^q)^{1/q}$ be the equation of the upper boundary of B_p for $0 \leq x \leq 1$. The Theorem of the Mean states that if $x' < x$, then

$$g_p(x) - g_p(x') = g'_p(z)(x - x'),$$

for some $z \in [x', x]$. In particular, since $g'_p(z)$ is bounded for $z \in [1/3, 2/3]$

$$\text{if } x, x' \in [1/3, 2/3] \quad \Rightarrow \quad g(x) = g(x') + \Theta(x - x').$$

Now let $g(x)$ be the curve of the upper boundary of $B_p + \delta B_q$. By construction $g(x)$ is monotonically decreasing for $x \in [0, 1]$ and is concave down. Because of the construction, we also know that $g(x)$ is within a distance $\Theta(1/\sqrt{n})$ of $g_p(x)$ in the following sense: there exists a $c > 0$ such that

$$g_p(x) \leq g(x) \leq g_p(x) + c/\sqrt{n} \quad \text{and} \quad g_p^{-1}(y) \leq g^{-1}(y) \leq g_p^{-1}(y) + c/\sqrt{n}.$$

Now set $\beta = (3 + c)/\sqrt{n}$, $t = \lfloor 1/3\beta \rfloor$ and, for $i = 0, 1, \dots, t - 1$

$$\begin{aligned} x_i &= \frac{1}{3} + i\beta, \\ y_i &= g_p(x_i) - \frac{1}{\sqrt{n}}, \\ p_i &= (x_i, y_{i+1}). \end{aligned}$$

Note that by construction, $p_i \in B_p$ and, furthermore $g_p(x_i) - y_{i+1} = \Theta(1/\sqrt{n})$ and $g^{-1}(y_{i+1}) - x_i = \Theta(1/\sqrt{n})$. Thus $\text{Area}(P(p_i)) \cap B_p = \Omega(1/n)$ and, thus, by Lemma 4 (c), $\mu(P(p_i)) = \Omega(1/n)$. Since, again by construction, if $i \neq j$, $P(p_i) \cap P(p_j) = \emptyset$ we can apply Lemma 2 to derive

$$\mathbf{E}[|MAX(S_n)|] = \Omega(t) = \Omega(\sqrt{n}),$$

and are done. □

6 Analysis of $\mathbf{B}_1 + \delta \mathbf{B}_1$

This section derives cell (ii)(c) in Theorem 1, that is, if n points are chosen from $\mathbf{D} = \mathbf{B}_1 + \delta \mathbf{B}_1$ and $\frac{1}{\sqrt{n}} \leq \delta \leq 1$ then $\mathbf{E}[M_n] = \Theta\left(\frac{n^{1/3}}{\delta^{1/3}}\right)$. Applying Lemma 9 for $1 \leq \delta \leq \sqrt{n}$ yields cell (ii)(d), i.e., that $\mathbf{E}[M_n] = \Theta(\delta^{1/3} n^{1/3})$.

This section follows the general structure laid out in Section 4.2 and will need the following special definitions.

Definition 10 Let $D_1 = (B_1 + \delta B_1) \cap Q_1$. Set

$$\begin{aligned} I &= D_1 \cap B_1, & U &= D_1 \cap \{u \in \mathbb{R}^2 : u.y \geq 1\}, \\ M &= D_1 \setminus (I \cup U \cup B), & B &= D_1 \cap \{u \in \mathbb{R}^2 : u.x \geq 1\}. \end{aligned}$$

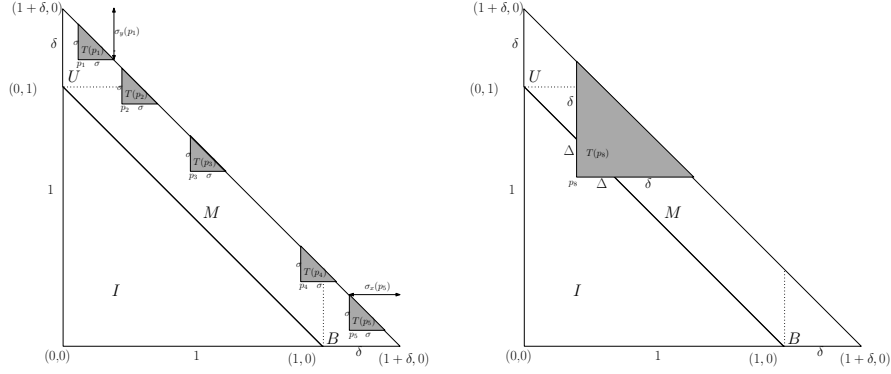


Figure 6: Illustration of Definition 10 and Lemma 13.

In the following Lemma, we deal with $D = B_1 + \delta B_1$ and $D_1 = D \cap Q_1$ as given in Definition 10, and for all $p \in D_1$ we consider $P(p)$ introduced in Definition 7 (see Fig. 6). Notice $P(p)$ is the isosceles right triangle with base p and hypotenuse flush with the upper-right border of $B_1 + \delta B_1$. For notational convenience, to emphasise that $P(p)$ is a triangle, set $T(p) = P(p)$. In particular notice that $T(p)$ is a dominant region. Further set

$$\begin{aligned} \sigma(p) &= 1 + \delta - p.x - p.y && \text{side length of triangle } T(p), \\ \Delta_x(p) &= 1 + \delta - p.x && p\text{'s } x\text{-distance to rightmost point in } B_1 + \delta B_1, \\ \Delta_y(p) &= 1 + \delta - p.y && p\text{'s } y\text{-distance to highest point in } B_1 + \delta B_1. \end{aligned}$$

Lemma 13 *Let D , D_1 and for every $p \in D_1$, $T(p)$ defined as above. For p chosen from $\mathbf{D} = \mathbf{B}_1 + \delta \mathbf{B}_1$, the formula for calculating $\mu(T(p))$ differs by location of p , as below:*

1. If $p \in U \Rightarrow \mu(T(p)) = \Theta\left(\Delta_y(p) \frac{\sigma^3(p)}{\delta^2}\right) = O\left(\frac{\sigma^3(p)}{\delta}\right)$.
2. If $p \in B \Rightarrow \mu(T(p)) = \Theta\left(\Delta_x(p) \frac{\sigma^3(p)}{\delta^2}\right) = O\left(\frac{\sigma^3(p)}{\delta}\right)$.
3. If $p \in M \Rightarrow \mu(T(p)) = \Theta\left(\frac{\sigma^3(p)}{\delta}\right)$.
4. If $p \in I \Rightarrow \mu(T(p)) = \Theta(\sigma^2(p))$.

In Fig. 6, p_1 is an example for case 1, p_5 for case 2, p_2, p_3 and p_4 for case 3, and p_8 for case 4.

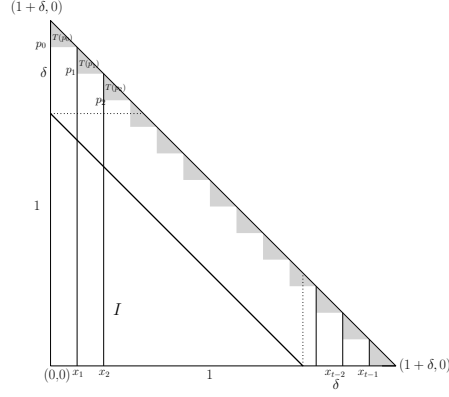


Figure 7: The lower bound construction for $\mathbf{B}_1 + \delta \mathbf{B}_1$. Note that the $T(p_i)$ (in gray) are pairwise disjoint dominant regions. Their horizontal and vertical sides all have length σ which is chosen to force $\mu(T(p_i)) = \Theta(1/n)$.

Before proving the Lemma, we use it to derive the upper and lower asymptotic bounds on $\mathbf{E}[M_n]$.

Lemma 14 *Let S_n be n points chosen from the distribution $\mathbf{D} = \mathbf{B}_1 + \delta \mathbf{B}_1$ with $\frac{1}{\sqrt{n}} \leq \delta \leq 1$. Then*

$$\mathbf{E}[M_n] = \Omega\left(\frac{n^{1/3}}{\delta^{1/3}}\right).$$

Proof: The proof creates $m = \Omega\left(\frac{n^{1/3}}{\delta^{1/3}}\right)$ points p_i such that, $\forall i \neq j$, $T(p_i) \cap T(p_j) = \emptyset$ and, $\forall i$, $\mu(T(p_i)) = \Theta(\frac{1}{n})$. See Figure 7. Since the $T(p_i)$ are all dominant regions, Lemma 2 then immediately implies that $\mathbf{E}[M_n] = \Omega(m) = \Omega\left(\frac{n^{1/3}}{\delta^{1/3}}\right)$.

Start by fixing $\sigma = \left(\frac{\delta}{n}\right)^{1/3}$, setting $m' = \lfloor \frac{1+\delta}{\sigma} \rfloor$ and, for $i = 0, \dots, m'$ defining

$$x_i = i\sigma, \quad y_i = 1 + \delta - x_i - \sigma, \quad p_i = (x_i, y_i).$$

Note that $m'\sigma \leq 1 + \delta$ so, if $i \leq m'$, $p_i \in D_1$. Note that

$$\sigma(p_i) = 1 + \delta - x_i - y_i = \sigma.$$

Thus, by the construction, if $i \neq j$ then $T(p_i) \cap T(p_j) = \emptyset$. Next note that, by definition,

$$\frac{\sigma^3}{\delta} = \frac{1}{n}. \quad (12)$$

Finally note that because $\frac{1}{\sqrt{n}} \leq \delta$, $\sigma^3 \leq \frac{\delta}{n} \leq \delta^3$ so $\sigma \leq \delta$. Thus, for $i \leq m' - 1$, $p_i(x) \in B \cup U \cup M$. Now let i satisfy $\frac{m'}{3} \leq i \leq \frac{2m'}{3}$. Then

$$\frac{1}{3}(1 + \delta) \leq x_i \leq \frac{2}{3}(1 + \delta), \quad (13)$$

and one of the following must be true

- $p(x_i) \in M$.
- $p(x_i) \in B$. Then $\Delta_x(p_i) = 1 + \delta - x_i \geq \frac{1}{3}(1 + \delta) \geq \frac{1}{3}$.
Also, by definition, $\Delta_x(p_i) \leq \delta \leq 1$.
Together these imply $\frac{\Delta_x(p_i)}{\delta} = \Theta(1)$.
- $p(x_i) \in U$. A symmetric analysis shows $\frac{\Delta_y(p_i)}{\delta} = \Theta(1)$.

Applying Cases 1-3 in Lemma 13 then shows that for all i satisfying Eq. 13, $\mu(T(p_i)) = \Omega(1/n)$.

There are $m = m/3$ values of i satisfying Eq. 13, and thus, from Lemma 2.

$$\mathbf{E}[M_n] = \Omega\left(\frac{m'}{3}\right) = \Omega\left(\left(\frac{n}{\delta}\right)^{1/3}\right).$$

Note: The motivation for the restriction (13) is that, if δ is close to 1, then almost no p_i would be in M . It is therefore necessary to analyze the measure of $T(p_i)$ for $p_i \in B, U$. But, in those cases, if i is close to 0 or m' then $\mu(T(p_i))$ could be $o(1/n)$. The proof therefore needs to bound away from those corners. $\square$

The upper bound is more technical.

Lemma 15 *Let S_n be n points chosen from the distribution $\mathbf{D} = \mathbf{B}_1 + \delta\mathbf{B}_1$ with $\frac{1}{\sqrt{n}} \leq \delta \leq 1$. Then*

$$\mathbf{E}[M_n] = O\left(\frac{n^{1/3}}{\delta^{1/3}}\right).$$

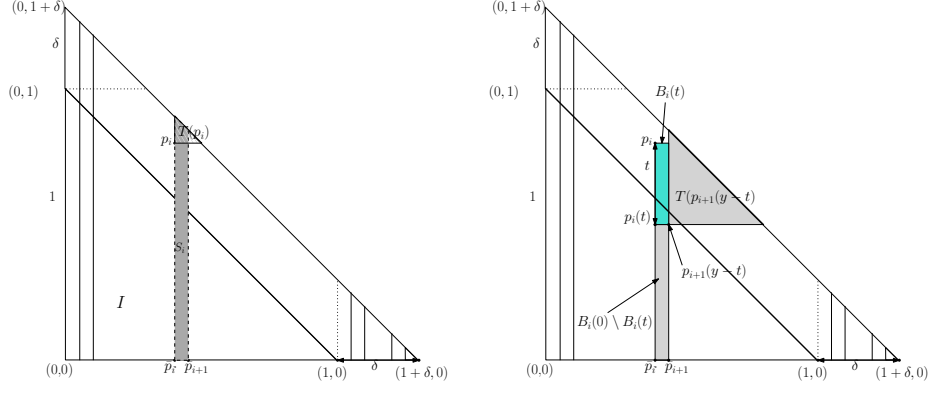


Figure 8: Illustration of the proof of Lemma 15. S_i denotes Strip_i .

Proof: The proof will assume that $\delta > \frac{8}{\sqrt{n}}$. The case $\delta \leq \frac{8}{\sqrt{n}}$ follows directly from Lemma 11.

As noted, $\mathbf{E}[M_n] = \mathbf{E}[|\text{MAX}(S_n) \cap D_1|] + O(1)$. By the x, y symmetry of $\mathbf{D}$,

$$\mathbf{E}[|\text{MAX}(S_n) \cap B|] = \mathbf{E}[|\text{MAX}(S_n) \cap U|].$$

Thus it is not actually necessary to analyze the number of maxima in region B because

$$\mathbf{E}[M_n] = \Theta\left(\mathbf{E}[|\text{MAX}(S_n) \cap (U \cup M \cup I)|]\right).$$

Set $\bar{\sigma} = \left(\frac{\delta}{n}\right)^{1/3}$, $m = \lceil \frac{1}{\bar{\sigma}} \rceil$ and $\sigma = \frac{1}{m}$. Note that this implies

$$\sigma \leq \bar{\sigma} = \left(\frac{\delta}{n}\right)^{1/3} \leq \left(\delta \cdot \frac{\delta^2}{64}\right)^{1/3} = \frac{\delta}{4}.$$

and $\sigma = \Theta(\bar{\sigma})$.

The proof now works by defining appropriate regions $\bar{B}_i = B_i \cup B'_i$, and A_i for $i = 1, \dots, m$ and then applying the Strip lemma.

For $i = 0, \dots, m$ define

$$\begin{aligned} x_i &= i, \sigma & p_i(y) &= (x_i, y), \\ y_i &= 1 + \delta - x_i - 2\sigma, & p_i &= p_i(y_i) = (x_i, y_i), \\ & & \bar{p}_i &= p_i(0) = (x_i, 0). \end{aligned}$$

and

$$\text{Strip}_i = (U \cup M \cup I) \cap \{u \in \mathfrak{R}^2 : x_i \leq u.x < x_{i+1}\}.$$

Further define the regions for the Strip Lemma:

$$\begin{aligned} B_i(t) &= \text{Strip}_i \cap \{u : y_i - t \leq u.y \leq y_i\}, & B_i &= B_i(y_i), & B'_i &= T(p_i), \\ A_i(t) &= T(p_{i+1}(y_i - t)), & A_i &= A_i(y), & \bar{B}_i &= B_i \cup B'_i \end{aligned}$$

By definition,

$$U \cup M \cup I = \bigcup_{i=0}^{m-1} \bar{B}_i.$$

Thus, if we could show both

(A) $\mu(B'_i) = O(1/n)$ and

(B) A_i continuously dominates B_i (Definition 9)

we could immediately apply the Strip Lemma (Lemma 8) to show that

$$\mathbf{E}[M_n] = \Theta \left(\mathbf{E} \left[\left| \text{MAX}(S_n) \cap \bigcup_{i=0}^{m-1} \bar{B}_i \right| \right] \right) = O(m) = O \left(\frac{n^{1/3}}{\delta^{1/3}} \right)$$

completing the proof.

To prove (A) we note the following straightforward observations that are true for $\forall i < m$ and $\forall t \leq y_i$:

$$\begin{aligned} \sigma(p_i) &= 1 + \delta - x_i - y_i = 2\sigma, & \sigma(p_{i+1}(y_i)) &= \sigma(p_i) - \sigma = \sigma, \\ \sigma(p_i(y_i - t)) &= \sigma(p_i) + t = 2\sigma + t, & \sigma(p_{i+1}(y_i - t)) &= \sigma(p_{i+1}) + t = \sigma + t, \end{aligned}$$

$$\begin{aligned} \Delta_x(p_i) &= \Delta_x(p_i(y - t)), & \Delta_y(p_i) &= \Delta_y(p_{i+1}(y_i)), \\ \Delta_x(p_{i+1}) &= \Delta_x(p_i) + \sigma, & \Delta_y(p_{i+1}(y_i - t)) &= \Delta_y(p_{i+1}) + t, \end{aligned}$$

and

$$\begin{aligned} (m-i)\sigma &\leq \Delta_x(p_i) \leq (m-i+1)\sigma, \\ (i+2)\sigma &\leq \Delta_y(p_i) \leq (i+3)\sigma. \end{aligned} \tag{14}$$

Since $\sigma \leq \delta/4$ the observations above yield

$$\sigma(p_i) = 2\sigma \leq \frac{\delta}{2},$$

so $p_i \notin I$ and thus $p_i \in M \cup U$. Cases 1 or 3 in Lemma 13 then apply and

$$\mu(B'_i) = \mu(T(p_i)) = O\left(\frac{\sigma^3}{\delta}\right) = O\left(\frac{1}{n}\right).$$

To prove **(B)** we first note that all points in $B_i(y_i) \setminus B_i(t)$ ARE dominated by all points in $A_i(t)$. Furthermore

$$B_i(t) \subset T(p_i(y-t)) \quad \Rightarrow \quad \mu(B_i(t)) \leq \mu(T(p_i(y-t))).$$

So, proving

$$\forall t \in [0, y_i], \quad \mu(T(p_i(y-t))) = \Theta(\mu(T(p_{i+1}(y_i-t))))), \quad (15)$$

will immediately imply that A_i continuously dominates B_i and complete the proof of the Lemma.

It thus only remains to prove Eq. 15. This will require splitting into multiple cases depending upon the value of x_i .

- $\delta \leq x_i \leq 1$: If $t \leq \delta - 2\sigma$ then $p_i(y_i - t) \in M$;
If $\delta - 2\sigma \leq t$ then $p_i(y_i - t) \in I$. This yields

$$\mu(T(p_i(y_i - t))) = \begin{cases} \Theta\left(\frac{(2\sigma+t)^3}{\delta}\right) & \text{if } 0 \leq t \leq \delta - 2\sigma, \\ \Theta((2\sigma+t)^2) & \text{if } \delta - 2\sigma \leq t. \end{cases}$$

- $x_i \leq \delta$: If $t \leq \delta - (i+2)\sigma$ then $p_i(y_i - t) \in U$;
If $\delta - (i+2)\sigma < t \leq \delta - 2\sigma$ then $p_i(y_i - t) \in M$;
If $\delta - 2\sigma \leq t$ then $p_i(y_i - t) \in I$.

Using (14) and working through the details gives

$$\mu(T(p_i(y_i - t))) = \begin{cases} \Theta\left((i+2)\sigma \frac{(2\sigma+t)^3}{\delta^2}\right) & \text{if } 0 \leq t \leq \delta - \sigma(i+2), \\ \Theta\left(\frac{(2\sigma+t)^3}{\delta}\right) & \text{if } \delta - \sigma(i+2) \leq t \leq \delta - 2\sigma, \\ \Theta((2\sigma+t)^2) & \text{if } \delta - 2\sigma \leq t. \end{cases}$$

We now work through similar calculations for $p_{i+1}(y_i - t)$ to prove

- $\delta - \sigma \leq x_{i+1} \leq 1$:

$$\mu(T(p_{i+1}(y_i - t))) = \begin{cases} \Theta\left(\frac{(\sigma+t)^3}{\delta}\right) & \text{if } 0 \leq t \leq \delta - \sigma, \\ \Theta((\sigma+t)^2) & \text{if } \delta - \sigma \leq t. \end{cases}$$

- $x_{i+1} < \delta - \sigma$:

$$\mu(T(p_{i+1}(y_i - t))) = \begin{cases} \Theta\left((i+3)\sigma\frac{(\sigma+t)^3}{\delta^2}\right) & \text{if } 0 \leq t \leq \delta - \sigma(i+2), \\ \Theta\left(\frac{(\sigma+t)^3}{\delta}\right) & \text{if } \delta - \sigma(i+2) \leq t \leq \delta - \sigma, \\ \Theta((\sigma+t)^2) & \text{if } \delta - \sigma \leq t. \end{cases}$$

We now prove Eq. 15.

Note that, $i+3 = \Theta(i+2)$ and $\sigma+t = \Theta(2\sigma+t)$ with the constants independent of i and t . This will be used below.

- $\delta - 2\sigma \leq x_i < x_{i+1} \leq 1$: We divide $t \in [0, y_i]$ into three ranges and write the values of the functions in each of those ranges.

$t \in$	$[0, \delta - 2\sigma]$	$[\delta - 2\sigma, \delta - \sigma]$	$[\delta - \sigma, y_i]$
$\mu(T(p_i(y_i - t))) =$	$\Theta\left(\frac{(2\sigma+t)^3}{\delta}\right)$	$\Theta((2\sigma+t)^2)$	$\Theta((2\sigma+t)^2)$
$\mu(T(p_{i+1}(y_i - t))) =$	$\Theta\left(\frac{(\sigma+t)^3}{\delta}\right)$	$\Theta\left(\frac{(\sigma+t)^3}{\delta}\right)$	$\Theta((\sigma+t)^2)$

Note that in the 1st and 3rd ranges the values in the two rows are within a constant factor of each other. For the 2nd range recall that $\sigma \leq \delta/4$ so, if $\delta - 2\sigma \leq t < \delta - \sigma$ then $\sigma+t = \Theta(\delta)$ so in the 2nd range as well the two rows are within a constant factor of each other, proving Eq. 15.

- $x_{i+1} < \delta - 2\sigma$: We divide $t \in [0, y_i]$ into four ranges and write the values of the functions in each of those ranges.

$t \in$	$[0, \delta - \sigma(i+2)]$	$[\delta - \sigma(i+2), \delta - 2\sigma]$	$[\delta - 2\sigma, \delta - \sigma]$	$[\delta - \sigma, y_i]$
$\mu(T(p_i(y_i - t))) =$	$\Theta\left((i+2)\sigma\frac{(2\sigma+t)^3}{\delta^2}\right)$	$\Theta\left(\frac{(2\sigma+t)^3}{\delta}\right)$	$\Theta(2(\sigma+t)^2)$	$\Theta(2(\sigma+t)^2)$
$\mu(T(p_{i+1}(y_i - t))) =$	$\Theta\left((i+3)\sigma\frac{(\sigma+t)^3}{\delta^2}\right)$	$\Theta\left(\frac{(\sigma+t)^3}{\delta}\right)$	$\Theta\left(\frac{(\sigma+t)^3}{\delta}\right)$	$\Theta((\sigma+t)^2)$

Note that in the 1st, 2nd and 4th ranges the values in the two rows are within a constant factor of each other. For the 3rd range, recall again that $\sigma \leq \delta/4$ so, if $\delta - 2\sigma \leq t < \delta - \sigma$ then $\sigma+t = \Theta(\delta)$ so in the 3rd range as well the two rows are within a constant factor of each other, proving Eq. 15 $\forall t \in [0, y_i]$.

We have just proven that Eq. 15 is valid for all cases and the proof of the Lemma is now complete. $\square$

We now return to prove Lemma 13.

Proof: of Lemma 13.

Recall that the goal of the proof is to analyze $\mu(T(p))$ for $p \in D_1$. For notational simplicity, we set $\sigma = \sigma(p)$. Recall

$$\mu(T(p)) = \int_{u \in T(p)} f(u) du, \quad (16)$$

where $f(u)$ is the density as defined by Definition 6.

Next set

$$T'(p) = T(p) \cap \{u : u.x + u.y \leq 1 + \delta - \sigma/2\}.$$

Note that $T'(p)$ is the upper right isosceles triangle with lower-left corner p and sidelengths $\sigma/2$. Since $T'(p) \subseteq T(p)$, $\mu(T'(p)) \leq \mu(T(p))$.

Finally, let

$$\begin{aligned} U(p) &= \{u \in B_1 : (u + \delta B_1) \cap T(p) \neq \emptyset\}, \\ U'(p) &= \{u \in B_1 : T'(p) \subseteq u + \delta B_1\}. \end{aligned}$$

Notice $U(p)$ is the preimage of $T(p)$ in $\mathbf{B}_1$ while $U'(p)$ is the intersection of the preimages of all of the points in $T'(p)$.

From Eq. 1 in Lemma 3,

$$\forall u \in T(p), \quad f(u) = O\left(\frac{\text{Area}(U(p))}{\delta^2}\right). \quad (17)$$

Plugging into Eq. 16 then gives

$$\mu(T(p)) \leq \text{Area}(T(p)) \max_{u \in T(p)} f(u) = O\left(\frac{\text{Area}(U(p))\sigma^2}{\delta^2}\right). \quad (18)$$

In the other direction we similarly find that

$$\forall u \in T'(p), \quad f(u) = \Omega\left(\frac{\text{Area}(U'(p))}{\delta^2}\right).$$

Again plugging into Eq. 16 gives

$$\mu(T(p)) \geq \mu(T'(p)) \geq \text{Area}(T'(p)) \min_{u \in T'(p)} f(u) = \Omega\left(\frac{\text{Area}(U'(p))\sigma^2}{\delta^2}\right). \quad (19)$$

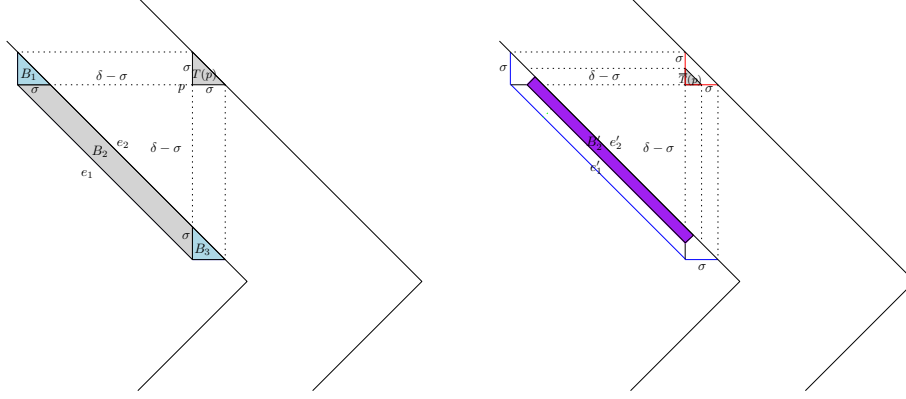


Figure 9: Case 3 in which $T(p) \subseteq M$.

We now work through the four cases in the Lemma, deriving the appropriate values of $\text{Area}(U(p))$ and $\text{Area}(U'(p))$ and plugging them into Eqs. 18 and 19 to complete the proof.

Case 3: $p \in M$.

First assume that $T(p) \subset M$, e.g., point p_3 in Figure 6.

Consider the left diagram in Figure 9. Simple geometric arguments show that $U(p) = B_1 \cup B_2 \cup B_3$. Then $\sigma \leq \delta$ immediately yields

$$\text{Area}(U(p)) = \Theta(\sigma\delta).$$

Consider the right diagram in Figure 9. This illustrates the fact (again derivable from direct geometric arguments) that $U'(p) = B_2'$ and

$$\text{Area}(U'(p)) = \Theta(\sigma\delta).$$

Plugging these formulas into Eqs. 18 and 19 we get

$$\mu(T(p)) = \Theta\left(\frac{\sigma^3}{\delta}\right).$$

Assume that $p \in M$ but $T(p) \not\subseteq M$, e.g., points p_2, p_4 in Figure 6. Again, simple geometric calculations (see Figure 10) show that

$$\text{Area}(U(p)) = \Theta(\sigma\delta), \quad \text{and} \quad \text{Area}(U'(p)) = \Theta(\sigma\delta).$$

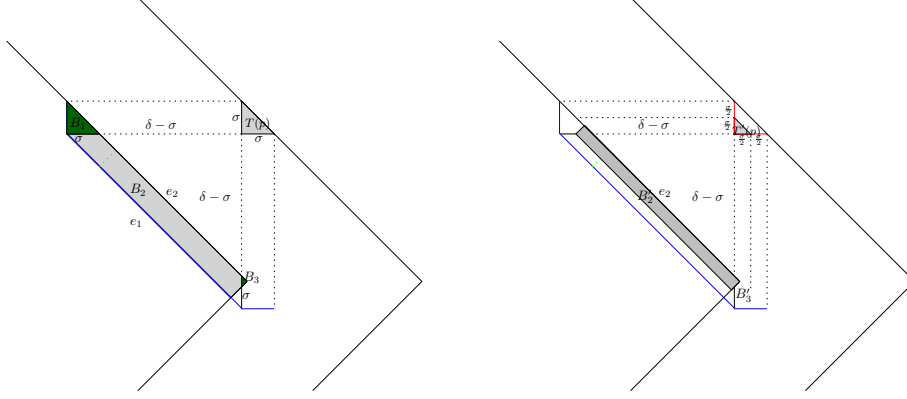


Figure 10: Case 3 in which $T(p) \not\subseteq M$.

(The intuition is that the areas are at least half the size of the preimages in the first subcase). Using again Eqs. 18 and 19 we get

$$\mu(T(p)) = \Theta\left(\frac{\sigma^3}{\delta}\right).$$

Case 2: $p \in B$.

An example is p_5 in Fig. 6.

In this case, $1 < p.x$. Set $h = p.y$. Note that, by definition, $h + \sigma \leq \delta$.
next note that, in Fig. 11, $U(p) = B_1 \cup B_2$ which, by calculation satisfies

$$\text{Area}(U(p)) = \Theta((h + \sigma)\sigma).$$

Similarly, in the other direction, similar geometric arguments also show

$$\text{Area}(U'(p)) = \Theta((h + \sigma)\sigma).$$

Plugging these formulas into Eqs. 18 and 19 and using the fact that $h + \sigma = \Delta_x(p)$ immediately gives

$$\mu(T(p)) = \Theta\left(\Delta_x(p) \frac{\sigma^3}{\delta^2}\right).$$

Case 1: $p \in U$.

An example is p_1 in Figure 6. The proof for this case is totally symmetric to that of Case 2 and is therefore omitted.

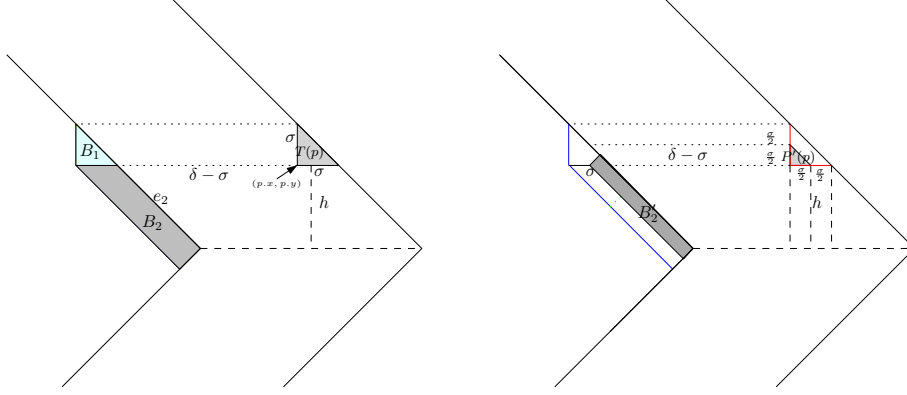


Figure 11: Case 2

Case 4: $p \in I$.

An example is p_8 in Fig. 6.

See Fig. 12. Set $\Delta = 1 - (p.x + p.y) = \sigma(p) - \delta$. We will prove that $\mu(T(p)) = \Theta(\Delta^2 + \delta^2)$. Since $\Delta^2 + \delta^2 = \Theta(\sigma^2(p))$ this completes the proof.

Given $p \in I$ set

$$p_1 = (p.x + \Delta, p.y), \quad p_2 = (p.x, p.y + \Delta), \quad p_3 = (p.x + \Delta, p.y + \Delta).$$

While p_3 is not necessarily in $\mathbf{B}_1 + \delta\mathbf{B}_1$, p_1, p_2 are on the boundary of $\mathbf{B}_1$ so $\sigma(p_1) = \sigma(p_2) = \delta$ and from Case 3

$$\mu(T(p_1)) = \mu(T(p_2)) = \Theta(\delta^2).$$

Let T be the triangle with vertices p, p_1, p_2 and T' the triangle with vertices p_1, p_2, p_3 . Since $T \subset B_1$, from Lemma 4(d) (with $\kappa = 1$)

$$\mu(T) = \Theta(\text{Area}(T)) = \Theta(\Delta^2).$$

From Lemma 4(c),

$$\mu(T') = O(\text{Area}(T')) = O(\delta^2).$$

Finally note that $T(p) \subseteq T(p_1) \cup T(p_2) \cup T \cup T'$ so

$$\mu(T(p)) \leq \mu(T(p_1)) + \mu(T(p_2)) + \mu(T) + \mu(T') = \Theta(\Delta^2 + \delta^2).$$

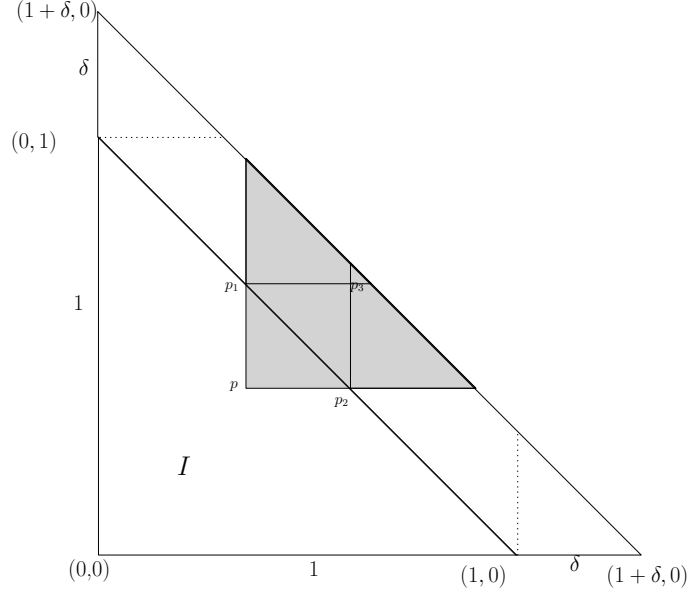


Figure 12: Case 4

From the other direction, T and $T(p_1)$ are disjoint with $T \cup T(p_1) \subset T(p)$ so

$$\mu(T(p)) \geq \mu(T(p_1)) + \mu(T) = \Theta(\Delta^2 + \delta^2).$$

Thus

$$\mu(T(p)) = \Theta(\Delta^2 + \delta^2).$$

□

7 Analysis of $\mathbf{B}_2 + \delta\mathbf{B}_2$

This section derives cell (iii)(c) in Theorem 1, that is, if n points are chosen from $\mathbf{D} = \mathbf{B}_2 + \delta\mathbf{B}_2$ and $\frac{1}{\sqrt{n}} \leq \delta \leq 1$ then $\mathbf{E}[M_n] = \Theta\left(\frac{n^{2/7}}{\delta^{3/7}}\right)$. Applying Lemma 9 for $1 \leq \delta \leq \sqrt{n}$ yields cell (iii)(d), i.e., that $\mathbf{E}[M_n] = \Theta(\delta^{3/7}n^{1/3})$.

This section follows the general structure laid out in Section 4.2. and will need the following special definitions

Definition 11 Let $D_1 = (B_2 + \delta B_2) \cap Q_1$ be the support in the first quadrant. The support restricted to the the first and second octants is

$$D_{1,1} = D_1 \cap \{(x, y) \in \mathbb{R}^2 : y \leq x\}, \quad D_{1,2} = D_1 \cap \{(x, y) \in \mathbb{R}^2 : x \leq y\}.$$

For all $v \in D$ define

$$\sigma(v) = 1 + \delta - \|v\|, \quad \theta(v) = \arctan\left(\frac{v.y}{v.x}\right).$$

$\sigma(v)$ is the distance from v to the boundary of D_1 and $\theta(v)$ is the angle formed by the x axis by the line connected the origin to v .

By construction, the induced measure is radially symmetric, i.e., $f(v) = g(\|v\|)$, for some function g . More specifically,

Lemma 16 Let $\mathbf{D} = \mathbf{B}_2 + \delta \mathbf{B}_2$. The density of $\mathbf{D}$ satisfies

$$\forall v \in D_1, \quad f(v) = \begin{cases} \Theta\left(\left(\frac{\sigma(v)}{\delta}\right)^{3/2}\right) & \text{if } \sigma(v) \leq \delta, \\ \Theta(1) & \text{if } \delta \leq \sigma(v) \leq 1 + \delta. \end{cases}$$

The proof of the Lemma is deferred to the end of this section.

Given the above, the lower bound is easy to derive:

Lemma 17 Let S_n be n points chosen from the distribution $\mathbf{D} = \mathbf{B}_2 + \delta \mathbf{B}_2$ with $\frac{1}{\sqrt{n}} \leq \delta \leq 1$. Then

$$\mathbf{E}[M_n] = \Omega\left(\frac{n^{2/7}}{\delta^{3/7}}\right).$$

Proof:

Let $v \in D_1$. The remainder of the proof assumes that (i) $\sigma(v) \leq \delta$ and (ii) $\theta(v) \in [\frac{\pi}{6}, \frac{\pi}{3}]$ (this could be replaced by any arbitrary fixed angular range bounded away from 0 and $\pi/2$ radians).

Refer to Figure 13 for the below. Recall $P(v)$ as introduced in Definition 7. Let v', v'' be the two other vertices of $P(v)$ with $v.x = v''.x$, $v.y = v'.y$, $\alpha(v) = v''.y - v.y$ and $\beta(v) = v'.x - v.x$. Note that $\pi/6 \leq \theta(v) \leq \pi/3$ immediately implies $\beta(v) = \Theta(\sigma(v))$ and $\alpha(v) = \Theta(\sigma(v))$. Let $c > 0$ be such that $\beta(v) \leq c\sigma(v)$ for all such v .

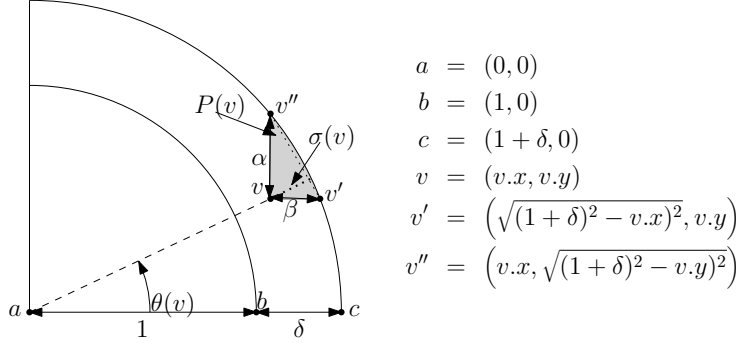


Figure 13: Illustration of the proof of Lemma 17.

Since $P(v)$ contains the triangle with vertices v, v', v'' and is contained in the rectangle that has those three points and $(v'.x, v''.y)$ as corners, $\frac{\alpha(v)\beta(v)}{2} \leq \text{Area}(P(v)) \leq \alpha(v)\beta(v)$. Thus

$$\text{Area}(P(v)) = \left(\frac{\sigma(v)}{\delta} \right)^{3/2}.$$

Let $u \in P(v)$. Note that $\sigma(u) \leq \sigma(v)$, so Lemma 16 immediately implies $f(u) = O(f(v))$ and

$$\begin{aligned} \mu(P(v)) &= \int_{u \in P(v)} f(u) du \leq \text{Area}(P(v)) \max_{u \in P(v)} f(u) \\ &= O \left(\sigma^2(v) \left(\frac{\sigma(v)}{\delta} \right)^{3/2} \right) = \Theta \left(\frac{\sigma^{7/2}(v)}{\delta^{3/2}} \right). \end{aligned}$$

Set

$$\bar{P}(v) = P(v) \cap \left\{ u \in \mathbb{R}^2 : \sigma(u) \geq \frac{1}{4}\sigma(v) \right\}.$$

Using basic geometric arguments it is straightforward that

$$\text{Area}(\bar{P}(v)) = \Theta(\text{Area}(P(v))) = \Theta(\sigma^2(v)).$$

From Lemma 16, for every $u \in \bar{P}(v)$, $f(v') = \Theta(f(v))$ and thus

$$\begin{aligned}
\mu(P(v)) &= \int_{u \in P(v)} f(u) du \geq \int_{u \in \bar{P}(v)} f(u) du \\
&\geq \text{Area}(\bar{P}(v)) \min_{u \in \bar{P}(v)} f(u) \\
&= \Theta \left(\sigma^2(v) \left(\frac{\sigma(v)}{\delta} \right)^{3/2} \right) = \Theta \left(\frac{\sigma^{7/2}(v)}{\delta^{3/2}} \right).
\end{aligned}$$

We have thus shown that, for all v satisfying the two conditions,

$$\mu(P(v)) = \Theta \left(\frac{\sigma^{7/2}(v)}{\delta^{3/2}} \right).$$

Set $\sigma = \frac{\delta^{3/7}}{n^{2/7}}$. Note that, for $\frac{1}{\sqrt{n}} \leq \delta$, this implies $\sigma \leq \delta$ and v satisfies the first condition. Thus, if $\theta(v) \in [\frac{\pi}{6}, \frac{\pi}{3}]$ and $\sigma(v) = \sigma$ then

$$\mu(P(v)) = \Theta \left(\frac{\sigma^{7/2}(v)}{\delta^{3/2}} \right) = \Theta \left(\frac{1}{n} \right). \quad (20)$$

We construct a collection of such v . Set $x_\ell = (1 + \delta) \cos(\pi/3)$ and $x_r = (1 + \delta) \cos(\pi/6)$. Note that $x_r - x_\ell = \Theta(1)$.

Let $m = \lfloor \frac{x_r - x_\ell}{c\sigma} \rfloor$. Note that since $x_r - x_\ell = \Theta(1)$, $m = \Theta(\frac{1}{\sigma}) = \Theta(\frac{n^{2/7}}{\delta^{3/7}})$. Now set, $x_0 = x_\ell$ and, for $i = 1, \dots, m$,

$$\begin{aligned}
x_i &= x_\ell + i\sigma, \\
v_i &= (x_i, \sqrt{(1 + \delta - \sigma)^2 - (x_i)^2}).
\end{aligned}$$

By construction, $\sigma(v_i) = \sigma$, $\theta(v_i) \in [\frac{\pi}{6}, \frac{\pi}{3}]$ and thus $\mu(P(v_i)) = \Theta(1/n)$. Furthermore, each $P(v_i)$ is a dominant region and, since by construction, $\forall i$, $x_i + \beta(v_i) \leq x_{i+1}$, the $P(v_i)$ are pairwise disjoint. Then, Lemma 2 then immediately proves the required

$$\mathbf{E}[M_n] = \Omega(t) = \Omega \left(\frac{n^{2/7}}{\delta^{3/7}} \right).$$

□

The upper bound proof will require an additional definition and lemma.

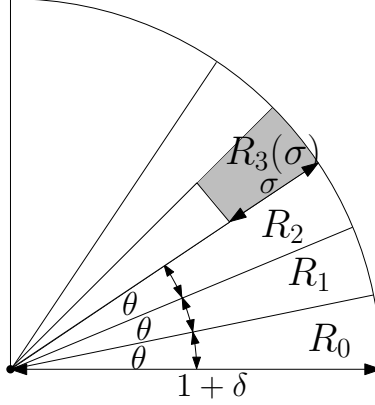


Figure 14: Illustration of Definition 12. $D = B_2 + \delta B_2$ (only the first quadrant is pictured) is partitioned into t radially symmetric sectors R_i . $R_i(\sigma)$ are the points in R_i within σ of the boundary of D .

Definition 12 Let t be a positive integer, $\theta = \frac{2\pi}{t}$ and $\theta_i = i\theta$, $i = 0, 1, \dots, t-1$. Now partition $D = B_2 + \delta B_2$ into t sectors, $i = 0, 1, \dots, t-1$,

$$R_i = D \cap \{v \in \mathbb{R}^2 : \theta_i \leq \theta(v) \leq \theta_{i+1}\},$$

and further define

$$R_i(\sigma) = \{v \in R_i : 1 + \delta - \sigma \leq \|v\|\}.$$

Lemma 18

$$\mu(R_i(\sigma)) = \begin{cases} \Theta\left(\frac{\sigma^{5/2}}{t\delta^{3/2}}\right) & \text{if } 0 \leq \sigma \leq \delta, \\ \Theta\left(\frac{\sigma}{t}\right) & \text{if } \delta \leq \sigma \leq 1 + \delta. \end{cases}$$

Proof:

Change into polar coordinates and write $v = (x, y) = (\|v\| \cos \gamma, \|v\| \sin \gamma)$. Recall from Lemma 16 that $f(v) = g(1 + \delta - \|v\|)$ so we can integrate to get

$$\begin{aligned}
\mu(R_i(\sigma)) &= \int \int_{(x,y) \in R_i(\sigma)} f((x,y)) \, dx dy \\
&= \int_{\gamma=\theta_i}^{\theta_{i+1}} \int_{r=1+\delta-\sigma}^{1+\delta} r \, g(1+\delta-r) \, dr \\
&= \int_{\gamma=\theta_i}^{\theta_{i+1}} \int_{r=0}^{\sigma} (1+\delta-\sigma) \, g(r) \, dr = \frac{2\pi}{t} \int_{r=0}^{\sigma} (1+\delta-r) \, g(r) \, dr.
\end{aligned}$$

For all $r \leq \delta$, $g(r) = \Theta\left(\left(\frac{r}{\delta}\right)^{3/2}\right)$ so

$$\mu(R_i(\sigma)) = \Theta\left(\frac{1}{t} \int_{r=0}^{\sigma} \frac{r^{3/2}}{\delta^{3/2}} \, dr\right) = \Theta\left(\frac{\sigma^{5/2}}{t\delta^{3/2}}\right).$$

Recall from Lemma 16 that, for $\delta \leq \sigma \leq 1+\delta$, $f(v) = \Theta(1)$, so

$$\mu(R_i(\sigma)) = \Theta\left(\frac{1}{t} \int_{r=0}^{\delta} \frac{r^{3/2}}{\delta^{3/2}} \, dr + \frac{1}{t} \int_{r=\delta}^{\sigma} \, dr\right) = \Theta\left(\frac{t}{\delta}\right) + \Theta\left(\frac{\sigma-\delta}{t}\right) = \Theta\left(\frac{\sigma}{t}\right).$$

□

As before, the upper bound is more technical.

Lemma 19 *Let S_n be n points chosen from the distribution $\mathbf{D} = \mathbf{B}_1 + \delta\mathbf{B}_1$ with $\frac{1}{\sqrt{n}} \leq \delta \leq 1$. Then*

$$\mathbf{E}[M_n] = O\left(\frac{n^{2/7}}{\delta^{3/7}}\right).$$

Proof: Chose integer t in Definition 12 and set m so that

$$t = \min \left\{ t' \geq \frac{n^{2/7}}{\delta^{3/7}} \text{ and } t' \text{ is a multiple of } 8 \right\} \quad \text{and} \quad m = \frac{t}{8}.$$

Recall from Corollary 7 that

$$\mathbf{E}[M_n] = \Theta(\mathbf{E}[|\text{MAX}(S_n) \cap D_{1,1}|]), \quad (21)$$

where $D_{1,1} = D \cap O_1$. Since $D_{1,1} = \bigcup_{i=1}^{t/8} R_i$,

$$\mathbf{E}[|\text{MAX}(S_n) \cap D_{1,1}|] = \sum_{i=1}^m \mathbf{E}[|\text{MAX}(S_n) \cap R_i|]. \quad (22)$$

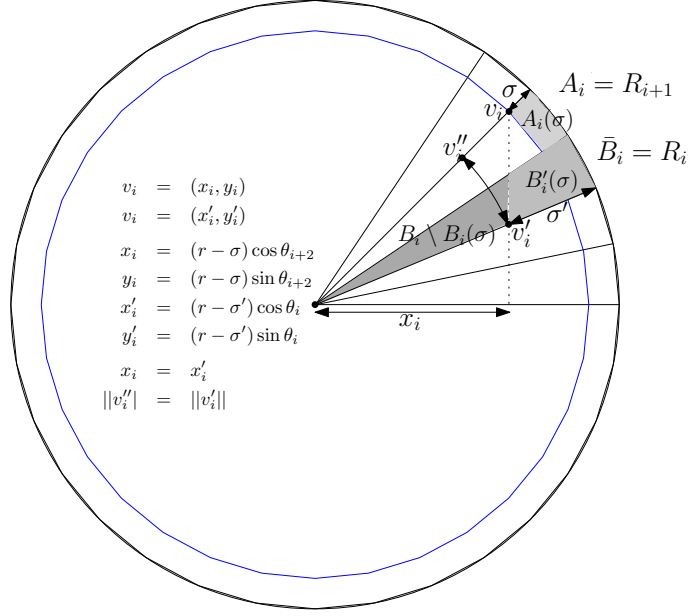


Figure 15: Illustration of the proof of Lemma 19. v_i , v'_i and v''_i denote $v_i(\sigma)$, $v'_i(\sigma)$ and $v''_i(\sigma)$.

We now set $\bar{B}_i = R_i$ and define appropriate regions A_i, B_i to apply the Strip Lemma. Combined with Eqs. 21 and 21 this will imply

$$\mathbf{E}[M_n] = O(m) = O\left(\frac{n^{2/7}}{\delta^{3/7}}\right), \quad (23)$$

proving the lemma.

Consider Figure 15. Set $r = 1 + \delta$ to be the full radius of $B_2 + \delta B_2$. Let $A_i(\sigma) = R_{i+1}(\sigma)$. Set

$$v_i(\sigma) = ((r - \sigma) \cos \theta_{i+2}, (r - \sigma) \sin \theta_{i+2}).$$

This is the leftmost point of $A_i(\sigma)$ on the boundary line between R_{i+2} and R_{i+3} . Now drop a vertical line from $v_i(\sigma)$ to the x -axis and let $v'_i(\sigma)$ be the point at which it intersects the boundary between R_i and R_{i-1} . Let $\sigma' = \sigma(v'_i)$. By construction

$$(r - \sigma) \cos \theta_{i+2} = v(\sigma).x = v'(\sigma).x = (r - \sigma') \cos \theta_i.$$

But

$$\begin{aligned}
\cos \theta_{i+2} &= \cos \left(\theta_i + \frac{2}{t} \right) \\
&= \cos \theta_i \cos(2/t) + \sin \theta_i \sin(2/t) \\
&= (1 + O(1/t) \cos \theta_i + O(1/t) \\
&= \cos \theta_i + O(1/t).
\end{aligned}$$

Because the R_i are in the first octant, θ_i is bounded away from $\pi/2$ so $\cos \theta_i$ is bounded away from 0. Thus, $\sigma' - \sigma = O(1/t)$. Set $\bar{\sigma} = \frac{1}{t}$ and define

$$B'_i(\sigma) = \{v \in R_i : v_i(\sigma).x \leq v.x\}$$

and then

$$\begin{aligned}
\bar{B}_i &= R_i, & B'_i &= \{v \in R_i : v_i(\bar{\sigma}).x \leq v.x\} = B'_i(\bar{\sigma}), \\
A_i &= R_{i+1}, & A_i(\sigma) &= R_{i+1}(\sigma), \\
B_i &= \bar{B}_i \setminus B'_i & B_i(\sigma) &= \{v \in B_i : v_i(\sigma).x \leq v.x\} \\
& & &= \{v \in R_i : v_i(\sigma).x \leq v.x \leq v_i(\bar{\sigma}).x\}
\end{aligned}$$

By construction

- $B'_i \subset R_i(\bar{\sigma}') \Rightarrow \mu(B'_i) \leq \mu(R_i(\bar{\sigma}'))$.
- Since $\bar{\sigma} = \frac{1}{t} \Rightarrow \bar{\sigma}' = \frac{1}{t} + O(\frac{1}{t}) = \Theta(\bar{\sigma})$.
- $\frac{1}{\sqrt{n}} \leq \delta \Rightarrow \frac{1}{n^2} \leq \delta^4 \Rightarrow \frac{\delta^3}{n^2} \leq \delta^7 \Rightarrow \bar{\sigma} = \frac{1}{t} \leq \frac{\delta^{3/7}}{n^{2/7}} \leq \delta$.

$$\Rightarrow \mu(B'_i) \leq \mu(R_i(\bar{\sigma}')) = \Theta \left(\frac{\bar{\sigma}^{5/2}}{t\delta^{3/2}} \right) = \Theta(1/n).$$

Furthermore

- Every point in $B_i \setminus B_i(\sigma)$ is dominated by every point in $A_i(\sigma)$.
- If $\sigma \leq \bar{\sigma} \Rightarrow B_i(\sigma) = \emptyset$.
- If $\sigma > \bar{\sigma} \Rightarrow B_i(\sigma) \subset B_i(\sigma') \subset R_i(\sigma')$.

Thus, if $\sigma \leq \bar{\sigma}$ then

$$\mu(B_i(\sigma)) = 0 = O(\mu(A(\sigma))),$$

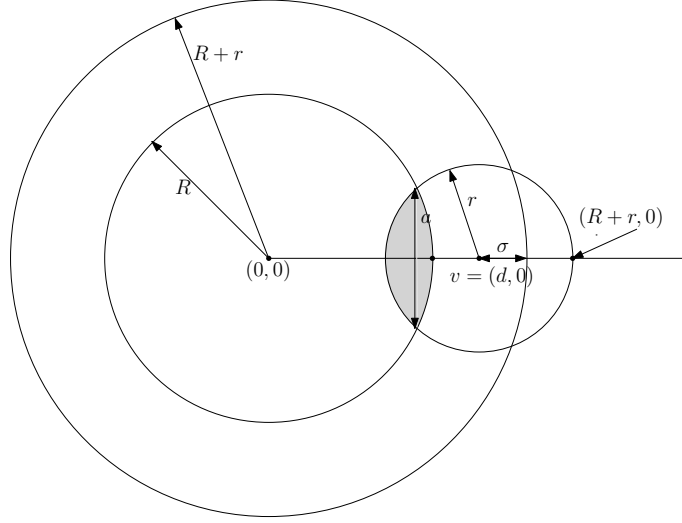


Figure 16: Illustration of proof of Lemma 16. $R = 1$, $r = \delta$. The gray area is $P'(v)$, the preimage of v in B_2 , i.e., the intersection of B_2 and $B_2(v, \delta)$.

while if $\sigma > \bar{\sigma}$ then

$$\begin{aligned}
 \mu(B_i(\sigma)) &= O(\mu(R_i(\sigma'))) \\
 &= O(\mu(R_{i+1}(\sigma'))) \\
 &= O(\mu(R_{i+1}(\sigma + O(\bar{\sigma})))) \\
 &= O(\mu(R_{i+1}(\sigma))),
 \end{aligned}$$

where the last inequality comes from plugging in the values from Lemma 18. We have therefore shown that the A_i, B_i, B_i satisfy the conditions of the Strip Lemma (Lemma 8), implying Eq. 23 and are therefore done. $\square$

Proof: of Lemma 16.

Set $\sigma = \sigma(v)$. The radial symmetry of $\mathbf{D}$ follows from the radial symmetry of $\mathbf{B}_2$. Because of the radial symmetry we may assume that $v = (1 + \delta - \sigma, 0)$, i.e., v lies on the non-negative x axis. First note that if $\sigma \geq \delta$, then $v \in B_2$ so, by Lemma 4(b) (with $\kappa = 1$) $f(v) = \Theta(1)$. If $\sigma > 1 + \delta$, then $v \notin B_2 + \delta B_2$. The remainder of the proof therefore assumes $0 \leq \sigma < \delta$.

From Eq. 1 in Lemma 3,

$$f(v) = \Theta\left(\frac{\text{Area}(P'(v))}{\delta^2}\right) \quad (24)$$

where

$$P'(v) = B_2 \cap B_2(v, \delta),$$

i.e. the preimage of v , is the intersection of two circular balls. The first has radius $R = 1$ with center $(0, 0)$ and the second has radius $r = \delta$ with center $v = (d, 0)$, where $d = 1 + \delta - \sigma$. The intersection region is shaped like an asymmetric lens (Fig. 16). The width of this lens is $R - (d - r) = \sigma$. Let a denote the height of the lens.

It is known, e.g., [23], that such an intersection satisfies

$$a = \frac{1}{d} \sqrt{(-d + r - R)(-d - r + R)(-d + r + R)(d + r + R)}. \quad (25)$$

Since $\sigma \leq \delta = r \leq 1$, $d = \Theta(1)$ and

$$\begin{aligned} -d + r - R &= (-1 - \delta + \sigma) + \delta - 1 = -2 + \delta = -\Theta(1), \\ -d - r + R &= (-1 - \delta + \sigma) - \delta + 1 = -2\delta + \sigma = -\Theta(\delta), \\ -d + r + R &= (-1 - \delta + \sigma) + \delta + 1 = \sigma, \\ d + r + R &= (1 + \delta - \sigma) + \delta + 1 = 2 + 2\delta + \sigma = \Theta(1). \end{aligned}$$

Plugging these values into (25) yields $a = \Theta(\sqrt{\sigma\delta})$. Because $P'(v)$ is the intersection of two circles, $P'(v)$ is convex and thus contains the quadrilateral defined by the four corners

$$\left(1 - \frac{\sigma}{2}, 0\right), \left(0, \frac{\alpha}{2}\right), \left(1 + \frac{\sigma}{2}, 0\right), \left(0, \frac{\alpha}{2}\right).$$

$P'(v)$ is also, by definition, contained in the rectangle

$$\{(x, y) : 1 - \sigma/2 \leq x \leq 1 + \sigma/2 \text{ and } -a/2 \leq y \leq a/2\}.$$

Thus, $\text{Area}(P'(v)) = \Theta(\sigma a) = \Theta(\sigma^{3/2} \sqrt{\delta})$. Plugging into Eq. 24 yields

$$f(v) = \Theta\left(\frac{\text{Area}(P'(v))}{\delta^2}\right) = \Theta\left(\left(\frac{\sigma}{\delta}\right)^{3/2}\right).$$

□

8 Analysis of $\mathbf{B}_1 + \delta \mathbf{B}_2$

This section derives cell $v(c)$ and $v(d)$ in Theorem 1, that is, if n points are chosen from $\mathbf{D} = \mathbf{B}_1 + \delta \mathbf{B}_2$ and $\frac{1}{\sqrt{n}} \leq \delta \leq n^{1/26}$ then $\mathbf{E}[M_n] = \Theta\left(\frac{n^{2/7}}{\delta^{3/7}}\right)$,

For $v \in C$ define

$$\begin{aligned} r(v) &= \sqrt{(v.x - 1)^2 + (v.y)^2}, \\ \sigma(v) &= \delta - r(v), \\ \theta(v) &= \arctan\left(\frac{v.x - 1}{v.y}\right). \end{aligned}$$

$$\sigma(v) = 1 + \delta - \|v\|, \quad \theta(v) = \arctan\left(\frac{v.y}{v.x}\right).$$

Note that in both cases $\sigma(v)$ is the distance from v to the boundary of $D_{1,1}$. If $v \in C$, $\theta(v)$ is the angle formed with the x axis by the line connecting $(1,0)$ to v .

Because it is the convolution of two very different distributions the density of $\mathbf{D}$ does not have a clean description. The next lemma encapsulates properties that can be used to derive the behavior of $\mathbf{E}[M_n]$.

Lemma 20 Let $\mathbf{D} = \mathbf{B}_1 + \delta \mathbf{B}_2$ and $v \in D_{1,1}$.

- If $v \in A$, then

$$f(v) = \begin{cases} \Theta(1) & \text{if } \delta \leq 1, \\ \Theta\left(\frac{1}{\delta^2}\right) & \text{if } \delta > 1. \end{cases}$$

- If $v \in B$, then

$$f(v) = \begin{cases} \Theta\left(\frac{1}{\delta^2}\right) & \text{if } \sigma(v) = \Omega(1), \\ \Theta\left(\frac{\sigma(v)}{\delta^2}\right) & \text{if } \delta\sigma(v) = \Omega(1) \text{ and } \sigma(v) = O(1), \\ \Theta\left(\left(\frac{\sigma(v)}{\delta}\right)^{3/2}\right) & \text{if } \delta\sigma(v) = O(1) \text{ and } \sigma(v) = O(1). \end{cases} \quad (26)$$

- If $v \in C$, then

1. If $\theta(v) = \frac{\pi}{4}$ then $f(v)$ is as defined in Eq. 26
2. Fix $d > 0$ and set $\gamma(v) = \frac{\pi}{4} - \theta(v)$. If $\sigma(v) < d\delta$, then

$$f(v) = \begin{cases} \Theta\left(\frac{\min(\sigma(v), 1) \cdot \min(\sqrt{\delta\sigma(v)}, 1)}{\delta^2}\right) & \text{if } \sigma(v) = \Omega(\delta(\gamma(v))^2), \\ \Theta\left(\frac{\min(\sigma(v), 1) \cdot \min(\frac{\sigma(v)}{\gamma(v)}, 1)}{\delta^2}\right) & \text{if } \sigma(v) = O(\delta(\gamma(v))^2). \end{cases} \quad (27)$$

This immediately implies

Corollary 21 Fix $d > 0$. If $v \in C$ and $\sigma \leq d\delta$ then $f(v) = \Theta(g(\theta(v), \sigma(v)))$, where $g(\theta, \sigma)$ satisfies

- $\forall \sigma$, if $\theta_1 \leq \theta_2$ then $g(\theta_1, \sigma) = O(g(\theta_2, \sigma))$
- $\forall \theta$, $g(\theta, 2\sigma) = O(g(\theta, \sigma))$,
where the constant in the $O()$ is independent of σ, δ .

The lower bound is easy to derive:

Lemma 22 Let S_n be n points chosen from the distribution $\mathbf{D} = \mathbf{B}_1 + \delta \mathbf{B}_2$ with $\frac{1}{\sqrt{n}} \leq \delta \leq \sqrt{n}$. Then

$$\mathbf{E}[M_n] = \begin{cases} \Omega\left(\frac{n^{2/7}}{\delta^{3/7}}\right), & \text{if } \frac{1}{\sqrt{n}} \leq \delta \leq n^{1/26}, \\ \Omega\left(\sqrt{\delta} n^{1/4}\right), & \text{if } n^{1/26} \leq \delta \leq \sqrt{n}. \end{cases}$$

Proof:

(a) $\mathbf{E}[|\text{MAX}(S_n \cap (A \cup B))|]$ when $\frac{1}{\sqrt{n}} \leq \delta \leq n^{1/5}$:

Fix σ with value to be determined later. Set $m = \lfloor 1/\sigma \rfloor$.

For $i = 0, \dots, m-1$ define

$$x_i = \frac{1}{2} + \frac{\delta}{\sqrt{2}} + i\sigma, \quad y_i = \frac{1}{2} + \frac{\delta}{\sqrt{2}} - x_i - \sigma, \quad p_i = (x_i, y_i).$$

The p_i are m equally spaced points along the line $\sigma(x) = 1 + \sqrt{2}\delta - \sigma$, which is parallel and distance σ from the line $x+y = 1 + \sqrt{2}\delta$ on the boundary of D . Thus the $P(p_i)$ are isosceles triangles. Note that $\forall i \neq j$, $P(p_i) \cap P(p_j) = \emptyset$. Since all $P(p_i)$ are dominant regions, if we could show $\forall i$, $\mu(P(p_i)) = \Theta(\frac{1}{n})$ Lemma 2 implies that $\mathbf{E}[M_n] = \Omega(m)$.

Set

$$\hat{P}(v) = P(v) \cap \{v' \in A \cup B : \sigma(v') \geq \sigma(v)/2\}.$$

Then, from Eq. 26,

$$\forall v' \in P(v), f(v') = O(f(v)) \quad \text{and} \quad \forall v' \in \hat{P}(v), f(v') = \theta(f(v)).$$

Since $\text{Area}(P(p_i)) = \frac{1}{2}\sigma^2$ and $\text{Area}(\hat{P}(p_i)) = \frac{1}{8}\sigma^2$, this gives

$$\mu(P(p_i)) = \Theta\left(\text{Area}(P(p_i)) f((p_i))\right) = \Theta\left(\sigma^2 f((p_i))\right).$$

Now set $\sigma = \frac{\delta^{3/7}}{n^{2/7}}$. Then $1 \leq \delta \leq n^{1/5}$ implies

$$\sigma = \frac{\delta^{3/7}}{n^{2/7}} \leq n^{\frac{3}{35} - \frac{10}{35}} \leq 1 \quad \text{and} \quad \delta\sigma = \frac{\delta^{10/7}}{n^{2/7}} \leq 1.$$

This satisfies the final condition in Eq. 26, so

$$\mu(P(p_i)) = \Theta\left(\sigma^2 f((p_i))\right) = \Theta\left(\sigma^2 \left(\frac{\sigma}{\delta}\right)^{3/2}\right) = \Theta\left(\frac{\sigma^{7/2}}{\delta^{3/2}}\right) = \Theta\left(\frac{\delta^{3/2}}{n} \frac{1}{\delta^{3/2}}\right) = \Theta\left(\frac{1}{n}\right).$$

Then Lemma 2 implies that

$$\mathbf{E}[M_n] = \Omega(m) = \Omega\left(\frac{1}{n}\right) = \Omega\left(\frac{n^{2/7}}{\delta^{3/7}}\right).$$

(b) $\mathbf{E}[\text{MAX}(S_n \cap C)]$ when $1 \leq \delta \leq \sqrt{n}$:

Set $\sigma = \frac{\sqrt{\delta}}{2n^{1/4}}$.

Let $v \in C$ be such that $\sigma(v) \leq \sigma$ and $\theta(v) \leq \pi/8$ so $\pi/8 \leq \gamma(v) \leq \pi/4$. Note that this implies $\sigma(v) \leq \min(1, \delta/2)$ and $\gamma(v) = \Theta(1)$ so, from Eq. 27

$$f(v) = \Theta\left(\frac{(\sigma(v))^2}{\delta^2}\right).$$

The construction is very similar of the proof of Lemma 17 so we only sketch the details.

First let $v \in C$ be such that $\sigma(v) = \sigma$ and $\theta(v) \leq \pi/8$.

Recall $P(v)$ as introduced in Definition 7. Let v', v'' be the two other vertices of $P(v)$ with $v.x = v''.x$, $v.y = v'.y$, $\alpha(v) = v''.y - v.y$ and $\beta(v) = v'.x - v.x$. Note that $\pi/8 \leq \theta(v) \leq \pi/4$ immediately implies $\beta(v) = \Theta(\sigma(v))$ and $\alpha(v) = \Theta(\sigma(v))$. Let $c > 0$ be such that $\beta(v) \leq c\sigma(v)$ for all such v .

Following the same steps as in the derivation of Eq. 20,

$$\mu(P(v)) = \Theta\left((\sigma(v))^2 \frac{(\sigma(v))^2}{\delta^2}\right) = \Theta\left(\frac{\sigma^4}{\delta^2}\right) = \Theta\left(\frac{1}{n}\right).$$

Set $x_\ell = 1 + \delta \cos(\pi/8)$ and $x_r = 1 + \delta \cos(\pi/4)$. Note that $x_r - x_\ell = \Theta(\delta)$.

Let $m = \lfloor \frac{x_r - x_\ell}{c\sigma} \rfloor$. Since $x_r - x_\ell = \Theta(1)$, $m = \Theta\left(\frac{\delta}{\sigma}\right) = \Theta\left(\sqrt{\delta} n^{1/4}\right)$.

Now set, $x_0 = x_l$ and, for $i = 1, \dots, m$,

$$\begin{aligned} x_i &= x_l + i\sigma, \\ v_i &= (x_i, \sqrt{(\delta - \sigma)^2 - (x_i)^2}). \end{aligned}$$

By construction, $\sigma(v_i) = \Theta(\sigma)$, $\theta(v_i) \in [\frac{\pi}{8}, \frac{\pi}{4}]$ and thus $\mu(P(v_i)) = \Theta(1/n)$. Furthermore, each $P(v_i)$ is a dominant region and, since by construction, $\forall i, x_i + \beta(v_i) \leq x_{i+1}$, the $P(v_i)$ are pairwise disjoint. Lemma 2 then proves the required

$$\mathbf{E}[M_n] = \Omega(m) = \Omega\left(\sqrt{\delta}n^{1/4}\right).$$

To complete the proof of the lemma simply note that

$$\sqrt{\delta}n^{1/4} \leq \frac{n^{2/7}}{\delta^{3/7}} \Leftrightarrow \delta^{26/28} = \delta^{13/14} = \delta^{1/2 + 3/7} \leq n^{2/7 - 1/4} = n^{1/28} \Leftrightarrow \delta \leq n^{1/26}.$$

□

As usual, the upper bound is more technical.

Lemma 23 *Let S_n be n points chosen from the distribution $\mathbf{D} = \mathbf{B}_1 + \delta\mathbf{B}_2$ with $\frac{1}{\sqrt{n}} \leq \delta \leq 1$. Then*

$$\mathbf{E}[M_n] = \begin{cases} O\left(\frac{n^{2/7}}{\delta^{3/7}}\right), & \text{if } \frac{1}{\sqrt{n}} \leq \delta \leq n^{1/26}, \\ O\left(\sqrt{\delta}n^{1/4}\right), & \text{if } n^{1/26} \leq \delta \leq \sqrt{n}. \end{cases} \quad (28)$$

Proof: The proof will be split into two parts: the first an upper bound on $v \in A \cup B$ and the second an upper bound on $v \in C$. This will proceed via case analyses that show

$$(a) \mathbf{E}[|\text{MAX}(S_n) \cap (A \cup B)|] = \begin{cases} O\left(\frac{n^{2/7}}{\delta^{3/7}}\right) & \text{If } \frac{1}{\sqrt{n}} \leq \delta \leq \frac{1}{\sqrt{2}}, \\ O\left(\frac{n^{2/7}}{\delta^{3/7}}\right) & \text{If } \frac{1}{\sqrt{2}} \leq \delta \leq n^{1/5}, \\ O\left(\frac{n^{1/3}}{\delta^{2/3}}\right) & \text{If } n^{1/5} \leq \delta \leq \frac{\sqrt{n}}{\log n}, \\ O(\log^4 n) & \text{If } \frac{\sqrt{n}}{\log n} \leq \delta \leq \sqrt{n}, \end{cases}$$

and

$$(b) \mathbf{E}[|\text{MAX}(S_n) \cap C|] = \begin{cases} O\left(\frac{n^{2/7}}{\delta^{3/7}}\right) & \text{If } \frac{1}{\sqrt{n}} \leq \delta \leq 1, \\ O\left(\sqrt{\delta}n^{1/4}\right) & \text{If } 1 \leq \delta \leq \sqrt{n}. \end{cases}$$

Since $\frac{n^{1/3}}{\delta^{2/3}} = O\left(\sqrt{\delta}n^{1/4}\right)$ if $n^{1/5} \leq \delta$ and $\frac{n^{2/7}}{\delta^{3/7}} > \sqrt{\delta}n^{1/4}$ if and only if $\delta < n^{1/26}$, the proof will follow.

(a) $v \in A \cup B$:

First note, that from Lemma 20, $\forall v, v' \in A \cup B$,

$$\text{If } \sigma(v') \leq \sigma(v) \Rightarrow f(v') = O((f(v))). \quad (29)$$

$$\text{If } \frac{1}{2}\sigma(v) \leq \sigma(v') \leq \sigma(v) \Rightarrow f(v') = \Theta((f(v))). \quad (30)$$

Now set $\sigma_b = \frac{\delta^{3/7}}{n^{2/7}}$ and $\sigma_c = \frac{\delta^{2/3}}{n^{1/3}}$.

If $\frac{1}{\sqrt{n}} \leq \delta \leq n^{1/5}$, then

$$\sigma_b = \frac{\delta^{3/7}}{n^{2/7}} \leq n^{\frac{3}{35} - \frac{10}{35}} \leq 1 \quad \text{and} \quad \delta\sigma_b = \frac{\delta^{10/7}}{n^{2/7}} \leq 1.$$

This satisfies the final condition in Eq. 26 so, if $\sigma(v) = \Theta(\sigma_b)$,

$$\Theta(\sigma^2(v)f((v))) = \Theta\left(\sigma_b^2 \left(\frac{\sigma_b}{\delta}\right)^{3/2}\right) = \Theta\left(\frac{\sigma_b^{7/2}}{\delta^{3/2}}\right) = \Theta\left(\frac{\delta^{3/2}}{n} \frac{1}{\delta^{3/2}}\right) = \Theta\left(\frac{1}{n}\right). \quad (31)$$

If $n^{1/5} < \delta$, then combining with the constraint $\delta \leq \sqrt{n}$ implies

$$\sigma_c = \frac{\delta^{2/3}}{n^{1/3}} \leq 1 \quad \text{and} \quad \delta\sigma_c = \frac{\delta^{5/3}}{n^{1/3}} > 1.$$

This satisfies the middle condition in Eq. 26 so, if $\sigma(v) = \Theta(\sigma_c)$,

$$\Theta(\sigma^2(v)f((v))) = \Theta\left(\sigma_c^2 \frac{\sigma_c^3}{\delta^2}\right) = \Theta\left(\frac{\delta^2}{n} \frac{1}{\delta^2}\right) = \Theta\left(\frac{1}{n}\right). \quad (32)$$

The analysis will be similar to that used to prove Lemma 15. $A \cup B$ will be partitioned into m strips and the Strip Lemma will then be used to provide that $A \cup B$ only contains an expected $O(m)$ maxima.

Due to changes in the geometry of the support and distribution of $\mathbf{B}_1 + \delta\mathbf{B}_2$ as δ grows, the analysis is split into three cases: (a)(i) $\frac{1}{\sqrt{n}} \leq \delta \leq \frac{1}{\sqrt{2}}$,

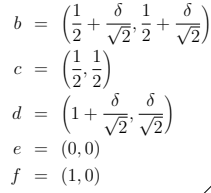
(a)(ii) $\frac{1}{\sqrt{2}} \leq \delta \leq \frac{\sqrt{n}}{\log n}$ and (a)(iii) $\frac{\sqrt{n}}{\log n} \leq \delta \leq \sqrt{n}$.

(a)(i) $\frac{1}{\sqrt{n}} \leq \delta \leq \frac{1}{\sqrt{2}}$:

Fix $\sigma = \sigma_b$ and define

$$m_1 = \left\lceil \frac{1}{\sigma} \left(\frac{1}{2} + \frac{\delta}{\sqrt{2}} \right) \right\rceil \quad \text{and} \quad m_2 = \left\lceil \frac{1}{2\sigma} \right\rceil,$$

$$\sigma_1 = \frac{1}{m_1} \left(\frac{1}{2} + \frac{\delta}{\sqrt{2}} \right) \quad \text{and} \quad \sigma_2 = \frac{1}{2m_2}.$$



Define (illustrated in Fig. 18)

$$\forall i \leq m_2, \quad x'_i = \frac{1}{2} + \frac{\delta}{\sqrt{2}} + i\sigma_2, \quad y'_i = 1 + \sqrt{2}\delta - x_i - 2\sigma_2, \quad p'_i = (x'_i, y'_i),$$

Note that $x_{m+1} = x'_0$. Also define

$$\begin{aligned} \forall i \leq m_1 - 1, \quad \text{Strip}_i &= \{\{u \in (A \cup B) : u.x \in [x_i, x_{i+1}]\}\}, \\ \forall i \leq m_2 - 1, \quad \text{Strip}'_i &= \{u \in A \cup B : u.x \in [x'_i, x'_{i+1}]\}, \quad \overline{\text{Strip}'_i} = \{u \in \text{Strip}'_i : u.y \leq y'_i\}. \end{aligned}$$

For $0 \leq i < m_1$ set

$$\begin{aligned} A_i(t) &= \{u \in \text{Strip}_{i+1} : u.y \geq y_{i+1} - t\}, & A_i &= A_i(y_{i+1}), \\ B(t) &= \{u \in \text{Strip}_i : u.y \geq y_{i+1} - t\}, & B_i &= B_i(y_{i+1}), \\ B'_i &= \emptyset, & \bar{B}_i &= B_i. \end{aligned}$$

From Eqs. 29 and 30 it is straightforward to see that $\forall t, \mu(B_i(t)) = O(\mu(A_i(t)))$, so A_i continuously dominates B_i and thus, from the Strip Lemma,

$$\sum_{i=0}^{m_1-1} \mathbf{E} [\text{MAX}(S_n) \cap \text{Strip}_i] = O(m_1). \quad (33)$$

Next, for $0 \leq i < m_2 - 1$ set

$$\begin{aligned} A_i(t) &= \{u \in \text{Strip}'_{i+1} : u.y \geq y_{i+1} - t\}, & A_i &= A_i(y_{i+1}) \\ B(t) &= \{u \in \overline{\text{Strip}'_i} : u.y \geq y_{i+1} - t\}, & B_i &= B_i(y_{i+1}) \\ B'_i &= P(p'_i) & \bar{B}_i &= B_i \cup B'_i \end{aligned}$$

and, for $i = m_2 - 1$, set

$$A_{m_2-1} = B_{m_2-1} = \emptyset \quad \text{and} \quad \bar{B}_{m_2-1} = B'_{m_2-1} = \text{Strip}'_{m_2-1}.$$

Because $\sigma(p'_i) = \Theta(\sigma_b)$, Eq. 31 gives $\mu(B'_i) = \mu(P(p'_i)) = O(\sigma_b^2 f(p'_i)) = O(1/n)$.

Again, from Eqs. 29 and 30 it is straightforward to see that for $0 \leq i < m_2 - 1$, $\forall t$, $\mu(B(t)) = O(\mu(A_i(t)))$. Since for $i = m_2 - 1$, $A_i = B_i = \emptyset$ it is trivially correct that (no matter what parameterization is used) $\mu(B_i(t)) = O(\mu(A_i(t)))$. Thus A_i continuously dominates B_i and, from the Strip Lemma,

$$\sum_{i=0}^{m_2-1} \mathbf{E} [|\text{MAX}(S_n) \cap \text{Strip}'_i|] = O(m_2). \quad (34)$$

Since the Strip_i and Strip'_i partition $A \cup B$, combining Eqs. 33 and 34 yields

$$\begin{aligned} \mathbf{E} [|\text{MAX}(S_n) \cap (A \cup B)|] &\leq \sum_{i=0}^{m_1-1} \mathbf{E} [|\text{MAX}(S_n) \cap \text{Strip}_i|] + \sum_{i=0}^{m_2-1} \mathbf{E} [|\text{MAX}(S_n) \cap \text{Strip}'_i|] \\ &= O(m_1 + m_2) = O\left(\frac{1}{\sigma_b}\right) = O\left(\frac{n^{2/7}}{\delta^{3/7}}\right). \end{aligned}$$

(a)(ii) $\frac{1}{\sqrt{2}} \leq \delta \leq \frac{\sqrt{n}}{\log n}$:

Set

$$b = \left(\frac{1}{2} + \frac{\delta}{\sqrt{2}}, \frac{1}{2} + \frac{\delta}{\sqrt{2}}\right), \quad d = \left(1 + \frac{\delta}{\sqrt{2}}, \frac{\delta}{\sqrt{2}}\right), \quad h = \left(\frac{1}{2} + \frac{\delta}{\sqrt{2}}, \frac{\delta}{\sqrt{2}}\right).$$

See Fig. 19. The proof works in two steps. The first is to show that

$$\mathbf{E} [|\text{MAX}(S_n) \cap (A \cup B)|] = \mathbf{E} [|\text{MAX}(S_n) \cap P(h)|] + O(1). \quad (35)$$

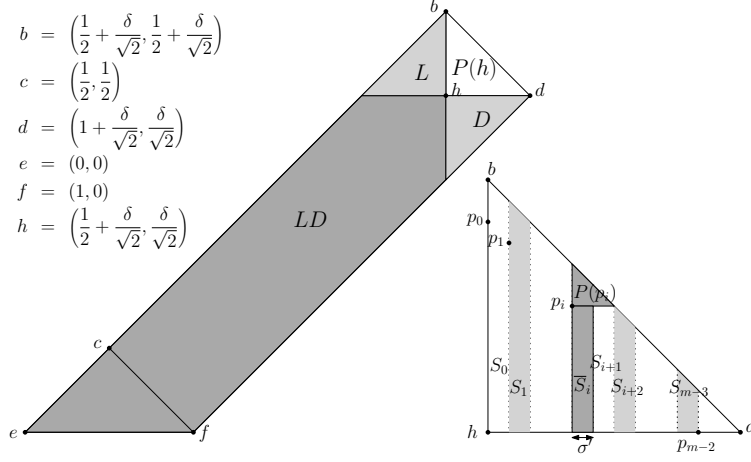


Figure 19: Illustration of the derivation of Lemma 23 for case (a)(ii). The right hand figure is a blown up version of $P(h)$. S_i , and $\bar{S}_i$ denote, respectively, Strip_i and $\bar{\text{Strip}}_i$.

The second is to prove

$$\mathbf{E} [|\text{MAX}(S_n) \cap P(h)|] = \begin{cases} O\left(\frac{n^{2/7}}{\delta^{3/7}}\right) & \text{if } \frac{1}{\sqrt{2}} \leq \delta \leq n^{1/5}, \\ O\left(\frac{n^{1/3}}{\delta^{2/3}}\right) & \text{if } n^{1/5} \leq \delta \leq \frac{\sqrt{n}}{\log n}. \end{cases} \quad (36)$$

Set

$$\begin{aligned} L &= \left\{ u \in A \cup B : u.x \leq \frac{1}{2} + \frac{\delta}{\sqrt{2}}, u.y \geq \frac{\delta}{\sqrt{2}} \right\}, \\ D &= \left\{ u \in A \cup B : u.x \geq \frac{1}{2} + \frac{\delta}{\sqrt{2}}, u.y \leq \frac{\delta}{\sqrt{2}} \right\}, \\ LD &= \left\{ u \in A \cup B : u.x \leq \frac{1}{2} + \frac{\delta}{\sqrt{2}}, u.y \leq \frac{\delta}{\sqrt{2}} \right\}. \end{aligned}$$

Since $h \in A \cup B$ and $\sigma(h) = \Theta(\delta)$, Lemma 20 implies that $f(h) = \Theta\left(\frac{1}{\delta^2}\right)$. Standard techniques show that

$$\mu(P(h)) = \Theta(\text{Area}(P(h)) \cdot f(h)) = \Theta\left(\frac{1}{\delta^2}\right).$$

From Lemma 1(c)

$$\Pr((S_n \cap P(h)) = \emptyset) = (1 - \mu(P(h)))^n \leq e^{-c \log^2 n} \quad (37)$$

for some constant $c > 0$. Since *any* point in $P(h)$ dominates *all* points in LD , $(S_n \cap P(h)) \neq \emptyset$ implies $\text{MAX}(S_n) \cap LD = \emptyset$. Thus

$$\mathbf{E} [|\text{MAX}(S_n) \cap LD|] \leq n \Pr((S_n \cap P(h)) = \emptyset) = o(1). \quad (38)$$

Now set

$$A(t) = \left\{ u \in P(h) : u.y \leq \frac{1}{2} + \frac{\delta}{\sqrt{2}} - t \right\}, \quad B(t) = \left\{ u \in L : u.y \leq \frac{1}{2} + \frac{\delta}{\sqrt{2}} - t \right\}.$$

It is straightforward to show that $\forall t \geq 0$, $\mu(B(t)) = O(\mu(A(t)))$, so the Sweep Lemma yields that

$$\mathbf{E} [|\text{MAX}(S_n) \cap L|] = O(1). \quad (39)$$

Similarly, setting

$$A(t) = \left\{ u \in P(h) : u.x \leq 1 + \frac{\delta}{\sqrt{2}} - t \right\}, \quad B(t) = \left\{ u \in D : u.y \leq 1 + \frac{\delta}{\sqrt{2}} - t \right\},$$

and noting that $\forall t \geq 0$, $\mu(B(t)) = O(\mu(A(t)))$ permits using the sweep Lemma to prove that

$$\mathbf{E} [|\text{MAX}(S_n) \cap D|] = O(1). \quad (40)$$

Combining Eqs. 38, 39 and 40 proves Eq. 35.

Eq. 36 will be proven using the Strip Lemma. Fix σ to be a value to be specified later and set

$$m = \left\lceil \frac{1}{2\sigma} \right\rceil \quad \text{and} \quad \sigma' = \frac{1}{2m}.$$

Note that $\sigma = \Theta(\sigma')$. Now set

$$\forall i \leq m, \quad x_i = \frac{1}{2} + \frac{\delta}{\sqrt{2}} + i\sigma', \quad y_i = 1 + \sqrt{2}\sigma - x_i - 2\sigma, \quad p_i = (x_i, y_i).$$

$$\forall i \leq m-2, \quad \text{Strip}_i = \{u \in P(h) : u.x \in [x_i, x_{i+1}]\}, \quad \overline{\text{Strip}}_i = \{u \in \text{Strip}_i : u.y \leq y_i\}.$$

Now, for all $i \leq m - 3$ set

$$\begin{aligned} A_i(t) &= \{u \in \text{Strip}_{i+1} : u.y \leq y_i - t\}, & A_i &= A_i(y_i), \\ B_i(t) &= \{u \in \overline{\text{Strip}}_i : u.y \leq y'_i - t\}, & B_i &= B_i(y_i), \\ B'_i &= P(p_i), & \bar{B}_i &= B'_i \cup B_i, \end{aligned}$$

and for $i = m - 2$,

$$A_{m-2} = B_{m-2} = \emptyset \quad \text{and} \quad \bar{B}_{m-2} = B'_{m-2} = P(p_{m-2}).$$

Again, from Eqs. 29 and 30 it is straightforward to see that, for $i \leq m - 3$. $\forall t$, $\mu(B(t)) = O(\mu(A(t)))$. Since for $i = m - 2$, $A_i = B_i = \emptyset$ it is trivially correct that (no matter what parameterization is used) $\mu(B_i(t)) = O(\mu(A_i(t)))$. Thus A_i continuously dominates B_i for all $i \leq m - 2$.

To complete the proof note that $\mu(P(p_i)) = \Theta(\sigma^2(p_i)f(p_i))$.

If $\delta \leq n^{1/5}$, set $\sigma = \sigma_b$. Since $\sigma(p_i) = \Theta(\sigma_b)$, Eq. 31 gives

$$\mu(P(p_i)) = \Theta\left(\frac{1}{n}\right).$$

If $\delta > n^{1/5}$ set $\sigma = \sigma_c$. Since $\sigma(p_i) = \Theta(\sigma_b)$, Eq. 32 gives

$$\mu(P(p_i)) = \Theta\left(\frac{1}{n}\right).$$

Substituting these values of σ in and applying the Strip Lemma with the defined $A_i, B_i, B'_i, \bar{B}_i$, noting that $P(h) = \cup_{i=0}^{m-2} \bar{B}_i$, yields

$$\mathbf{E}[|\text{MAX}(S_n) \cap P(h)|] = O(m) = O\left(\frac{1}{\sigma}\right) = \begin{cases} O\left(\frac{1}{\sigma_b}\right) = O\left(\frac{n^{2/7}}{\delta^{3/7}}\right), & \text{if } \delta \leq n^{1/5}, \\ O\left(\frac{1}{\sigma_c}\right) = O\left(\frac{n^{1/3}}{\delta^{2/3}}\right) & \text{if } \delta > n^{1/5}. \end{cases}$$

(a)(iii) $\frac{\sqrt{n}}{\log n} \leq \delta \leq \sqrt{n}$:

Note that the method used for case (a)(ii) fails in this interval because Eq. 37 is no longer valid. Moreover, $\mu(p(h))$ can be so small, that $\Pr(S_n \cap P(h) = \emptyset)$ is non-negligible and thus LD might contain some maximal points.

Refer to Fig. 20 for definitions partitioning $A \cup B$ into three regions U, V, W . Note that $\text{Area}(U) = \log^2 n$ and more than half of the points v in U have $\sigma(v) > 1$. For these points v Lemma 20 implies that $f(v) = \Theta\left(\frac{1}{\delta^2}\right)$ and thus $\mu(A) = \Theta\left(\frac{\log^2 n}{\delta^2}\right) = \Omega\left(\frac{\log^2 n}{n}\right)$ and also $\mu(A) = O\left(\frac{\log^4 n}{n}\right)$.

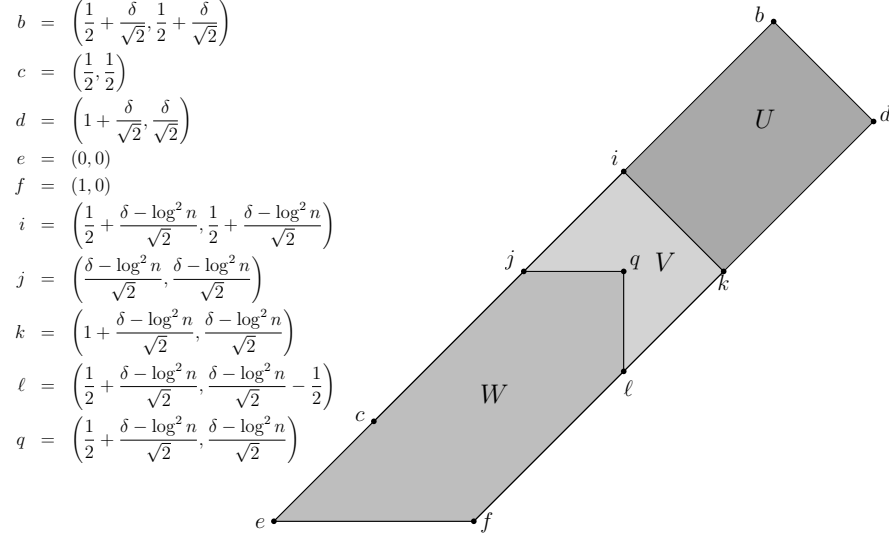


Figure 20: Illustration of the derivation of Lemma 23 for case (a)(iii).

Further note that $\text{Area}(V) = O(1)$ so $\mu(V) = O\left(\frac{1}{\delta^2}\right) = O\left(\frac{\log^2 n}{n}\right)$.

From Lemma 1(c)

$$\Pr((S_n \cap U) = \emptyset) = (1 - \mu(U))^n \leq e^{-c \log^2 n},$$

for some constant $c > 0$. Since *any* point in U dominates *all* points in W , $(S_n \cap U) \neq \emptyset$ implies $\text{MAX}(S_n) \cap U = \emptyset$. Thus

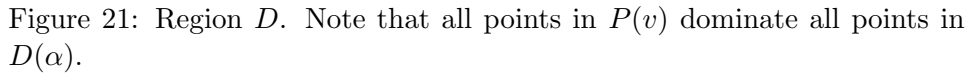
$$\mathbf{E}[|\text{MAX}(S_n) \cap W|] \leq n \Pr((S_n \cap U) = \emptyset) = o(1).$$

This implies

$$\begin{aligned} \mathbf{E}[|\text{MAX}(S_n) \cap (A \cup B)|] &= \mathbf{E}[|\text{MAX}(S_n) \cap U|] + \mathbf{E}[|\text{MAX}(S_n) \cap V|] + \mathbf{E}[|\text{MAX}(S_n) \cap W|] \\ &\leq \mathbf{E}[|S_n \cap U|] + \mathbf{E}[|S_n \cap V|] + o(1) \\ &= n\mu(U) + n\mu(V) + o(1) = O(\log^4 n). \end{aligned}$$

(b) $v \in C$:

Before starting the main bound we first note how to restrict the analysis to $v \in C$ with $\sigma(v) \leq d\delta$ for some constant $d > 0$.


$$\begin{aligned} D(\alpha) &= \left\{ u \in C : \sigma(u) \geq \left(\frac{1}{\sqrt{2}} - \alpha \right) \delta \right\}, \\ E(\alpha) &= \left\{ u \in C : \sigma(u) \leq \left(\frac{1}{\sqrt{2}} - \alpha \right) \delta \right\}. \end{aligned}$$
$$\mathbf{E} [|\text{MAX}(S_n) \cap D(\alpha)|] = O(\log^2 n). \quad (41)$$
$$\mu(D(\alpha)) = O(\text{Area}(D(\alpha))) = O(\delta^2) = O\left(\frac{\log^2 n}{n}\right).$$
$$\mathbf{E} [|\text{MAX}(S_n) \cap D(\alpha)|] \leq \mathbf{E} [|S_n \cap D(\alpha)|] = n\mu(D(\alpha)) = O(\log^2 n).$$

For the case $\delta > \frac{\log n}{\sqrt{n}}$, see Figure 21 for the definitions of points and regions. From Eqs. 29 and 30 together with the fact that $\sigma(v) = \Theta(\delta)$,

$$\mu(P(v)) = \Theta\left(\text{Area}(P(v))f(v)\right) = \Theta(\delta^2 f(v)).$$

Recall that $\mu(P(v)) = \Theta(\text{Area}(P(v))f(v)) = \Theta(\delta^2 f(v))$. From Eq. 26 in Lemma 20: If $\delta \leq 1$ then $f(v) = \Theta(1)$. If $\delta > 1$ then $f(v) = \Theta\left(\frac{1}{\delta^2}\right)$. In both cases this implies $\mu(P(v)) = \Omega\left(\frac{\log^2 n}{n}\right)$.

From Lemma 1(c)

$$\Pr((S_n \cap P(v)) = \emptyset) = (1 - \mu(P(v)))^n \leq e^{-c \log^2 n}, \quad (42)$$

for some constant $c > 0$. Since *any* point in $P(v)$ dominates *all* points in $D(\alpha)$, $(S_n \cap P(v)) \neq \emptyset$ implies $\text{MAX}(S_n) \cap D(\alpha) = \emptyset$. Thus

$$\mathbf{E} [|\text{MAX}(S_n) \cap D(\alpha)|] \leq n \Pr((S_n \cap P(v)) = \emptyset) = o(1), \quad (43)$$

and we have completed the proof of Eq. 41 for all cases. This implies that, for any fixed $\alpha \leq \frac{1}{\sqrt{2}}$,

$$\mathbf{E} [|\text{MAX}(S_n) \cap C|] = \mathbf{E} [|\text{MAX}(S_n) \cap E(\alpha)|] + O(\log^2 n). \quad (44)$$

Fix $\alpha \leq \frac{1}{\sqrt{2}}$ and set $d = 1 - \left(\frac{1}{\sqrt{2}} - \alpha\right) > 0$. This restricts the analysis to $v \in C$ with $\sigma(v) \leq d\delta$.

Our analysis will require the following definition and Lemma which are generalizations of the the structures and proofs techniques introduced in Lemma 19.

Definition 15 Let $0 \leq \theta_1 \leq \theta_2 \leq \frac{\pi}{4}$. Define the $\bar{\theta}_1, \bar{\theta}_2$ wedge as

$$W(\bar{\theta}_1, \bar{\theta}_2) = \{v \in E(\alpha) : \theta_1 < \theta \leq \theta_2\}.$$

See Fig. 23 for an example.

Lemma 24 Let $W = W(\bar{\theta}_1, \bar{\theta}_2)$ where $0 \leq \bar{\theta}_1 \leq \bar{\theta}_2 < \frac{\pi}{4}$.

Given $m > 0$, set $\hat{\theta} = (\bar{\theta}_2 - \bar{\theta}_1)/m$. For $i \in [0, m]$, set $\theta_i = \bar{\theta}_1 + i\hat{\theta}$ and,

$$\forall i \in [0, m-1], \quad R_i = W(\theta_i, \theta_{i+1}) \quad \text{and} \quad R_i(\sigma) = \{v \in R_i : \sigma(v) \leq \delta\}.$$

Now set $\bar{\sigma} = \hat{\theta}\delta$. Given that $\bar{\sigma} \leq d\delta$ and $\forall i, \quad \mu(R_i(\bar{\sigma})) = O(1/n)$, then

$$\mathbf{E} [|\text{MAX}(S_n) \cap W|] = O(m) + O(\log^2 n).$$

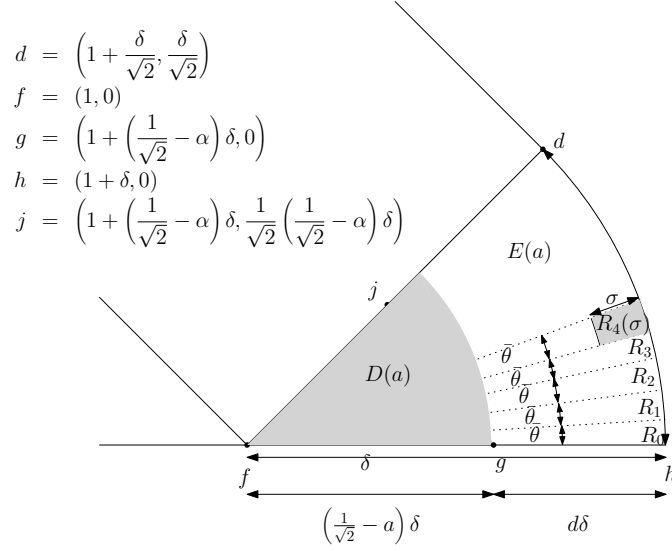


Figure 22: Illustration of the decomposition of $E(a)$ into the R_i in the proof of b(i).

The proof of this lemma is deferred until the end of this section.

We now analyze $\mathbf{E}[|\text{MAX}(S_n) \cap E(\alpha)|]$ by splitting into two cases: (i) $\delta \leq 1$ and (ii) $\delta \geq 1$.

b(i) $\delta \leq 1$:

Set $\bar{\theta}_1 = 0$, $\bar{\theta}_2 = \frac{\pi}{4}$ and $m = \left\lfloor \frac{n^{2/7}}{\delta^{3/7}} \right\rfloor$.

Let R_i and $R_i(\theta)$ be as defined by Lemma 24. Since for large enough n , $\bar{\sigma} \leq d\delta$, to use that lemma, we only need to show that $\mu(R_i(\bar{\sigma})) = \Theta(1/n)$.

To see this, let $v \in B \cap C$, i.e., $\theta(v) = \pi/4$. Then, from Lemma 20

$$f(v) = O\left(\left(\frac{\sigma(v)}{\delta}\right)^{3/2}\right). \quad (45)$$

Corollary 21 then implies that

$$\forall v \in E(\alpha), \quad f(v) = O\left(\left(\frac{\sigma(v)}{\delta}\right)^{3/2}\right). \quad (46)$$

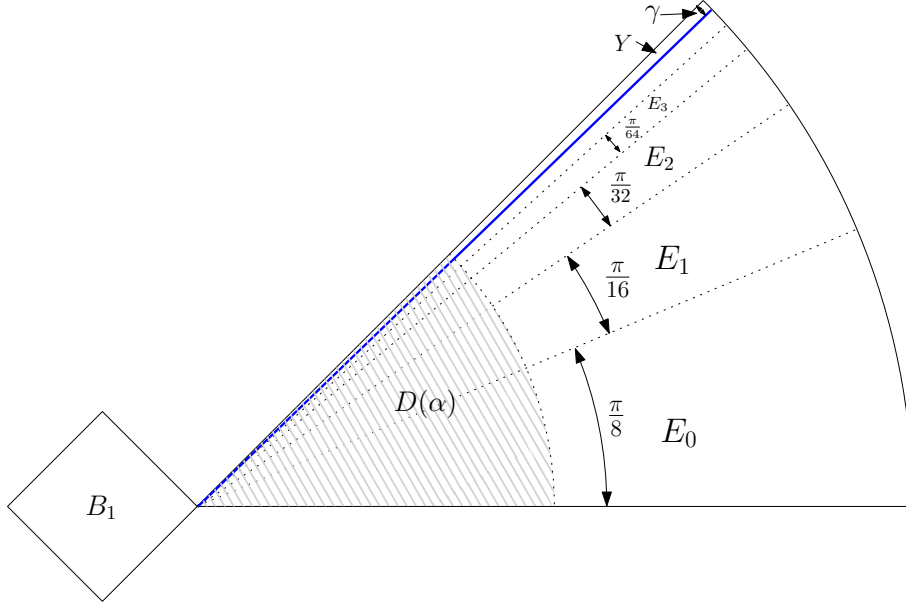


Figure 23: Illustrates the further partition of $E(\alpha)$ into smaller wedges Y and $E_0, E_1, E_2, E_3, \dots$ where $E_i = W\left(\frac{\pi}{4} - \frac{\pi}{2^i+2}, \frac{\pi}{4} - \frac{\pi}{2^i+3}\right)$.

The same integration as in the proof of Lemma 19 now yields

$$\mu(R_i(\sigma)) = O\left(\frac{1}{m} \int_{r=0}^{\sigma} \frac{r^{3/2}}{\delta^{3/2}} dr\right) = O\left(\frac{\sigma^{5/2}}{m\delta^{3/2}}\right). \quad (47)$$

In particular, for $\bar{\sigma} = \Theta(1/m)$ this evaluates out to

$$\mu(R_i(\bar{\sigma})) = \Theta(1/n). \quad (48)$$

Applying Lemma 24 then proves that

$$\mathbf{E} \left[\left| MAX(S_n) \cap W\left(0, \frac{\pi}{4} - \frac{\pi}{4m}\right) \right| \right] = O(m) + O(\log^2 n) = O\left(\frac{n^{2/7}}{\delta^{3/7}}\right).$$

b(ii) $\delta \geq 1$:

Set $\gamma_0 = \delta^{-2/7} n^{-1/7}$. Further define angles θ_i and wedges E_i by

$$\theta_0 = 0, \theta_1 = \frac{\pi}{8}, \quad \forall i \geq 1, \theta_{i+1} = \theta_i + \frac{1}{2^i} \cdot \frac{\pi}{8} = \frac{\pi}{4} - \frac{\pi}{2^{i+2}}, \quad \theta'_i = \theta_i + \frac{1}{2^i} \frac{\pi}{8},$$

$$E_i = W(\theta_i, \theta_{i+1}).$$

Finally, set

$$k = \max \left\{ i : \theta_i < \frac{\pi}{4} - \gamma_0 \right\} \quad \text{and} \quad Y = W \left(\theta_{k+1}, \frac{\pi}{4} \right).$$

Then

$$\mathbf{E} [| \text{MAX}(S_n) \cap E(\alpha) |] \leq \sum_{i=0}^k \mathbf{E} [| \text{MAX}(S_n) \cap E_i |] + \mathbf{E} [| \text{MAX}(S_n) \cap Y |]. \quad (49)$$

Note: The reason for the inequality instead of an equality is that E_k and Y might overlap.

We now analyze each item on the right hand side separately and then add them all together to complete the proof.

E_0 : Apply Lemma 24 to E_0 with $\bar{\theta}_1 = \theta_0$, $\bar{\theta}_2 = \theta_1$ and $m = \left\lceil \sqrt{\delta} n^{1/4} \right\rceil$, for some constant $c > 0$. Note that $\hat{\theta} = \Theta \left(\frac{1}{\sqrt{\delta} n^{1/4}} \right)$ so $\bar{\sigma} = \hat{\theta} \delta = \Theta \left(\frac{d\sqrt{\delta}}{n^{1/4}} \right)$.

Note that $\delta \leq \sqrt{n}$ implies $\bar{\sigma} \leq 1$ and also, for later use, $\bar{\sigma} \leq d\delta$.

For $v \in E_1$, $\gamma = \Omega(1)$. Let $\sigma = \sigma(v)$. If $\sigma \leq 1$ then Eq. 27 of Lemma 20 implies

$$f(v) = \Theta \left(\frac{\sigma^2}{\delta^2} \right).$$

The same type of integration as in the proof of Lemma 19 now yields (assuming $\sigma \leq d\delta$)

$$\mu(R_i(\sigma)) = O \left(\frac{1}{m} \int_{r=0}^{\sigma} (\delta - r) \frac{r^2}{\delta^2} dr \right) = O \left(\frac{\sigma^3}{m\delta} \right). \quad (50)$$

This yields $\mu(R_i(\bar{\sigma})) = O(1/n)$. Applying Lemma 24 to E_1 yields

$$\mathbf{E} [| \text{MAX}(S_n) \cap E_i |] = O(m) + O(\log^2 n) = O \left(\sqrt{\delta} n^{1/4} \right). \quad (51)$$

$E_i, i > 0$:

Let $v \in E(\alpha)$. For simplicity in the remainder of the proof, set $\gamma = \gamma(v) = \pi/4 - \theta(v)$ and $\sigma = \sigma(v)$.

Let $x_1 > 0$ (with value to be specified later). Set

$$\sigma_\gamma = \frac{\sqrt{\delta}}{n^{1/4}} (x_1 \gamma)^{1/4}.$$

To ensure that the second case in Eq. 27 of Lemma 20 holds for σ_γ we need $\sigma_\gamma \delta = O((\delta \gamma)^2)$, i.e, $\sigma_\gamma \leq x_2 \delta \gamma^2$ for some fixed $x_2 > 0$. Note

$$\begin{aligned} \sigma_\gamma \leq x_2 \delta \gamma^2 &\Leftrightarrow x_1^{1/4} \frac{\sqrt{\delta}}{n^{1/4}} \delta \gamma^{1/4} \leq x_2 \delta \gamma^2 \\ &\Leftrightarrow x_1^{1/4} \frac{1}{\sqrt{\delta} n^{1/4}} \leq x_2 \gamma^{7/4} \\ &\Leftrightarrow \left(\frac{x_1}{x_2^4} \right)^{1/7} \frac{1}{\delta^{2/7} n^{1/7}} \leq \gamma. \end{aligned}$$

Assume further that x_1, x_2 satisfy $\frac{x_1}{x_2^4} < 1$. Now, if $v \in W\left(\frac{\pi}{8}, \frac{\pi}{4} - \gamma_0\right)$ then

$$\gamma_0 \leq \gamma \leq \frac{\pi}{4}.$$

In particular, suppose $\sigma < \sigma_\gamma$. Then the above implies

$$\begin{aligned} \gamma > \left(\frac{x_1}{x_2^4} \right)^{1/7} \gamma_0 &\Rightarrow \gamma \geq \left(\frac{x_1}{x_2^4} \right)^{1/7} \frac{1}{\delta^{2/7} n^{1/7}} \\ &\Rightarrow \sigma \leq \sigma_\gamma \leq x_2 \delta \gamma^2 \\ &\Rightarrow \text{the second case in Eq. 27 applies} \\ &\Rightarrow f(v) = \Theta\left(\frac{\sigma^2}{\delta^2} \frac{1}{\gamma}\right). \end{aligned}$$

Recall that $E_i = W(\theta_i, \theta_{i+1})$ where $\theta'_{i+1} - \theta_i = \frac{\pi}{2^{i+3}}$.

Let $v \in E_i$ and set $\gamma = \frac{\pi}{4} - \theta(v)$. By definition

$$\frac{\pi}{2^{i+3}} \leq \gamma \leq \frac{\pi}{2^{i+2}}.$$

Set

$$\sigma_i = \frac{\sqrt{\delta}}{n^{1/4}} \left(\frac{\pi}{2^{i+2}} \right)^{1/4}.$$

Then

$$\sigma_i = \frac{\sqrt{\delta}}{n^{1/4}} (x_1 \gamma)^{1/4},$$

where $1 \leq x_1 \leq 2$.

Now fix $x_2 = 2$. By the argument above, all $v \in E_i$ with $\sigma \leq \sigma_i$, satisfy the v in the second case of Eq. 27. So

$$f(v) = \Theta\left(\frac{\sigma^2}{\delta^2} \frac{1}{\gamma}\right) = \Theta\left(\frac{\sigma^2}{\delta^2} 2^i\right). \quad (52)$$

Apply Lemma 24 to E_i with $\bar{\theta}_1 = \theta_i$, $\bar{\theta}_2 = \theta_{i+1}$ and $m = m_i = \left\lceil c \frac{1}{2^{3i/4}} \sqrt{\delta} n^{1/4} \right\rceil$, for some constant $c > 0$. Note that

$$\hat{\theta} = \Theta\left(\frac{\theta_{i+1} - \theta_i}{m}\right) = \Theta\left(\frac{1}{2^{i/4} \sqrt{\delta} n^{1/4}}\right),$$

therefore $\bar{\sigma} = \hat{\theta} \delta = \Theta\left(\frac{\sqrt{\delta}}{2^{1/4} n^{1/4}}\right)$. Choose c large enough so that $\bar{\sigma} \leq \frac{d\sqrt{\delta}}{n^{1/4}}$.

Following the same logic as in Eq. 50

$$\mu(R_i(\sigma)) = O\left(\hat{\theta} \int_{r=0}^{\sigma} (\delta - r) \frac{r^2}{\delta^2} 2^i dr\right) = O\left(\hat{\theta} \frac{\sigma^3}{\delta} 2^i\right). \quad (53)$$

Using $\bar{\sigma} = \hat{\theta} \delta$ permits evaluating

$$\mu(R_i(\bar{\sigma})) = O\left(\hat{\theta} \frac{\bar{\sigma}^3}{\delta} 2^i\right) = O\left(\frac{\bar{\sigma}^4}{\delta^2} 2^i\right) = O\left(\frac{1}{n}\right).$$

Applying Lemma 24 to E_i yields

$$\mathbf{E}[|\text{MAX}(S_n) \cap E_i|] = O(m_i) + \log^2 n = O\left(2^{-3i/4} \sqrt{\delta} n^{1/4}\right) + O(\log^2 n). \quad (54)$$

Y :

Recall that $Y = W\left(\frac{\pi}{4} - \gamma_0, \frac{\pi}{4}\right)$, where $\gamma_0 = \delta^{-2/7} n^{-1/7}$. We apply Lemma 24 to Y with $\bar{\theta}_1 = \frac{\pi}{4} - \gamma_0$ and $\bar{\theta}_2 = \frac{\pi}{4}$. There are two cases to be analyzed separately: (A) $1 \leq \delta \leq n^{1/5}$ and (B) $n^{1/5} \leq \delta \leq \sqrt{n}$.

(A) $1 \leq \delta \leq n^{1/5}$: Set

$$m = \left\lceil c n^{1/7} \delta^{2/7} \right\rceil, \quad \hat{\theta} = \frac{\theta_2 - \theta_1}{m} = \frac{\gamma_0}{m} = \Theta\left(\delta^{-4/7} n^{-2/7}\right), \quad \bar{\sigma} = \hat{\theta} \delta = \Theta\left(\frac{\delta^{3/7}}{n^{2/7}}\right),$$

for some constant $c > 0$.

First observe that because $\delta \leq n^{-1/5}$, if c is large enough, then $\bar{\sigma} < 1$. Next observe that

$$\delta \gamma_0^2 = \delta \delta^{-4/7} n^{-2/7} = \bar{\sigma}.$$

Therefore, if $v \in Y$ with $\sigma(v) = \Theta(\bar{\sigma})$, then this is the second case in Eq. 27 of Lemma 20. Since $\delta \leq n^{1/5}$,

$$\sqrt{\delta\bar{\sigma}} = \sqrt{\frac{\delta^{10/7}}{n^{2/7}}} = \left(\frac{\delta^5}{n}\right)^{1/7} \leq 1.$$

Plugging into Eq. 27 gives

$$f(v) = \Theta\left(\frac{\bar{\sigma}}{\delta^2} \sqrt{\delta\bar{\sigma}}\right) = \Theta\left(\frac{\bar{\sigma}^{3/2}}{\delta^{3/2}}\right).$$

Thus, for all $v \in Y$ with $\sigma(v) \leq \bar{\sigma}$,

$$f(v) = O\left(\frac{\bar{\sigma}^{3/2}}{\delta^{3/2}}\right).$$

Repeating the standard integration gives

$$\mu(R_i(\bar{\sigma})) = O\left(\hat{\theta} \int_{r=0}^{\bar{\sigma}} (\delta - r) \frac{\bar{\sigma}^{3/2}}{\delta^{3/2}} dr\right) = O\left(\hat{\theta} \delta \bar{\sigma} \frac{\bar{\sigma}^{3/2}}{\delta^{3/2}}\right) = O\left(\frac{\bar{\sigma}^{7/2}}{\delta^{3/2}}\right) = O(1/n).$$

Thus

$$\mathbf{E}[|MAX(S_n) \cap Y|] = O(m) + O(\log^2 n) = O\left(n^{1/7} \delta^{2/7}\right) = O\left(\sqrt{\delta} n^{1/4}\right).$$

(B) $n^{1/5} \leq \delta \leq \sqrt{n}$: Set

$$m = \left\lceil \delta^{1/21} n^{4/21} \right\rceil, \quad \hat{\theta} = \frac{\gamma_0}{m} = \Theta\left(\delta^{-1/3} n^{-1/3}\right), \quad \bar{\sigma} = \hat{\theta} \delta = \Theta\left(\frac{\delta^{2/3}}{n^{1/3}}\right),$$

for some constant $c > 0$. Note that if c is chosen large enough, then $\bar{\sigma} \leq 1$ and $\bar{\sigma} \leq d\delta$.

Next observe that

$$\begin{aligned} \delta \gamma^2 = \frac{\delta^{3/7}}{n^{2/7}} \leq \frac{\delta^{2/3}}{n^{1/3}} = \bar{\sigma} &\Leftrightarrow n^{1/21} = n^{\frac{1}{3} - \frac{2}{7}} \leq \delta^{\frac{2}{3} - \frac{3}{7}} = 2\delta^{5/21} \\ &\Leftrightarrow n^{1/5} \leq \delta. \end{aligned}$$

So, if $v \in Y$ with $\sigma(v) = \Theta(\bar{\sigma})$, this is again the second case in Eq. 27 of Lemma 20. Unlike in case (A) above, though, $\delta \geq n^{1/5}$ implies that

$$\sqrt{\delta\sigma} = \sqrt{\frac{\delta^{5/3}}{n^{1/3}}} = \left(\frac{\delta^5}{n}\right)^{1/6} = \Omega(1).$$

Plugging into Eq. 27, then gives

$$f(v) = \Theta\left(\frac{\bar{\sigma}}{\delta^2}\right).$$

Thus, for all $v \in Y$ with $\sigma(v) \leq \bar{\sigma}$,

$$f(v) = O\left(\frac{\bar{\sigma}}{\delta^2}\right).$$

Repeating the standard integration gives

$$\mu(R_i(\bar{\sigma})) = O\left(\hat{\theta} \int_{r=0}^{\bar{\sigma}} (\delta - r) \frac{\bar{\sigma}}{\delta^2} dr\right) = O\left(\hat{\theta} \delta \bar{\sigma} \frac{\bar{\sigma}}{\delta^2}\right) = O\left(\frac{\bar{\sigma}^3}{\delta^2}\right) = O(1/n).$$

Thus

$$\mathbf{E}[|MAX(S_n) \cap Y|] = O(m) + O(\log^2 n) = O\left(\delta^{1/21} n^{4/21}\right) = O\left(\sqrt{\delta} n^{1/4}\right).$$

Combining the pieces:

We now complete the analysis in part (b) (ii) by combining the pieces together, plugging into Eq. 49 and using the fact that $k = O(\log n)$ to yield

$$\begin{aligned} EXP|MAX(S_n) \cap E(\alpha)| &\leq \sum_{i=0}^k \mathbf{E}[|MAX(S_n) \cap E_i|] + \mathbf{E}[|MAX(S_n) \cap Y|] \\ &= \sum_{i=0}^k \left(O\left(2^{-3i/4} \sqrt{\delta} n^{1/4}\right) + O(\log^2 n) \right) + O\left(\sqrt{\delta} n^{1/4}\right) \\ &= O\left(\sqrt{\delta} n^{1/4}\right) + O(k \log^2 n) + O\left(\sqrt{\delta} n^{1/4}\right) \\ &= O\left(\sqrt{\delta} n^{1/4}\right). \end{aligned}$$

□

Proof: of Lemma 20.

From Lemma 3,

$$f(v) = \Theta\left(\frac{\text{Area}(B_2(v, \delta) \cap B_1)}{\delta^2}\right). \quad (55)$$

For each region the proof performs a case-by-case analysis to derive the value of $\text{Area}(B_2(v, \delta) \cap B_1)$.

$v \in A$: This is a just a restatement of of Lemma 4(b).

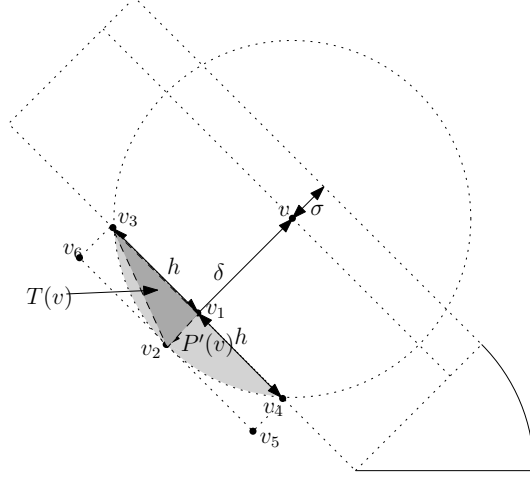


Figure 24: Illustration of the derivation of Lemma 20 when $v \in B$.

$v \in B$: Consider Figure 24. For simplicity, let σ denote $\sigma(v)$. Let v_3 and v_4 be the upper and lower intersection of $B_2(v, \delta)$ with the line $x + y = 1$ (the upper-right boundary of B_1). Now set $v_1 = \frac{v_3 + v_4}{2}$ to be their midpoint, and $h = \|v_1 - v_3\|$. Draw the radius of $B_2(v, \delta)$ that passes through v_1 . Label its other endpoint as v_2 . Finally, let T be the triangle with vertices v_1, v_2, v_3 . Straightforward geometric arguments show that

$$\text{Area}(B_2(v, \delta) \cap B_1) = \Theta(\text{Area}(T \cap B_1)) = \Theta\left(\min\left(h, \frac{1}{\sqrt{2}}\right) \cdot \min(\sigma, \sqrt{2})\right).$$

Next, note that

$$h^2 = \delta^2 - (\delta - \sigma)^2 = \delta^2 \left(1 - \left(1 - \frac{\sigma}{\delta}\right)^2\right).$$

Since $\sigma \leq \delta$,

$$\begin{aligned} h &= \delta \sqrt{1 - \left(1 - \frac{\sigma}{\delta}\right)^2} \\ &= \delta \sqrt{\frac{2\sigma}{\delta} - \frac{\sigma^2}{\delta^2}} \\ &= \Theta\left(\delta \sqrt{\frac{\sigma}{\delta} \left(2 - \frac{\sigma}{\delta}\right)}\right) = \Theta(\sqrt{\delta\sigma}). \end{aligned}$$

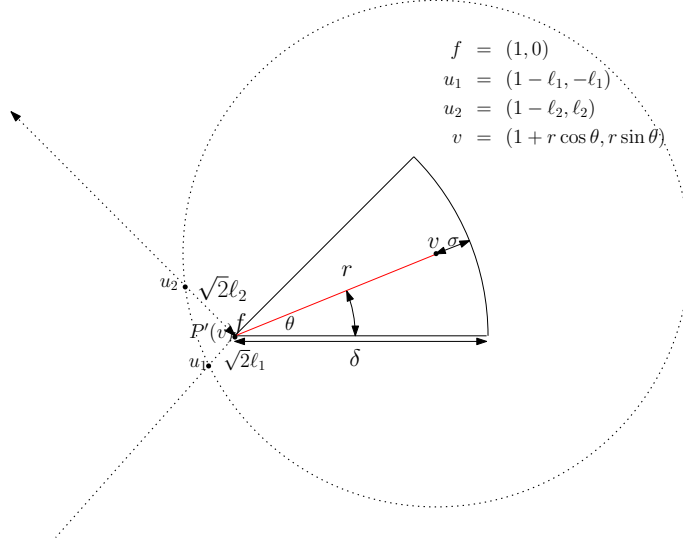


Figure 25: Illustration of the derivation of Lemma 20 when $v \in C$. This is the simple case in which $\ell_1, \ell_2 \leq 1$.

Again since $\sigma \leq \delta$, if $\sigma \geq \sqrt{2}$, then $\delta = \Omega(1)$ and $h = \Omega(1)$. Thus, $\text{Area}(B_2(v, \delta) \cap B_1) = \Theta(1)$. Working through the other cases similarly and plugging the derived values into Eq. 55 yields

$$f(v) = \Theta \left(\frac{\text{Area}(T \cap B_1)}{\delta^2} \right) = \begin{cases} \Theta \left(\frac{1}{\delta^2} \right) & \text{if } \sigma = \Omega(1), \\ \Theta \left(\frac{\sigma}{\delta^2} \right) & \text{if } \delta\sigma = \Omega(1) \text{ and } \sigma = O(1), \\ \Theta \left(\frac{\sqrt{\delta\sigma}}{\delta^2} \right) = \Theta \left(\left(\frac{\sigma}{\delta} \right)^{3/2} \right) & \text{if } \delta\sigma = O(1) \text{ and } \sigma = O(1). \end{cases}$$

$v \in C$: Fix $v \in C$. Set $\sigma = \sigma(v)$, $r = r(v) = \delta - \sigma(v)$, and $\theta = \theta(v)$. By definition, $0 \leq \theta(v) \leq \pi/4$. Also, from the assumption that $\sigma < d\delta$ for some $\delta < 1$, we know that $r = \Theta(\delta)$.

See Fig. 25. Let u_1, u_2 be the points of intersection of $B_2(v, \delta)$ with the boundary lines $x + y = 1$ and $x - y = 1$ of B_1 and $f = (1, 0)$. Let $\ell_1 = \frac{1}{\sqrt{2}} \|u_1 - f\|$ and $\ell_2 = \frac{1}{\sqrt{2}} \|u_2 - f\|$.

Set $P'(v) = B_2(v, \delta) \cap B_1$ and T to be the triangle formed by u_1, u_2, f . First note that if $\ell_1, \ell_2 \leq 1$, then $T \subset B_1$ and, since

$$\text{Area}(T) < \text{Area}(P'(v)) < 2\text{Area}(T),$$

we find $\text{Area}(P'(v)) = \Theta(\text{Area}(T)) = \Theta(\ell_1 \ell_2)$. Note that if $\ell_1 > 1$ or $\ell_2 > 1$ then $T \not\subseteq B_1$. This can be fixed by capping the lengths of the triangle to stay within B_1 , i.e, if $\ell_1 \geq 1$, replacing v_1 by $(0, -\sqrt{2})$ and if $\ell_2 \geq 1$, replacing v_2 by $(0, \sqrt{2})$. Straightforward geometric arguments show that

$$f(v) = \Theta\left(\frac{\text{Area}(P'(v))}{\delta^2}\right) = \Theta\left(\frac{\min(\ell_1, 1) \cdot \min(\ell_2, 1)}{\delta^2}\right). \quad (56)$$

Now note that

$$\begin{aligned} \delta^2 &= \|v - u_1\|^2 \\ &= (r \cos \theta + \ell_1)^2 + (r \sin \theta + \ell_1)^2 \\ &= r^2 \cos^2 \theta + 2r\ell_1 \cos \theta + \ell_1^2 + r^2 \sin^2 \theta + 2r\ell_1 \sin \theta + \ell_1^2 \\ &= r^2 + 2r\ell_1(\cos \theta + \sin \theta) + 2\ell_1^2. \end{aligned}$$

Similarly

$$\begin{aligned} \delta^2 &= \|v - u_2\|^2 \\ &= (r \cos \theta + \ell_2)^2 + (r \sin \theta - \ell_2)^2 \\ &= r^2 \cos^2 \theta + 2r\ell_2 \cos \theta + \ell_2^2 + r^2 \sin^2 \theta - 2r\ell_2 \sin \theta + \ell_2^2 \\ &= r^2 + 2r\ell_2(\cos \theta - \sin \theta) + 2\ell_2^2. \end{aligned}$$

These can be rewritten as

$$\Theta(\delta\sigma) = \frac{1}{2}(2\delta - \sigma)\sigma = \frac{\delta^2 - r^2}{2} = r\ell_1\beta_\theta + \ell_1^2, \quad (57)$$

$$\Theta(\delta\sigma) = \frac{1}{2}(2\delta - \sigma)\sigma = \frac{\delta^2 - r^2}{2} = r\ell_2\alpha_\theta + \ell_2^2, \quad (58)$$

where

$$\beta_\theta = \cos \theta + \sin \theta \quad \text{and} \quad \alpha_\theta = \cos \theta - \sin \theta.$$

Since $\alpha_\theta, \beta_\theta \geq 0$ and $\sigma \leq \delta$, Eqs. 57 and 58 immediately imply $\ell_1, \ell_2 = O(\delta)$. Since $0 \leq \theta(v) \leq \pi/4$, $\beta_\theta = \Theta(1)$ so Eq. 57 can be rewritten as

$$\Theta(\delta\sigma) = \Theta(r\ell_1 + \ell_1^2), \quad (59)$$

which combined with $r = \Theta(\delta)$, implies

$$\ell_1 = \Theta(\sigma).$$

The analysis of ℓ_2 is more complicated. Unlike β_θ , α_θ *decreases* monotonically to 0 as $\theta \uparrow \pi/4$. In particular, $\alpha_\theta \neq \Theta(1)$ so, for values of θ close

enough to $\pi/4$, we have $\ell_2 \neq \Theta(\sigma)$. Its analysis requires a more careful derivation.

Set

$$B(\sigma) = r(\sigma)\alpha_\theta, \quad C(\sigma) = \frac{1}{2}(2\delta - \sigma)\sigma, \quad \gamma(\theta) = \pi/4 - \theta.$$

We will write B, C, γ when σ, θ values are fixed.

Set $\gamma = \pi/4 - \theta$. Taking Taylor series around $\pi/4$ yields

$$\alpha_\theta = \cos \theta - \sin \theta = \sqrt{2}\gamma + \Theta(\gamma^2),$$

where the constant implicit in the $\Theta()$ is again only dependent upon $\hat{\Theta}$. This implies

$$B = \Theta(\delta\alpha_\theta) = \theta(\delta\gamma) \quad \text{and} \quad C = \Theta(\delta\sigma).$$

From Eq. 58, ℓ_2 is the solution to the quadratic equation

$$\ell_2^2 + B\ell_2 - C = 0,$$

therefore

$$\ell_2 = \frac{-B + \sqrt{B^2 + 4C}}{2}. \tag{60}$$

Fix any arbitrary $c > 0$. There are now two possibilities.

1. If $\delta\sigma \geq c(\delta\gamma)^2$: Eq. 60 then yields $\ell_2 = \Theta(\sqrt{\delta\sigma})$.
2. If $\delta\sigma < c(\delta\gamma)^2$: Then Eq. 60 yields

$$\begin{aligned} \ell_2 &= \frac{1}{2} \left(-B + B\sqrt{1 + \frac{4C}{B^2}} \right) \\ &= \Theta \left(-\delta\gamma + \delta\gamma \left(1 + \Theta \left(\frac{4\delta\sigma}{(\delta\gamma)^2} \right) \right) \right) \\ &= \Theta \left(\frac{\sigma}{\gamma} \right). \end{aligned}$$

The above can be rewritten as

$\sigma \leq \delta$ so $\delta\sigma < \delta^2 \leq c\delta^2\gamma^2$ where $c = 1/(\hat{\theta})^2$. Thus, for all cases of $\delta\sigma \leq c(\delta\gamma)^2$, $\Theta \left(\frac{\sigma}{\gamma} \right)$. This implies

$$\ell_2 = \begin{cases} \Theta(\sqrt{\delta\sigma}) & \text{if } \sigma \geq c(\delta\gamma)^2, \\ \Theta \left(\frac{\sigma}{\gamma} \right) & \text{if } \sigma < c(\delta\gamma)^2. \end{cases}$$

Note that if this equation is correct for *any* $c > 0$, it is correct for *all* $c > 0$. Thus, using the fact that $\ell_1 = \Theta(\sigma)$ and plugging into Eq. 56 we have proven

$$f(v) = \begin{cases} \Theta\left(\frac{\min(\sigma, 1) \cdot \min(\sqrt{\delta\sigma}, 1)}{\delta^2}\right) & \text{if } \sigma = \Omega(\delta\gamma^2), \\ \Theta\left(\frac{\min(\sigma, 1) \cdot \min(\frac{\sigma}{\gamma}, 1)}{\delta^2}\right) & \text{if } \sigma = O(\delta\gamma^2). \end{cases} \quad (61)$$

□

Proof: of Lemma 24.

Until otherwise stated, $i < m - 1$ is assumed

This part of the analysis will completely mimic the upper bound derivation for $\mathbf{B}_2 + \delta\mathbf{B}_2$ in Lemma 19, but with the $\Theta(\cdot)$ term replaced by an $O(\cdot)$ term in many places. In particular, we use the Strip Lemma almost exactly the same way as was done in the proof of Lemma 19. We therefore only state the main items and skip the details which are exactly the same as in Lemma 19.

Let $A_i(\sigma) = R_{i+1}(\sigma)$. Set

$$v_i(\sigma) = ((1 + (\delta - \sigma) \cos \theta_{i+2}, (\delta - \sigma) \sin \theta_{i+2}).$$

This is the leftmost point of $A_i(\sigma)$ on the boundary line between R_{i+1} and R_{i+2} . Now drop a vertical line from $v_i(\sigma)$ to the x -axis and let $v'_i(\sigma)$ be the point at which it intersects the boundary between R_i and R_{i-1} . Let $\sigma' = \sigma(v'_i(\sigma))$. By construction

$$(\delta - \sigma) \cos \theta_{i+2} = v_i(\sigma).x = v'_i(\sigma).x = (\delta - \sigma') \cos \theta_i.$$

But

$$\begin{aligned} \cos \theta_{i+2} &= \cos(\theta_i + 2\hat{\theta}) \\ &= \cos \theta_i \cos(2\hat{\theta}) - \sin \theta_i \sin(2\hat{\theta}) \\ &= (1 + O(\hat{\theta})) \cos \theta_i - O(\hat{\theta}) \\ &= \cos \theta_i - O(\hat{\theta}). \end{aligned}$$

Then

$$(\delta - \sigma) (\cos \theta_i - O(\hat{\theta})) = (\delta - \sigma') \cos \theta_i.$$

Because the R_i are in the first octant, θ_i is bounded away from $\pi/2$ so $\cos \theta_i$ is bounded away from 0. Thus, $\sigma' - \sigma = O(\delta\hat{\theta})$. Set $\bar{\sigma} = \delta\hat{\theta}$ and define

$$B'_i(\sigma) = \{v \in R_i : v_i(\sigma).x \leq v.x\}$$

and then

$$\begin{aligned}
\bar{B}_i &= R_i, & B'_i &= \{v \in R_i : v_i(\bar{\sigma}).x \leq v.x\} = B'_i(\bar{\sigma}), \\
A_i &= R_{i+1}, & A_i(\sigma) &= R_{i+1}(\sigma), \\
B_i &= \bar{B}_i \setminus B'_i & B_i(\sigma) &= \{v \in B_i : v_i(\sigma).x \leq v.x\} \\
& & &= \{v \in R_i : v_i(\sigma).x \leq v.x \leq v_i(\bar{\sigma}).x\}
\end{aligned}$$

By construction

- $B'_i \subset R_i(\bar{\sigma}') \Rightarrow \mu(B'_i) \leq \mu(R_i(\bar{\sigma}'))$.
- Since $\bar{\sigma} = \bar{\sigma} = \delta\hat{\theta} \Rightarrow \bar{\sigma}' = \Theta(\delta\hat{\theta}) = \Theta(\bar{\sigma})$.
- From Corollary 21 and the definition of the R_i ,
 $\bar{\sigma}' = \Theta(\bar{\sigma}) \Rightarrow \mu(R_i(\bar{\sigma}')) = O(\mu(R_i(\bar{\sigma})))$.

Combining these points yields

$$\Rightarrow \mu(B'_i) \leq \mu(R_i(\bar{\sigma}')) = O(\mu(R_i(\bar{\sigma}))) = O(1/n)$$

(Where the last equality is given by the Lemma's conditions). Furthermore

- Every point in $B_i \setminus B_i(\sigma)$ is dominated by every point in $A_i(\sigma)$.
- If $\sigma \leq \bar{\sigma} \Rightarrow B_i(\sigma) = \emptyset$.
- If $\sigma > \bar{\sigma} \Rightarrow B_i(\sigma) \subset B_i(\sigma') \subset R_i(\sigma')$.

Thus, if $\sigma \leq \bar{\sigma}$ then

$$\mu(B_i(\sigma)) = 0 = O(\mu(A(\sigma))),$$

while if $\sigma > \bar{\sigma}$ then, using the fact that $\sigma' - \sigma = O(\delta\hat{\theta}) = O(\bar{\sigma})$,

$$\begin{aligned}
\mu(B(\sigma)) &= O(\mu(R_i(\sigma'))) \\
&= O(\mu(R_{i+1}(\sigma'))) \\
&= O(\mu(R_{i+1}(\sigma + O(\bar{\sigma})))) \\
&= O(\mu(R_{i+1}(\sigma))),
\end{aligned}$$

where the last inequality comes from the fact that $\sigma > \bar{\sigma}$ so $\sigma + O(\bar{\sigma}) = O(\sigma)$.

The $A_i, \bar{B}_i, B_i$ therefore satisfy the conditions of the Strip Lemma (Lemma 8). Since $R_i, i = 0, 1, \dots, m-2$, partition $W(\theta_1, \theta_2 - \hat{\theta})$, this implies

$$\mathbf{E} \left[\left| MAX(S_n) \cap W(\theta_1, \theta_2 - \hat{\theta}) \right| \right] = \sum_{i=0}^{m-2} \mathbf{E} [|MAX(S_n) \cap R_i|] = O(m). \tag{62}$$

We now examine R_{m-1} . Define $\hat{R} = R_{m-1}$ and $\hat{R}(\sigma) = R_{m-1}(\sigma)$. To complete the proof of the lemma, it only remains to prove that

$$\mathbf{E} \left[\left| \text{MAX}(S_n) \cap \hat{R} \right| \right] = O(\log^2 n). \quad (63)$$

Set

$$v(\sigma) = ((1 + (\delta - \sigma) \cos \bar{\theta}_2, (\delta - \sigma) \sin \bar{\theta}_2).$$

See Figure 26. This is the leftmost point of $A_i(\sigma)$ on the top boundary line of $W(\bar{\theta}_1, \bar{\theta}_2)$. Similar to before, drop a vertical line from v to the x -axis and let $v'(\sigma)$ be the point at which it intersects the boundary between R_{m-1} and R_{m-2} . Let $\sigma' = \sigma(v'(\sigma))$. By construction every point in $\hat{R}(\sigma)$ dominates every point in $\hat{R} \setminus \hat{R}(\sigma')$.

$$(\delta - \sigma) \cos \bar{\theta}_2 = v.x = v'.x = (\delta - \sigma') \cos (\bar{\theta}_2 - \hat{\theta}).$$

But

$$\begin{aligned} \cos (\bar{\theta}_2 - \hat{\theta}) &= \cos \bar{\theta}_2 \cos(\hat{\theta}) + \sin \bar{\theta}_2 \sin(\hat{\theta}) \\ &= (1 + O(\hat{\theta})) \cos \bar{\theta}_2 - O(\hat{\theta}) \\ &= \cos \bar{\theta}_2 - O(\hat{\theta}). \end{aligned}$$

Then

$$(\delta - \sigma) \cos \bar{\theta}_2 = (\delta - \sigma') (\cos \bar{\theta}_2 - O(\hat{\theta})).$$

Because the R_i are in the first octant, θ_i is bounded away from $\pi/2$ so $\cos \theta_i$ is bounded away from 0. Thus, $\sigma' - \sigma = O(\delta \hat{\theta}) = O(\bar{\sigma})$.

If $\mu(\hat{R}) \leq \frac{\log^2 n}{n}$ then Eq. 63 is trivially correct because

$$\mathbf{E} \left[\left| \text{MAX}(S_n) \cap \hat{R} \right| \right] \leq \mathbf{E} \left[\left| S_n \cap \hat{R} \right| \right] = n\mu(\hat{R}) = O(\log^2 n).$$

We therefore assume that $\mu(\hat{R}) > \frac{\log^2 n}{n}$ and let $\hat{\sigma}$ be the unique value such that $\mu(\hat{R}(\hat{\sigma})) = \frac{\log^2 n}{n}$. We know that $\hat{\sigma} = \Omega(\bar{\sigma})$ because $\mu(\hat{R}(\sigma))$ is monotonically increasing in σ and $\mu(\hat{R}(\bar{\sigma})) = \Theta(1/n)$. Thus $\hat{\sigma}' = \Theta(\hat{\sigma})$ and, from Corollary 21,

$$\mu(\hat{R}(\hat{\sigma}')) = O\left(\mu(\hat{R}(\hat{\sigma}))\right) = O\left(\frac{\log^2 n}{n}\right).$$

Now

$$\begin{aligned}\mathbf{E} \left[\left| \text{MAX}(S_n) \cap \hat{R} \right| \right] &= \mathbf{E} \left[\left| \text{MAX}(S_n) \cap \hat{R}(\hat{\sigma}') \right| \right] + \mathbf{E} \left[\left| \text{MAX}(S_n) \cap \left(\hat{R} \setminus \hat{R}(\hat{\sigma}') \right) \right| \right] \\ &\leq \mathbf{E} \left[\left| S_n \cap \hat{R}(\hat{\sigma}') \right| \right] + \mathbf{E} \left[\left| \text{MAX}(S_n) \cap \left(\hat{R} \setminus \hat{R}(\hat{\sigma}') \right) \right| \right].\end{aligned}$$

From Lemma 1(b)

$$\mathbf{E} \left[\left| S_n \cap \hat{R}(\hat{\sigma}') \right| \right] = n\mu \left(\hat{R}(\hat{\sigma}') \right) = O(\log^2 n).$$

and from Lemma 1(c)

$$\Pr((S_n \cap \hat{R}(\hat{\sigma})) = \emptyset) = \left(1 - \mu \left(\hat{R}(\hat{\sigma}) \right) \right)^n \leq e^{-c \log^2 n},$$

for some constant $c > 0$. Since, as noted, *any* point in $\hat{R}(\hat{\sigma})$ dominates *all* points in $\hat{R} \setminus \hat{R}(\hat{\sigma})$, $(S_n \cap \hat{R}(\hat{\sigma})) \neq \emptyset$ implies $\text{MAX}(S_n) \cap \left(\hat{R} \setminus \hat{R}(\hat{\sigma}') \right) = \emptyset$. Thus

$$\mathbf{E} \left[\left| \text{MAX}(S_n) \cap \left(\hat{R} \setminus \hat{R}(\hat{\sigma}') \right) \right| \right] \leq n \Pr((S_n \cap \hat{R}(\hat{\sigma})) = \emptyset) = o(1).$$

This yields

$$\mathbf{E} \left[\left| \text{MAX}(S_n) \cap \hat{R} \right| \right] \leq O(\log^2 n) + o(1) = O(\log^2 n),$$

proving Eq. 63 and thus completing the proof of the lemma. $\square$

9 Analysis of $\mathbf{B}_\infty + \delta \mathbf{B}_q$

This section derives cells (iv)(c-d) in Theorem 1, that is, if n points are chosen from $\mathbf{D} = \mathbf{B}_\infty + \delta \mathbf{B}_q$, for any fixed $q \in [1, \infty)$ and $\frac{1}{\sqrt{n}} \leq \delta \leq 1$, then

$\mathbf{E}[M_n] = \Theta \left(\ln n + \sqrt{\delta} n^{1/4} \right)$. Note that this implies that $\mathbf{E}[M_n] = \Theta(\ln n)$ for $\frac{1}{\sqrt{n}} \leq \delta \leq \frac{\ln^2 n}{\sqrt{n}}$ and it only starts increasing as $\delta > \frac{\ln^2 n}{\sqrt{n}}$.

Applying Lemma 9 also provides a full analysis for $\mathbf{B}_q + \delta \mathbf{B}_\infty$.

This section follows the general structure laid out in Section 4.2 and will need the following special definitions.

Note that this section differs slightly from the previous ones in that it allows q to be *any* value in $[1, \infty]$. The Lemmas and Theorems are correct for all such q , but the constants implicit in the $\Theta()$, $O()$ and $\Omega()$ terms will depend explicitly upon q . We caution the reader that the diagrams are all drawn for the case $q = 2$ and other cases might look quite different.

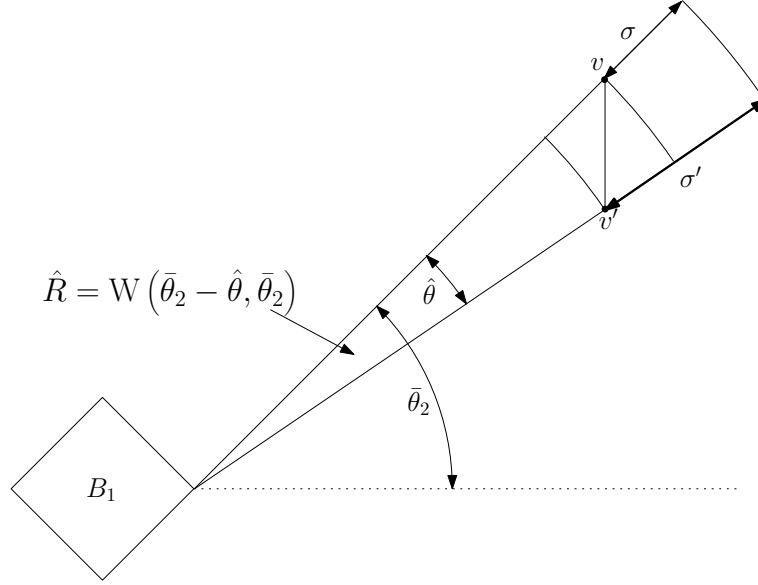


Figure 26: Illustrates the last piece of the proof of Lemma 24. The diagram is not to scale. $\hat{\theta}$ is actually very small, and decreasing with n . Note that any point in $\hat{R}(\sigma')$ will dominate *all* points in $\hat{R} \setminus \hat{R}(\sigma' + \sigma'')$.

Definition 16 Let $D_1 = (B_\infty + \delta B_q) \cap Q_1$ be the support in the first quadrant. Partition D_1 into the following four regions

$$\begin{aligned} A &= \{u \in D_1 : u.x \leq 1, u.y \leq 1\}, & C &= \{u \in D_1 : u.x \geq 1, u.y \leq 1\}, \\ B &= \{u \in D_1 : u.x \geq 1, u.y \geq 1\}, & C &= \{u \in D_1 : u.x \leq 1, u.y \geq 1\}. \end{aligned}$$

If $v \in B$, define

$$\alpha(v) = (\delta^q - (p.y)^q)^{1/q} - p.x \quad \text{and} \quad \beta(v) = (\delta^q - (p.x)^q)^{1/q} - p.y.$$

If $v \in C$, define

$$\alpha(v) = 1 + \delta - p.x \quad \text{and} \quad \beta(v) = (\delta^q - (\delta - \alpha(v))^q)^{1/q}.$$

$\alpha(v)$ is the horizontal distance from v to the right border of D . If $v \in B$, $\beta(v)$ is the distance from v to the vertical border but if $v \in C$, $\beta(v)$ is just a function of $\alpha(v)$.

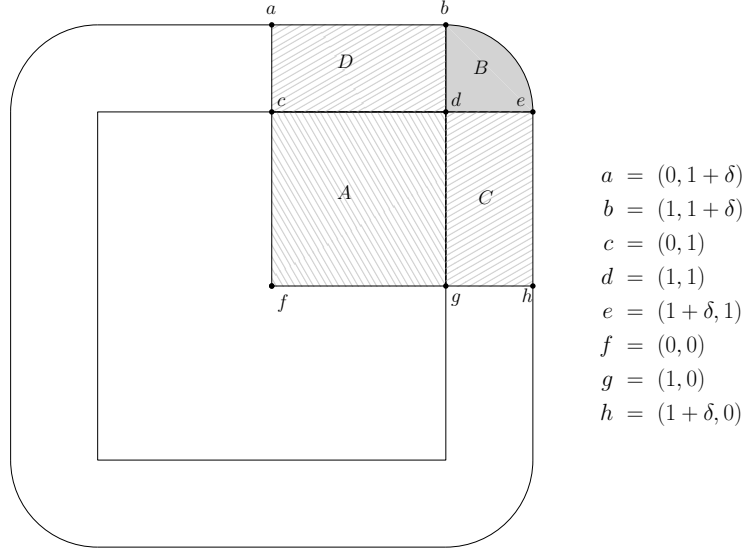


Figure 27: The regions introduced in Definition 16.

Lemma 25 *Let $\mathbf{D} = \mathbf{B}_\infty + \delta \mathbf{B}_q$ and $v \in D_1$.*

- *If $v \in A$, then*

$$f(v) = \begin{cases} \Theta(1) & \text{if } \delta \leq 1, \\ \Theta\left(\frac{1}{\delta^2}\right) & \text{if } \delta > 1. \end{cases} \quad (64)$$

- *If $v \in B$, then*

$$f(v) = \Theta\left(\frac{\min(\alpha(v), 1) \cdot \min(\beta(v), 1)}{\delta^2}\right). \quad (65)$$

$$\mu(P(v)) = \Theta(\alpha(v)\beta(v)f(v)). \quad (66)$$

- *If $v \in C$, then*

$$f(v) = \Theta\left(\frac{\min(\alpha(v), 1) \cdot \min(\beta(v), 1)}{\delta^2}\right). \quad (67)$$

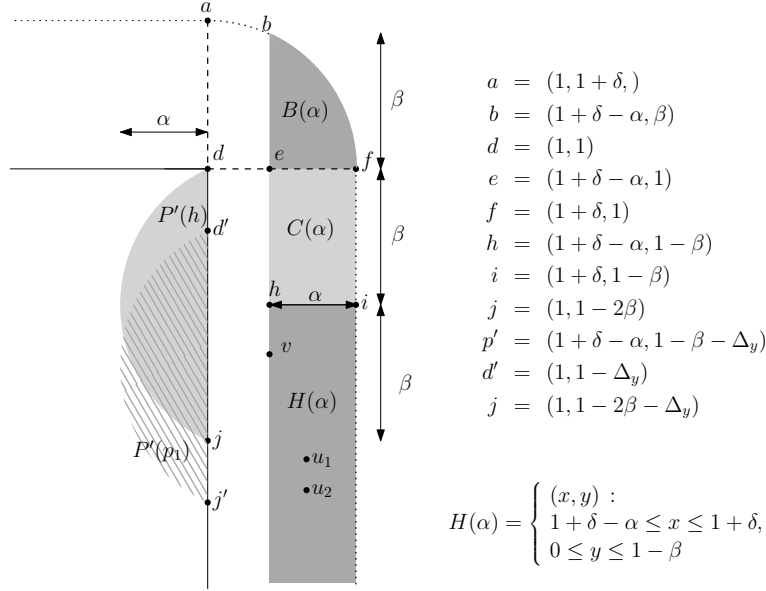


Figure 28: Illustration of Lemma 26 and its proof. Note that only the top of $H(\alpha)$ is shown. This illustrates the case $\delta \leq 1$; the case $\delta > 1$ looks different.

Lemma 26 Fix $\alpha \leq \delta$. Define $\beta = (\delta^q - (\delta - \alpha)^q)^{1/q}$. See Fig. 28.

Set $x = 1 + \delta - \alpha$ and points $e = (x, 1)$, $h = (x, \max(0, 1 - \beta))$. Further define regions

$$B(\alpha) = P(e),$$

$$C(\alpha) = P(h) \setminus P(e),$$

$$H(\alpha) = \{u \in C : \alpha(u) \leq \alpha(x), 1 + \delta - \alpha \leq u.x \leq 1 + \delta, u.y \leq \max(1 - \beta, 0)\}.$$

Note that if $\beta \geq 1$, then $H(\alpha) = \emptyset$.

(A) if $\beta \leq 1$, then

1. If $v = (x, v.y)$ where $-1 + \beta \leq p.y \leq 1 + \beta$ then $f(h) = f(v)$.

2. If $u_1, u_2 \in H(\alpha)$ with $u_1.x = u_2.x$, then $f(u_1) = f(u_2)$.

3. $\mu(C(\alpha)) = \Theta(\mu(B(\alpha))) = \Theta\left(\frac{\alpha^2 \beta^2}{\delta^2}\right)$.

(B) if $\beta > 1$, then

$$\mu(C(\alpha)) = O(\mu(B(\alpha))).$$

Proof: We first assume $\beta \leq 1$ and prove (A).

Recall from Lemma 3 that

$$f(u) = \frac{\text{Area}(P'(u))}{a_\infty a_q \delta^2}, \quad \text{where } P'(u) = B_q(u, \delta) \cap B_\infty.$$

By the definition of h and the fact that $\beta \leq 1$,

$$P'(h) = \{u : \|u - h\|_q \leq \delta, |u.x| \leq 1\},$$

which is the light gray sector in Fig. 28. By basic geometric arguments, $P'(v) = P'(h) - (0, h.y - v.y)$, i.e., $P'(h)$ is shifted down appropriately. Thus, $\text{Area}(P'(h)) = \text{Area}(P'(v))$ and $f(h) = f(v)$, proving 1.

Next set $\alpha' = 1 + \delta - u_1.x$ and $\beta' = (\delta^q - (\alpha')^q)^{1/q}$. Since $\alpha' \leq \alpha$, $\beta' \leq \beta$, we can define h' associated with α', β' and then apply part 1 twice to get $f(u_1) = f(h') = f(u_2)$, proving part 2.

For part 3 first note that from Eq. 66 in Lemma 25, $\mu(B(\alpha)) = \mu(P(e)) = \Theta\left(\frac{\alpha^2 \beta^2}{\delta^2}\right)$.

As $\text{Area}(C(\alpha)) = \alpha\beta$ and $\forall u \in C(\alpha), \alpha(u) \leq \alpha$ then, from Eq. 67 $f(u) = O\left(\frac{\alpha\beta}{\delta^2}\right)$. Thus

$$\mu(C(\alpha)) = \int_{u \in C(\alpha)} f(u) du \leq \text{Area}(C(\alpha)) \cdot \max_{u \in C(\alpha)} f(u) = O\left(\frac{\alpha^2 \beta^2}{\delta^2}\right).$$

For the other direction, set

$$C'(\alpha) = \{u \in C(\alpha) : \alpha(u) \geq \alpha/2\}.$$

For $\alpha' \in [0, \delta]$, $\beta(\alpha')$ is a monotonically increasing concave function. Since $\beta(0) = 0$, $\beta(\alpha/2) \geq \frac{1}{2}\beta(\alpha)$. Thus, $\forall \alpha' \in C'(\alpha)$, $\beta(\alpha') \geq \beta/2$. Then

$$\begin{aligned} \mu(C(\alpha)) &= \int_{u \in C(\alpha)} f(u) du = \int_{u \in C'(\alpha)} f(u) du \\ &\geq \text{Area}(C'(\alpha)) \cdot \min\{f(u) : u \in C'(\alpha)\} \geq \frac{\alpha\beta}{2} \frac{\alpha\beta}{4} = \Omega\left(\frac{\alpha^2 \beta^2}{\delta^2}\right), \end{aligned}$$

and the proof of (A) is complete.

Part (B) follows directly from Eqs. 66 and 67. $\square$

Lemma 27 *Let S_n be n points chosen from the distribution $\mathbf{D} = \mathbf{B}_\infty + \delta \mathbf{B}_q$ with $\frac{1}{\sqrt{n}} \leq \delta \leq \sqrt{n}$. Then*

$$\mathbf{E}[M_n] = \Omega\left(\ln n + \sqrt{\delta} n^{1/4}\right).$$

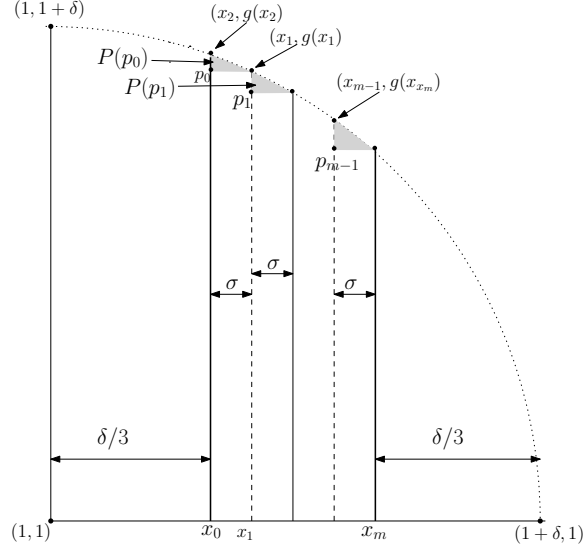


Figure 29: Illustration of case (a) of the upper bound.

Proof: The proof splits into two parts;

$$(a) \quad \forall \delta \text{ satisfying } \frac{1}{\sqrt{n}} \leq \delta \leq \sqrt{n}, \quad \mathbf{E}[|\text{MAX}(S_n) \cap B|] = \Omega\left(\sqrt{\delta} n^{1/4}\right).$$

$$(b) \quad \forall \delta \text{ satisfying } \frac{1}{\sqrt{n}} \leq \delta \leq \frac{\log^2}{\sqrt{n}}, \quad \mathbf{E}[|\text{MAX}(S_n) \cap C|] = \Omega(\ln n).$$

By symmetry, $\mathbf{E}[|\text{MAX}(S_n) \cap D|] = \mathbf{E}[|\text{MAX}(S_n) \cap B|]$. Combining this with (a) and (b) proves the lemma.

(a) $\mathbf{E}[|\text{MAX}(S_n) \cap B|]$ for $\frac{1}{\sqrt{n}} \leq \delta \leq \sqrt{n}$:

Fix σ as a value to be determined later. Set $m = \lfloor \delta/(3\sigma) \rfloor$ and, for $i \geq 0$,

$$x_i = 1 + \frac{\delta}{3} + i\sigma, \quad y_i = g(x_{i+1}), \quad p_i = (x_i, y_i).$$

Then

$$g(x) = 1 + (\delta^q - (x - 1)^q)^{1/q}$$

is the equation of the top boundary of B . We will show that for $0 \leq i < j \leq m - 1$, with an appropriate choice of σ , then $P(p_i) \cap P(p_j) = \emptyset$ and $\mu(P(p_i)) = \Omega(1/n)$. Lemma 2 would then imply that $\mathbf{E}[M_n] = \Omega(m)$.

By construction,

$$\alpha(p_i) = x_{i+1} - x_i = \sigma, \quad \beta(p_i) = y_i - y_{i+1} = g(x_i) - g(x_{i+1}),$$

and $P(p_i) \cap P(p_j) = \emptyset$.

The Theorem of the Mean states that, for $1 < x_i < x_{i+1} < 1 + \delta$,

$$\beta(p_i) = |g(x_{i+1}) - g(x_i)| = |g'(z)| |x_{i+1} - x_i| = |g'(z)| \sigma, \quad (68)$$

for some $x_i \leq z \leq x_{i+1}$. Direct differentiation yields

$$g'(x) = \frac{q(x-1)^{q-1}}{q} (\delta^q - (x-1)^q)^{\frac{1}{q}-1}. \quad (69)$$

Observe that $\forall x \in [1 + \delta/3, 1 + 2\delta/3]$, then

$$|g'x| = \Theta \left(\delta^{q-1} (\delta^q)^{\frac{1}{q}-1} \right) = \Theta(1),$$

and thus, by Eq. 68, $\beta(p_i) = \Theta(\sigma)$, where the constant in the $\Theta(\cdot)$ depends only upon q .

Set $\sigma = c' \sqrt{\delta}/n^{1/4}$ for some small $c' > 0$. For all $i \in [0, m-1]$, Eq. 66 in Lemma 25 yields³

$$\begin{aligned} \mu(P(p_i)) &= \Theta(\alpha(v)\beta(v)f(v)) \\ &= \Theta\left(\frac{\alpha^2(v)\beta^2(v)}{\delta^2}\right) = \Theta\left(\frac{\sigma^4}{\delta^2}\right) = \Theta\left(\frac{1}{n}\right). \end{aligned}$$

Lemma 2 then yields

$$\mathbf{E}[M_n] = \Omega(m) = \Omega\left(\frac{\delta}{\sigma}\right) = \Omega\left(\sqrt{\delta}n^{1/4}\right).$$

(b) $\mathbf{E}[|\text{MAX}(S_n) \cap C|]$ for $\frac{1}{\sqrt{n}} \leq \delta \leq \frac{\log^2}{\sqrt{n}}$:

For fixed $\alpha' \leq \delta$ define $\beta_{\alpha'} = (\delta^q - (\delta - \alpha')^q)^{1/q}$ and point $e(\alpha') = (1 + \delta - \alpha', 1)$. This is illustrated in Fig. 28.

Note that if $\alpha' = \delta$ then $\beta_{\alpha'} = \delta$ and $\frac{(\alpha')^2(\beta_{\alpha'})^2}{\delta^2} = \Omega\left(\frac{1}{n}\right)$. Since $\beta_{\alpha'}$ increases monotonically with α' we can, for suitably small but fixed c always find $\alpha \leq \delta$ such that $\frac{\alpha^2\beta^2}{\delta^2} = \frac{c}{n}$. Fix this α and set $\beta = \beta_{\alpha}$, $e = e(\alpha)$.

³This implicitly requires that $\beta < \min(1, \delta/3)$, but this is guaranteed by $\delta > 1/\sqrt{n}$ and appropriate choice of c' .

Set $B(\alpha) = P(e)$, and let $C(\alpha)$ and $H(\alpha)$ be as defined in Lemma 26.

To simplify the remainder of the proof, set $T(\alpha) = B(\alpha) \cup C(\alpha)$. From Lemma 26.

$$\mu(T(\alpha)) = \Theta(\mu(B(\alpha)) + \mu(C(\alpha))) = \Theta\left(\frac{\alpha^2 \beta^2}{\delta^2}\right) = \Theta\left(\frac{1}{n}\right).$$

Thus, from Lemma 1,

$$\Pr(T(\alpha) \cap S_n = \emptyset) = (1 - \mu(T(\alpha)))^n = \Theta(1).$$

Points in $H(\alpha)$ can only be dominated by points in $T(\alpha)$. So, if $S_n \cap T(\alpha)$ is empty then

$$\text{MAX}(S_n) \cap H(\alpha) = \text{MAX}(S_n \cap H(\alpha)).$$

implying

$$\begin{aligned} \mathbf{E}[|\text{MAX}(S_n)|] &\geq \mathbf{E}[|\text{MAX}(S_n \cap H(\alpha))|] \\ &\geq \mathbf{E}\left[|\text{MAX}(S_n \cap H(\alpha))| \mid S_n \cap T(\alpha) = \emptyset\right] \cdot \Pr[S_n \cap T(\alpha) = \emptyset] \\ &= \Omega\left(\mathbf{E}\left[|\text{MAX}(S_n \cap H(\alpha))| \mid S_n \cap T(\alpha) = \emptyset\right]\right). \end{aligned} \quad (70)$$

Next note that if $u \in C$, Eq. 67 states that $f(u) = \Theta(g(\alpha(u), \beta(u)))$, for some function $g(\cdot)$. Since $\beta(u)$ is a function of $\alpha(u)$, this implies that if $u_1, u_2 \in C$ with $u_1.x = u_2.x$, then $f(u_1) = \Theta(f(u_2))$. Since $C(\alpha)$ is an $\alpha \times \beta$ rectangle and $H(\alpha)$ is an $\alpha \times (1 - \beta)$ rectangle this yields

$$\mu(H(\alpha)) = \Theta\left(\frac{1 - \beta}{\beta} \mu(C(\alpha))\right) = \Theta\left(\frac{1}{\beta} \cdot \frac{1}{n}\right) = \Omega\left(\frac{1}{\sqrt{n} \log^2 n}\right) = \Omega\left(n^{-1/3}\right)$$

where the fact that $\beta \leq \delta \leq \log^2 n / \sqrt{n}$ is explicitly used.

Assume that the only information extra known is that $S_n \cap T(\Delta) = \emptyset$. Conditioned on this event, the new density for $v \in D \setminus T(\alpha)$ is $\hat{f}(v) = \frac{f(v)}{1 - \mu(T(\alpha))} > f(v)$. In particular, this implies that $\hat{\mu}(R_1(\Delta)) = \Omega(n^{-1/3})$. It also implies that if $u, v \in R$ and $u.x = v.x$ then $\hat{f}(u) = \hat{f}(v)$.

We now apply Lemma 10 with $R = H(\alpha)$ (this assumes that S_n is chosen from $\bar{\mathbf{D}}$ defined as the distribution on $D \setminus T(\alpha)$ with density $\hat{f}(u)$) to see that

$$\mathbf{E}\left[|\text{MAX}(S_n \cap H(\alpha))| \mid S_n \cap T(\alpha) = \emptyset\right] = \Omega(\ln n),$$

completing the proof. $\square$

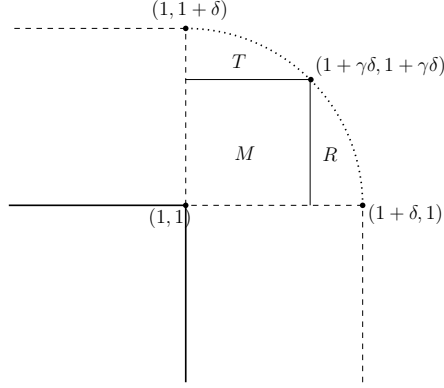


Figure 30: Illustration of Case (a) of Lemma 28 that partitions B into three regions M , T and R . The figure illustrates the $q = 2$ case.

Lemma 28 *Let S_n be n points chosen from the distribution $\mathbf{D} = \mathbf{B}_\infty + \delta \mathbf{B}_q$ with $\frac{1}{\sqrt{n}} \leq \delta \leq \sqrt{n}$. Then*

$$\mathbf{E}[M_n] = O\left(\ln n + \sqrt{\delta} n^{1/4}\right).$$

Proof: The proof splits into three parts that show

- (a) $\mathbf{E}[|\text{MAX}(S_n) \cap B|] = O\left(\ln n + \sqrt{\delta} n^{1/4}\right).$
- (b) $\mathbf{E}[|\text{MAX}(S_n) \cap C|] = O(\ln n).$
- (b) $\mathbf{E}[|\text{MAX}(S_n) \cap A|] = O(\ln n).$

By symmetry, $\mathbf{E}[|\text{MAX}(S_n) \cap D|] = \mathbf{E}[|\text{MAX}(S_n) \cap B|]$. Combining this with (a), (b) and (c) proves the lemma.

(a) $\mathbf{E}[|\text{MAX}(S_n) \cap B|]$:

The proof decomposes B into M , T , and R as illustrated in Figure 30. We prove

$$\mathbf{E}[|\text{MAX}(S_n) \cap (T \cup M)|] = O\left(\sqrt{\delta} n^{1/4}\right).$$

A symmetrical argument will prove that

$$\mathbf{E}[|\text{MAX}(S_n) \cap (T \cup R)|] = O\left(\sqrt{\delta} n^{1/4}\right),$$

and thus $\mathbf{E} [| \text{MAX}(S_n) \cap B |] = O \left(\sqrt{\delta} n^{1/4} \right)$.

Set γ to be the solution to $\gamma^q + (1 - \gamma)^q = 1$. Let $c > 0$ be some constant to be specified later and set

$$m = \left\lfloor c\sqrt{\delta} n^{1/4} \right\rfloor \quad \text{and} \quad \sigma = \frac{\gamma\delta}{m}.$$

Note this immediately implies

$$\frac{\sigma^4}{\delta^2} = \Theta \left(\frac{\delta^4}{\delta^2 m^4} \right) = \Theta \left(\frac{1}{n} \right). \quad (71)$$

Set $g(x) = 1 + (\delta^q - (x - 1)^q)^{1/q}$, the equation of the upper cap of B .

For $i \geq 0$ define

$$\begin{aligned} x_i &= 1 + i\sigma, & p_i &= (x_i, y_i), & p_i(t) &= (x_i, y_i - t), \\ y_i &= g(x_i + 2\sigma), & p'_i &= (x_{i+1}, y_i), & p'_i(t) &= (x_{i+1}, y_i - t), \end{aligned}$$

For later use we will need

$$(1 + \gamma\delta) + \sigma = x_m + \sigma = x_{m+1} \leq 1 + \frac{\gamma + 1}{2}\delta. \quad (72)$$

Note that

$$\begin{aligned} x_{m+1} \leq 1 + \frac{\gamma + 1}{2}\delta &\Leftrightarrow \sigma \leq \frac{1 - \gamma}{2}\delta \\ &\Leftrightarrow \frac{\gamma\delta}{m} \leq \frac{1 - \gamma}{2}\delta \\ &\Leftrightarrow \frac{2\gamma}{1 - \gamma} \leq m. \end{aligned}$$

Since $\frac{1}{\sqrt{n}} \leq \delta$ we can therefore ensure Eq. 72 by choosing large enough c (this depends upon q).

Now set

$$\begin{aligned} \text{Strip}_i &= \{u \in B : x_i \leq u.x \leq x_{i+1}\}, \\ \text{Strip}'_i(t) &= \{u \in \text{Strip}'_i : y_i - t \leq u.y \leq y_i\}, \quad \text{Strip}'_i = \text{Strip}'_i(y_i) = \text{Strip}_i \setminus P(p_i). \end{aligned}$$

Now set (See Fig. 31)

$$\begin{aligned} A_i(t) &= P(p'_i)(t), & A_i &= P(p'_i)(y_i) = P((x_i, 1)), \\ B_i(t) &= \text{Strip}'_i(t), & B_i &= B_i(y_i) = \text{Strip}'_i, \\ B'_i &= P(p_i), & \bar{B}_i &= B'_i \cup B_i. \end{aligned}$$

Since $T \cup M = \bigcup_{i=0}^{m-1} \bar{B}_i$ if we could show that $A_i, B_i, \bar{B}_i$ satisfy the conditions of the Strip Lemma we would have shown

$$\mathbf{E} [|\text{MAX}(S_n) \cap (T \cup M)|] = O(m) = O\left(\sqrt{\delta} n^{1/4}\right), \quad (73)$$

and this would complete the proof of part (a).

By construction every point in $B \setminus B_i(t)$ is dominated by every point in $A_i(t)$ and $T \cup M = \bigcup_{i=0}^{m-1} \bar{B}_i$.

As in the derivations of Eqs. 68 and 69, we have from the Theorem of the Mean that

$$\forall i, \quad g(x_{i+1}) - g(x_i) = g'(z)(x_{i+1} - x_i) = g'(z)\sigma,$$

where $z \in [x_i, x_{i+1}]$. Since $g'(z)$ is uniformly bounded for $z \in [1, 1 + \delta(\gamma + 1)/2]$, and, by Eq. 72, $[x_0, x_{m+1}] \subseteq [1, 1 + \delta(\gamma + 1)/2]$, this implies

$$\forall i, \quad g(x_{i+1}) - g(x_i) = O(\sigma).$$

In addition, because $g(x)$ is concave in the interval $[1, 1 + \delta]$,

$$g(x_{i+1}) - g(x_i) \leq g(x_{i+2}) - g(x_{i+1}).$$

From this and the definitions, $\forall i$ and $\forall t \geq 0$,

$$\begin{aligned} \alpha(p_i) &= x_{i+2} - x_i = 2\sigma, & \alpha(p'_i) &= x_{i+1} - x_i = \sigma \\ \alpha(p_i)(t) &= \alpha(p'_i)(t) + \sigma, & \Rightarrow \alpha(p_i)(t) &= \Theta(\alpha(p'_i)(t)), \\ \beta(p_i) &= g(x_i) - g(x_{i+2}) = O(\sigma), & \beta(p'_i) &= g(x_{i+1}) - g(x_{i+2}) = O(\sigma) \\ \beta(p_i)(t) &= \beta(p'_i)(t) + g(x_i) - g(x_{i+1}), & \Rightarrow \beta(p_i)(t) &= \Theta(\beta(p'_i)(t)). \end{aligned}$$

Thus from Eqs. 66 and 67 $\forall t > 0$,

$$\begin{aligned} f(P(p_i(t))) &= \Theta\left(\frac{\min(\alpha(P(p_i(t))), 1) \cdot \min(\beta(P(p_i(t))), 1)}{\delta^2}\right) \\ &= \Theta\left(\frac{\min(\alpha(P(p'_i(t))), 1) \cdot \min(\beta(P(p'_i(t))), 1)}{\delta^2}\right) \\ &= f(P(p'_i(t))). \end{aligned}$$

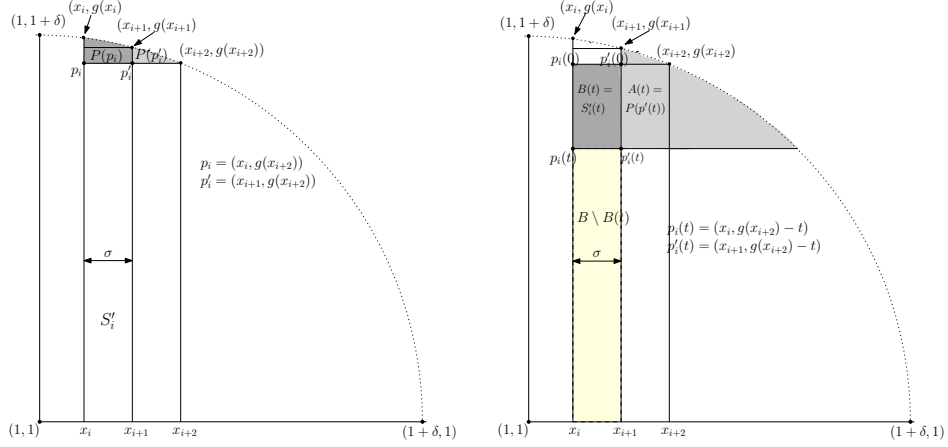


Figure 31: Illustration of case (a) of the Upper Bound. $g(x) = 1 + (\delta^q - (x-1)^q)^{1/q}$ is the equation of the upper cap. S'_i denotes Strip'_i . The figure illustrates the $q = 2$ case.

and

$$\mu(P(p_i(t))) = \Theta(\mu(P(p'_i(t)))). \quad (74)$$

We can now show that the conditions of the Strip Lemma hold.

First, since $\alpha(p_i) = 2\sigma$ and $\beta(p_i) = O(\sigma)$, from Eqs. 65 and 71

$$\forall i, \quad \mu(B'_i) = \mu(P(p_i)) = \Theta\left(\frac{\alpha^2(v)\beta^2(v)}{\delta^2}\right) = O\left(\frac{\sigma^4}{\delta^2}\right) = O\left(\frac{1}{n}\right).$$

Next note that every point in $B \setminus B(t)$ is dominated by every point in $A(t)$.

Finally, since

$$\text{Strip}'_i(t) \subset P(p_i(t)),$$

Eq. 74 also implies

$$\mu(B_i(t)) \leq \mu(P(p_i(t))) = \Theta(\mu(P(p_i(t)))) = \Theta(\mu(A_i(t))).$$

The above shows that the conditions of the Strip lemma hold so Eq. 73 holds and the proof of this section is complete.

(b) $\mathbf{E}[\text{MAX}(S_n) \cap C]$: Consider the regions $B(\alpha)$, $C(\alpha)$, $H(\alpha)$, defined in Lemma 26. Set

$$V(\alpha) = C \setminus ((C\alpha) \cup H(\alpha)).$$

Note that if $\delta > 1$ then, for large enough values of α , $H(\alpha) = V(\alpha) = \emptyset$.

By the decomposition, for every α ,

$$|\text{MAX}(S_n) \cap C| = |\text{MAX}(S_n) \cap C(\alpha)| + |\text{MAX}(S_n) \cap H(\alpha)| + |\text{MAX}(S_n) \cap V(\alpha)|.$$

Now define the random variable

$$\bar{\alpha} = \begin{cases} \max(1 + \delta - u.x : u \in) & \text{if } S_n \cap B \neq \emptyset, \\ \delta & \text{if } S_n \cap B = \emptyset. \end{cases}$$

From the definitions, $\forall \alpha \leq \delta$, any point in $B(\alpha)$ dominates any point in $V(\alpha)$. If $S_n \cap B(\alpha) \neq \emptyset$, then such a point exists and thus $|\text{MAX}(S_n \cap V(\bar{\alpha}))| = 0$. If not, then $V(\alpha) = \emptyset$, so trivially $|\text{MAX}(S_n \cap V(\bar{\alpha}))| = 0$. Thus, in all cases

$$|\text{MAX}(S_n \cap V(\bar{\alpha}))| = 0.$$

Using a similar argument to that in part (a) of Lemma 27 (the upper bound) if we let $X(\alpha) = |S_n \cap H(\alpha)|$, then, conditioning on $\bar{\alpha} = \alpha$

$$\mathbf{E} \left[|\text{MAX}(S_n \cap H(\bar{\alpha}))| \mid \bar{\alpha} = \alpha \right] = \mathbf{E} \left[H_{X_{\bar{\alpha}}} \mid \bar{\alpha} = \alpha \right] = \Theta \left(\mathbf{E} \left[\ln(X(\alpha)) \mid \bar{\alpha} = \alpha \right] \right).$$

But, since $X(\alpha) \leq n$ this immediately yields

$$\mathbf{E} [|\text{MAX}(S_n \cap H(\bar{\alpha}))|] = O(\ln n).$$

The last piece is to apply the Sweep Lemma to bound $\mathbf{E} [|\text{MAX}(S_n) \cap C(\alpha)|]$. Set

$$\begin{aligned} \bar{A} &= B = B(\delta), & \bar{B} &= C = C(\delta), \\ \bar{A}(\alpha) &= B(\alpha), & \bar{B}(\alpha) &= C(\alpha). \end{aligned}$$

By definition every point in $\bar{A}(\alpha)$ dominates every point in $\bar{B} \setminus \bar{B}(\alpha)$ and from Lemma 26, $\forall \alpha \leq \delta$, $\mu(\bar{B}(\alpha)) = O\mu(\bar{A}(\alpha))$. Applying the Sweep Lemma with $\bar{A}(t)$, $\bar{B}(t)$ and α as the sweep parameter yields

$$\mathbf{E} [|\text{MAX}(S_n) \cap C|] = O(1),$$

so

$$\mathbf{E} [|\text{MAX}(S_n) \cap C(\bar{\alpha})|] \leq \mathbf{E} [|\text{MAX}(S_n) \cap C|] = O(1).$$

Combining the items proved above yields

$$\begin{aligned} \mathbf{E} [|\text{MAX}(S_n) \cap C|] &= \mathbf{E} [|\text{MAX}(S_n) \cap C(\bar{\alpha})|] \\ &= O(1) + O(\ln n) = O(\ln n). \end{aligned}$$

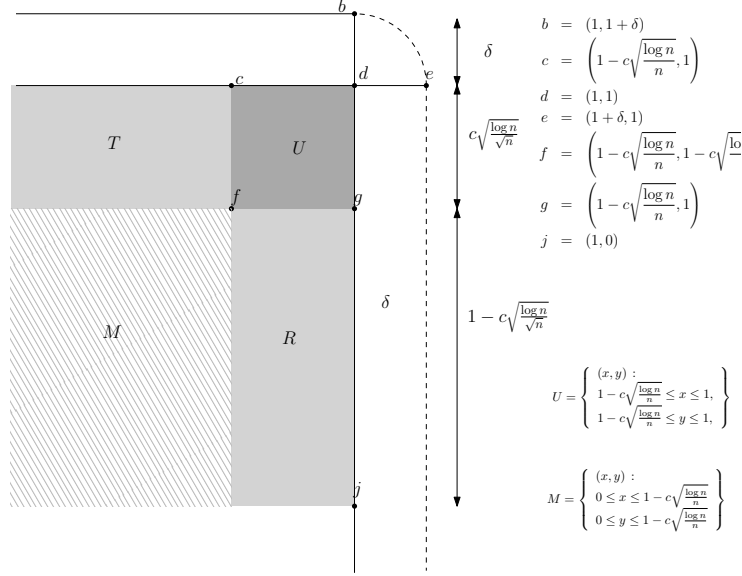


Figure 32: Illustration of Upper Bound case (c).

(c) $\mathbf{E}[|\text{MAX}(S_n) \cap A|]$:

Let $d = (1, 1)$. Then $B = P(d)$. See Fig. 32. Note that $\alpha(d) = \beta(d) = \delta$.

If $\delta > 1$ then Eqs. 65 and 66 give $\mu(B) = \Theta(1)$. If $c'\sqrt{\frac{\log n}{n}} \leq \delta \leq 1$ for large enough c' , then Eqs. 65 and 66 give $\mu(B) > 3\frac{\log n}{n}$. So, in both of these cases, for large enough n , $\mu(B) > 3\frac{\ln n}{n}$. Thus, from Lemma 1,

$$\Pr(B \cap S_n = \emptyset) = (1 - \mu(B))^n \leq \frac{1}{n}.$$

Since every point in B dominates every point in A , if $\Pr(B \cap S_n = \emptyset)$, then $\text{MAX}(S_n) \cap A = \emptyset$. Thus, if $c\sqrt{\frac{\log n}{n}} \leq \delta$,

$$\mathbf{E}[|\text{MAX}(S_n) \cap B|] \leq n\Pr(B \cap S_n = \emptyset) = O(1).$$

We therefore now assume that $\frac{1}{\sqrt{n}} \leq \delta \leq c\sqrt{\frac{\log n}{n}}$. Eq. 64 implies that, if $V \subseteq A$, then $\mu(V) = \Theta(\text{Area}(V))$.

Partition A into four subregions, U, T, R, M as illustrated in Figure 32 where U is a square with side-length $c\sqrt{\frac{\ln n}{n}}$ for some $c > 0$.

Then

$$\text{Area}(U) = \frac{c^2 \log n}{n} \quad \Rightarrow \quad \mu(U) = \Theta\left(\frac{c^2 \log n}{n}\right).$$

Choose $c > c'$ large enough so that $\mu(U) \geq \frac{3 \ln n}{n}$. Since every point in U dominates every point in M exactly the same type of analysis as performed above shows that

$$\mathbf{E}[|\text{MAX}(S_n) \cap M|] \leq n \Pr(U \cap S_n = \emptyset) = O(1).$$

Also

$$\mathbf{E}[|\text{MAX}(S_n) \cap U|] \leq \mathbf{E}[|S_n \cap U|] = n\mu(U) = O(\log n).$$

From the symmetry between T and R this yields that

$$\mathbf{E}[|\text{MAX}(S_n) \cap A|] \leq O(1) + O(\log n) + 2\mathbf{E}[|\text{MAX}(S_n) \cap R|]. \quad (75)$$

Because the side-length of U is greater than δ we find that for all $u_1, u_2 \in R$ with $u_1.x = u_2.x$,

$$\text{Area}(B_q(u_1, \delta) \cap D) = \text{Area}(B_q(u_2, \delta) \cap D),$$

and thus

$$f(u_1) = f(u_2).$$

Using a very similar analysis to that performed in part (a) this shows that

$$\mathbf{E}[|\text{MAX}(S_n \cap R)|] = \mathbf{E}[H_X] = \Theta(\mathbf{E}[\ln X]) = \Theta(\ln n),$$

where $X = |S_n \cap R|$ and the last equality come from the fact that $X \leq n$. Since

$$\mathbf{E}[|\text{MAX}(S_n) \cap R|] \leq \mathbf{E}[|\text{MAX}(S_n \cap R)|],$$

combining this with Eq. 75 proves

$$\mathbf{E}[|\text{MAX}(S_n \cap R)|] = O(\ln n).$$

□

Proof: of Lemma 25.

Physically, this mean that $B_q(p, \delta) \cap U$ is the mirror image of $P(p)$ flipped across the line $x = -y$ and then placed with the image of p at $(1, 1)$. See Fig. 33.

Let $T'(p)$ be the triangle with with corners $(1 - \alpha, 0)$, $(0, 1 - \beta)$ and $(1, 1)$ and $R(p)$ the rectangle with lower left corner $(1 - \alpha, 1 - \beta)$ and upper-right corner $(1, 1)$. The convexity of $B_q(p, \delta) \cap U$ implies

$$T'(p) \subseteq B_q(p, \delta) \cap U \subseteq R'(p),$$

therefore

$$\frac{1}{2}\alpha\beta = \text{Area}(T'(p)) \leq \text{Area}(B_q(p, \delta) \cap U) \leq \text{Area}(R'(p)) \leq \alpha\beta.$$

Note that B_∞ is the upper right hand corner of U .

If $\alpha, \beta \leq 1$ then $R'(p) \subset B_\infty$, therefore

$$P'(p) = B_q(p, \delta) \cap B_\infty = B_q(p, \delta) \cap U,$$

yielding $\text{Area}(P'(p)) = \Theta(\alpha\beta)$.

If $\alpha, \beta \geq 1$, then $\text{Area}(T(p) \cap B_1) \geq 1/2$ so $\text{Area}(P'(p)) = \Theta(1)$.

If $\alpha \geq 1$ and $\beta \leq 1$, then $\text{Area}(T(p) \cap B_1) = \Theta(\beta/2)$.

f $\alpha \leq 1$ and $\beta \geq 1$, then $\text{Area}(T(p) \cap B_1) = \Theta(\alpha/2)$.

Combining the four cases proves Eq. 65.

To prove Eq. 66 recall that $\text{Area}(P(v)) = \Theta(\alpha\beta)$ and

$$\mu(P(v)) = \int_{u \in P(v)} f(u) du.$$

Also note that $\forall u \in P(v)$, $\alpha(u) \leq \alpha(v) = \alpha$ and $\beta(u) \leq \beta(v) = \beta$. Eq. 65 then implies that $\forall u \in P(u)$, $f(u) = O(f(v))$. Thus

$$\mu(P(v)) = \int_{u \in P(v)} f(u) du \leq \text{Area}(P(v)) \cdot \max_{u \in P(v)} f(u) = O(\alpha\beta f(v)).$$

For the other direction define $\bar{P}(p)$ to be the upper right triangle with lower left corner p and horizontal, vertical side lengths $\alpha/2, \beta/2$. See the right hand side of Fig. 33. Note that $\forall u \in \bar{P}(v)$, $\alpha(u) \geq \alpha(v)/2 = \alpha/2$ and $\beta(u) \geq \beta(v)/2 = \beta/2$. Thus, $\forall u \in \bar{P}(u)$, $f(u) = \Omega(f(v))$. Finish by noting that because $\bar{P}(p) \subset P(p)$,

$$\begin{aligned} \mu(P(v)) &= \int_{u \in P(v)} f(u) du \geq \int_{u \in \bar{P}(v)} f(u) du \\ &\geq \text{Area}(\bar{P}(v)) \cdot \min_{u \in \bar{P}(v)} f(u) = \Omega(\alpha\beta f(v)). \end{aligned}$$

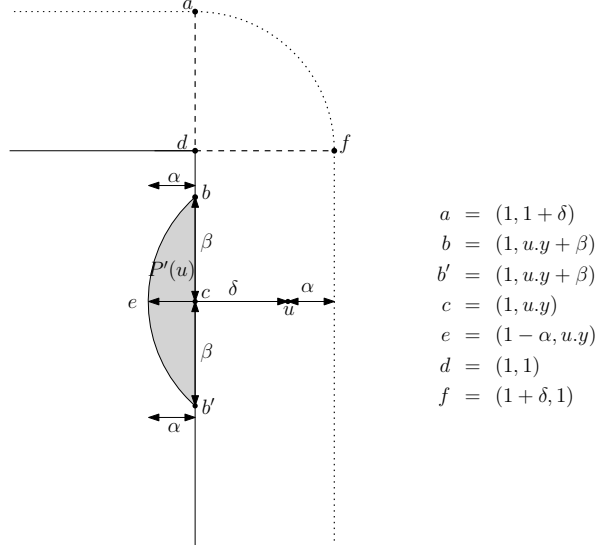


Figure 34: Illustration of the proof of Lemma 25 Eq. 67 when $p \in C$.

We have thus shown $\mu(P(v)) = \Theta(\alpha\beta f(v))$.

$p \in C$: By straightforward geometric arguments (see Fig. 34)

$$\text{Area}(P'(v)) = \Theta\left(\min(\alpha(v), 1) \cdot \min(\beta(v), 1)\right).$$

□

10 Conclusion

This paper developed a suite of tools for deriving $\mathbf{E}[M_n]$, the expected number of maximal points in a set of n points chosen IID from $\mathbf{B}_p + \delta\mathbf{B}_q$, which is the convolution of two distributions; the first is $\mathbf{B}_p$, the uniform distribution over the L_p ball and the second $\delta\mathbf{B}_q$, the uniform distribution over a δ -scaled L_q ball. For small δ , $\mathbf{B}_p + \delta\mathbf{B}_q$ could be considered as a *smoothed* version of $\mathbf{B}_p$ with $\delta\mathbf{B}_q$ error. This result seems to be the first analysis of $\mathbf{E}[M_n]$ for non-uniform and non-Gaussian distributions.

This paper is only a first step. Obvious next steps are

- The results in the paper were only proven for $p, q \in \{1, 2, \infty\}$ and $p = \infty, q \in [1, \infty]$. The next step would be to attempt to extend the results to *all* pairs $p, q \in [1, \infty]$.
- The results in this paper only derive first-order asymptotics. Another obvious direction would be to try to prove limit-theorems, e.g., paralleling the results in [3] for 2-dimensional uniform samples.
- There is a rich literature stretching back more than fifty years on the average number of points on the *convex hull* of points chosen IID from a uniform distribution in a planar region or a Gaussian distribution, e.g., [13, 19]. It would be interesting to see how the convex hull evolves in these convoluted distributions.
- Finally, we note that the results on $\mathbf{E}[M_n]$ for n points chosen IID from a uniform distribution over an L_p ball have analogues in higher dimensions, i.e., $\Theta(\log^{d-1} n)$ if $p = \infty$ and $\Theta(n^{1-\frac{1}{d}})$ if $p \in [1, \infty)$ [13, 4]. The next step would be to attempt to extend the results in this paper to higher distributions.

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