Random. Algorithms 2018 Final Exam

17 Dec. 2018, (10am-1pm) Total= 25 points

NAME:

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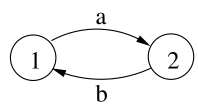
1. (4) We have a fair 6-side die. Let X be the number of times 6 occurs over n throws of the die. Let p be the probability of the event $X \geq n/4$ (i.e. having $n/4$ sixes). Compute $\mathbf{E}[X]$ and use Chernoff to prove that w.h.p. X is concentrated around $\mathbf{E}[X]$.

2. (4) We throw m balls in 3 bins , uniformly and independently. Give an upper bound on the probability that the first bin contain more balls than the total number in the other two.

3. (3) We perform a random walk on the integer line, starting from 0. At each step we flip a coin and go one unit up if it is heads and one unit down if it is tails. (For ex. if at the beginning we have 5 coin flips that are H, H, T, H, H then we would be at the positions $1, 2, 1, 2, 3$ after the respective steps). Compute the probability that after 100 steps we are at position 20.

4. (2) Consider the following Markov chain:



For which values of a and b do we obtain an absorbing Markov Chain?

5. Consider the following studies at the UPC, The ETSETB, the FIB ad the FM. Assume that among the descendants of the ETSETB's alumni, 80% also study in the ETSETB and 20% went to the FIB; 40% of the descendants from the FM graduates went to the FM and the rest split evenly between the ETSETB and the FIB; from the descendants of FIB alumni, 70% went to the FIB, 20% to the ETSETB and 10% to the FM.

   (a) (1) Pose this problem as a Markov Chain

   (b) (1) Find the probability that the grandson of an alumni from the ETSETB went to the ETSETB.

   (c) (2) Modify the above setting by assuming that the descendants of ETSETB students always went to the ETSETB. Again, find the probability that the grandson of an alumni from the ETSETB went to the ETSETB.

6. Consider the Markov chain with state $S = \{1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}$$

(a) (2) Show the digraph of the Markov chain

(b) (2) Show that the Markov chain is irreducible and aperiodic.

(c) (2) If the process starts in state 1, find the probability that it is in state 3 after two steps.

(d) (2) Find the stationary distribution and $\lim_{n\to\infty} P^n$.

Recall Chernoff's bounds

Let $X_1 \dots X_n$ be independent Bernuilli r.v. with $\Pr X_i = 1 = p_i$, and let $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbf{E}[X]$. Then:

1. For any $\delta > 0$,

$$\mathbf{Pr}[X \leq (1-\delta)\mu] < \left(\frac{e^{\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu},$$

$$\mathbf{Pr}[X \geq (1+\delta)\mu] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}.$$

2. For any $\delta \in (0,1]$

$$\mathbf{Pr}[X \leq (1-\delta)\mu] < e^{-\mu\delta^2/2},$$

$$\mathbf{Pr}[X \geq (1-\delta)\mu] < e^{-\mu\delta^2/4}.$$

3. If each for every $1 \leq i \leq n$, $0 \leq X_i \leq 1$, then for every $\delta > 0$

$$\mathbf{Pr}[|X-\mu| \geq \delta\mu] \leq 2e^{-\delta^2\mu/(2+\delta)}.$$