Geometric Constraint Solving in 2D

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Motivation

Feature-based CAD systems

+ Editable representation ($Erep$)

⇒ 2-dimensional constraint-based editor
A geometric constraint problem (GCP) consists of

- A set of geometric elements \( \{A, \cdots, L_{AB}, \cdots\} \).
- A set of values \( \{d, \alpha, h\} \)
- A set of dimensional variables \( \{x, y\} \).
- A set of external variables \( \{v\} \).
- A set of valuated and symbolic constraints.
- A set of equations.

\[
y = x \cdot v
\]
\[
v = 0.5
\]
The geometric constraints of a GCP can be represented by a constraint graph $G = (E, V)$.

The vertices in $V$ are two-dimensional geometric elements with two degrees of freedom.

The edges in $E$ are constraints that reduce by one the degrees of freedom.
A geometric constraint problem can be represented by a formula in first-order logic.

\[
\varphi(A, B, C, D, L_{AB}, L_{AC}, L_{BC}, x, y, v)
\]
\[
\equiv d(A, B) = d \land on(A, L_{AB}) \land on(B, L_{AB}) \land
\quad on(A, L_{AC}) \land on(C, L_{AC}) \land on(D, L_{AC}) \land
\quad on(B, L_{BC}) \land on(C, L_{BC}) \land
\quad h(C, L_{AB}) = h \land a(L_{AB}, L_{BC}) = \alpha \land
\quad d(A, C) = x \land d(C, D) = y \land
\quad y = x \cdot v \land v = 0.5
\]
Geometric Constraint Solving

Geometric constraint solving (GCS) consists in proving the truth of the formula

$$\exists A \exists B \exists C \exists D \exists L_{AB} \exists L_{AC} \exists L_{BC} \exists x \exists y \exists v \varphi(A, B, C, D, L_{AB}, L_{AC}, L_{BC}, x, y, v)$$

by finding the position of the geometric elements and the values of tags and external variables that satisfy the constraints.
Over-constrained Geometric Constraint Problem

Theorem 1 (Laman, 1970)
Let $G = (P, D)$ a geometric constraint graph where the vertices in $P$ are points in the two dimensional Euclidean space and the edges $D \subseteq P \times P$ are distance constraints. $G$ is generically well constrained if and only if for all $G' = (P', D')$, subgraph of $G$ induced by $P' \subseteq P$,

1. $|D'| \leq 2|P'| - 3$, and
2. $|D| = 2|P| - 3$. 

![Diagram of a geometric constraint graph](attachment:image.png)
Definition 1 A geometric constraint graph is structurally over-constrained if and only if exists an induced subgraph with $n$ vertices and $m$ edges such that $m > 2 \cdot n - 3$.

Definition 2 A geometric constraint graph is structurally well-constrained if and only if it is not structurally over-constrained and $|E| = 2 \cdot |V| - 3$.

Definition 3 A geometric constraint graph is structurally under-constrained if and only if it is not structurally over-constrained and $|E| \leq 2 \cdot |V| - 3$. 

Structurally over-constrained Geometric Constraint Problem
Approaches to Geometric Constraint Solving

• Solving systems of equations
  – Numerical Constraint Solvers
  – Symbolic Constraint Solvers
  – Propagation Methods
  – Structural analysis

• Constructive Constraint Solvers
  – Graph based
  – Rule based

• Degrees of freedom analysis

• Geometric theorem proving
Constructive technique: Ruler-and-compass constructibility

A point $P$ is constructible if there exists a finite sequence $P_0, P_1, \ldots, P_n = P$ of points in the plane with the following property. Let $S_j = \{P_0, P_1, \ldots, P_j\}$, for $1 \leq j \leq n$.

For each $2 \leq j \leq n$ is either

1. the intersection of two distinct straight lines, each joining two points of $S_{j-1}$, or

2. a point of intersection of a straight line joining two points of $S_{j-1}$ and a circle with centre a point of $S_{j-1}$ and radius the distance between two points of $S_{j-1}$, or

3. a point of intersection of two distinct circles, each with centre a point of $S_{j-1}$ and radius the distance between two points of $S_{j-1}$. 
Constructive technique: Constraints sets

- A **CA** set is a pair of oriented segments which are mutually constrained in angle.

- A **CD** set is a set of points with mutually constrained distances.

- A **CH** set is a point and a segment constrained by the perpendicular distance from the point to the segment.
Constructive technique: Geometric locus
Constructive technique: Set of rules

CR-CR

CR-CC

CR-RA

CR-RP

CR-RT
Constructive technique: Analysis and Construction phases

\[ \varphi(p_1, p_2, p_3) \equiv \text{Geometric Constraint Problem} \]

\[ d(p_1, p_2) = d_1 \wedge \]
\[ d(p_2, p_3) = d_2 \wedge \]
\[ d(p_3, p_1) = d_3 \]

\[ \psi(p_1, p_2, p_3) \equiv \]
\[ p_1 = (0, 0) \wedge \]
\[ p_2 = (d_1, 0) \wedge \]
\[ p_3 = \text{inter(circle}(p_1, d_3), \text{circle}(p_2, d_2)) \]

\[ \psi_v(p_1, p_2, p_3) \equiv \]
\[ p_1 = (0, 0) \wedge \]
\[ p_2 = (200, 0) \wedge \]
\[ p_3 = (140, 169.706) \]

\[ \psi \equiv \]
\[ d_1 = 200 \wedge \]
\[ d_2 = 180 \wedge \]
\[ d_3 = 220 \]
Constructive technique: Structure of the Analysis phase

\[ \varphi(p_1, p_2, p_3) \equiv \]
\[ d(p_1, p_2) = d_1 \land \]
\[ d(p_2, p_3) = d_2 \land \]
\[ d(p_3, p_1) = d_3 \]

\[ C_2(cd_1, p_1, p_2, d_1) \]
\[ C_2(cd_2, p_2, p_3, d_2) \]
\[ C_2(cd_3, p_1, p_3, d_3) \]

\[ E_0 \equiv \]
\[ cd_1 = \{p_1, p_2\} \land \]
\[ cd_2 = \{p_2, p_3\} \land \]
\[ cd_3 = \{p_1, p_3\} \]

\[ DDD(cd_4, p_1, p_2, p_3, cd_1, cd_2, cd_3) \]

\[ E_f \equiv cd_4 = \{p_1, p_2, p_3\} \]

\[ \psi(p_1, p_2, p_3) \equiv \]
\[ p_1 = (0, 0) \land \]
\[ p_2 = (d_1, 0) \land \]
\[ p_3 = \text{inter}(\text{circle}(p_1, d_3), \text{circle}(p_2, d_2)) \]
Constructive technique: Correctness (1)

- Let $R$ be a set of tuples $(D, X)$ where
- $D$ is a set of $\text{CD}$ sets, and
- $X$ is a set of $\text{CA}$ sets and $\text{CH}$ sets.
- Let $\rightarrow = \rightarrow_{\text{DDD}} \cup \rightarrow_{\text{DDX}} \cup \rightarrow_{\text{DXX}}$ be a reduction relation.
- We define the abstract reduction system $R = \langle R, \rightarrow \rangle$.
- We proof termination and confluence that implies canonicity and unique normal form property.
Constructive technique: Correctness (2)
Constructive technique:

Reduction rules (1)

\[(D, X) \rightarrow_{\text{DDD}} ((D \setminus \{d_1, d_2, d_3\}) \cup \{d_1 \cup d_2 \cup d_3\}, X)\]

if

\[
\begin{align*}
\{d_1, d_2, d_3\} \subseteq D & \land \\
\{d_1 \cap d_2\} = \{p_1\} & \land \\
\{d_2 \cap d_3\} = \{p_2\} & \land \\
\{d_1 \cap d_3\} = \{p_3\} & \land \\
p_1 \neq p_2 \neq p_3
\end{align*}
\]
Constructive technique: 
Reduction rules (2)

\[(D, X) \rightarrow_{DDX} ((D - \{d_1, d_2\}) \cup \{d_1 \cup d_2 \cup \text{punts}(x_1)\}, X - \{x_1\}) \]

\[
\begin{align*}
\{d_1, d_2\} &\subseteq D \\
d_1 \cap d_2 &= \{p_1\} \\
p &\in d_2 \\
x_1 &\in X \\
\text{punts}(x_1) - d_1 &= \{p\} \\
p &\neq p_1
\end{align*}
\]
Constructive technique:
Reduction rules (3)

\[(D, X) \rightarrow_{\text{DXX}} ((D - \{d_1\}) \cup \{d_1 \cup \text{punts}(x_1) \cup \text{punts}(x_2)\}, X - \{x_1, x_2\})\]

if
\[
\begin{align*}
d_1 & \in D \land \\
\{x_1, x_2\} & \subseteq X \land \\
\text{punts}(x_1) - d_1 & = \{p\} \land \\
\text{punts}(x_2) - d_1 & = \{p\}
\end{align*}
\]
Hybrid technique: Introduction

**Goal** Solve symbolic constraints keeping the two phases of the constructive technique for valuated constraints.

**Idea** Federate a constructive solver and an equational solver.

**Required** A technique of analysis of systems of equations.
Hybrid technique: Analysis of systems of equations

1. Represent the structure of the systems of equations by a bipartite graph (bigraph).

2. Compute the Dulmage-Mendelsohn decomposition of the bigraph. \( V_0 \) is the over-determined part, \( V_\infty \) is the under-determined part. \( V_1, \ldots, V_n \) are the consistent part.
Hybrid technique: Motivation (1)

A geometric constraint problem that can not be solved without considering geometric variables.

\[ \varphi(a, b, c, d, g, l, m, n, r) \]
\[ \equiv d(a, b) = d_1 \land d(b, c) = d_2 \land \]
\[ \text{on}(a, l) \land \text{on}(b, l) \land \text{on}(b, n) \land \]
\[ \text{on}(c, n) \land \text{on}(c, m) \land \text{on}(d, m) \land \]
\[ a(l, n) = \alpha_1 \land a(l, m) = \alpha_2 \land \]
\[ d(g, c) = r \land h(g, l) = r \land \]
\[ h(g, m) = r \land d(g, d) = r \]
Hybrid technique: Motivation (2)

Final state of the geometric analyzer.

Final state of the equational analyzer.

\[ d(g, a) = r \quad h(g, l) = r \quad h(g, m) = r \quad d(g, d) = r \]
Hybrid technique: Technique (1)

1. Represent geometric variables in the bi-graph \((B)\).

\[
\begin{align*}
\text{d}(g,a) &= r \\
\text{h}(g,l) &= r \\
\text{h}(g,m) &= r \\
\text{d}(g,d) &= r \\
\end{align*}
\]

2. Compute \(R(B,C_1)\), the restriction of bi-graph \(B\) by CD set \(C_1\). The equations are analyzed with respect to a coordinate system (CD set).

\[
\begin{align*}
\text{d}(g,a) &= r \\
\text{h}(g,l) &= r \\
\text{h}(g,m) &= r \\
\text{d}(g,d) &= r \\
\end{align*}
\]
3. For each solved dimensional variable, add a new constraint set to the state of the geometric analyzer.

4. For each pair of solved geometric variables \((v_x, v_y)\), add the geometric element \(v\) to the projection CD set \(C_1\).

5. Remove solved variables and equations from the bigraph \(B\).
Hybrid technique: Correctness (1)

- Let $S$ be a set of tuples $(D, X, B)$ where
  - $D$ is a set of CD sets,
  - $X$ is a set of CA sets and CH sets, and
  - $B$ is a bigraf representing symbolic geometric constraints and equations.
- Let $\rightarrow_{\rho'}$ be the constructive reduction relation.
- Let $\rightarrow_{\kappa}$ be the equational analysis reduction relation.
- We define the abstract reduction system $S = \langle S, \rightarrow_{\rho'} \cup \rightarrow_{\kappa}\rangle$.
- We proof termination and confluence that implies canonicity and unique normal form property.
Conclusions


- A clean phase structure.

- A correct hybrid method combining a constructive method and an equation analysis method.

- A prototype implementation.
Future work

- Study the domain of constructive methods.

- Extending the domain of our constructive method.

- Selection of the solution.

- Determine the range of values of a constraint.