Declarative Characterization of a
General Architecture for
Constructive Geometric Constraint Solvers

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Geometric constraint solvers

- Geometric constraint problems = geometric elements + constraints between them.

- Geometric constraint solvers analyze geometric problems and construct realizations.
Constructive geometric constraint solvers

- Constructive geometric constraint solvers describe the solution as a sequence of ruler-and-compass geometric operations.

- Constructive geometric constraint solvers share a common architecture.
Objective: characterize a general architecture for constructive geometric constraint solvers

- Identify the relevant functional units.
- Identify the data entities that the functional units manipulate.
- Characterize the functional units independently of its implementation.
- State the semantics of the data entities.
Architecture overview: a data flow diagram relating data entities and functional units

- Abstract problem
  - Analyzer
    - Abstract plan
    - Index selector
      - Index assign.
      - Constructor
        - Realization
  - Parameters ass.
Abstract problems

- An abstract problem \( A = \langle G, C, P \rangle \) describes the geometric elements \( G \), the constraints \( C \) and the parameters \( P \) of the problem.

\[
G = \{ p_1, p_2, p_3, p_4, l_1, l_2, l_3, l_4 \} \\
P = \{ d_1, d_2, a_1, a_2, h_1 \} \\
C = \{ \text{onPL}(p_1, l_1), \text{onPL}(p_1, l_2), \text{onPL}(p_2, l_1), \text{onPL}(p_2, l_3), \text{onPL}(p_3, l_3), \text{onPL}(p_3, l_4), \text{onPL}(p_4, l_2), \text{onPL}(p_4, l_4), \text{distPP}(p_2, p_3, d_1), \text{distPP}(p_3, p_4, d_2), \text{distPL}(p_1, l_3, h_1), \text{angleLL}(l_3, l_1, a_2), \text{angleLL}(l_3, l_4, a_1) \} 
\]
What does an abstract problem mean?

• Characteristic formula $\Psi$.

$$\Psi(A) \equiv \left( onPL(p_1, l_1) \right)$$
$$\land \left( onPL(p_1, l_2) \right)$$
$$\land \left( onPL(p_2, l_1) \right)$$
$$\land \left( onPL(p_2, l_3) \right)$$
$$\land \left( onPL(p_3, l_3) \right)$$
$$\land \left( onPL(p_3, l_4) \right)$$
$$\land \left( onPL(p_4, l_2) \right)$$
$$\land \left( onPL(p_4, l_4) \right)$$
$$\land \left( distPP(p_2, p_3, d_1) \right)$$
$$\land \left( distPP(p_3, p_4, d_2) \right)$$
$$\land \left( distPL(p_1, l_3, h_1) \right)$$
$$\land \left( angleLL(l_3, l_1, a_2) \right)$$
$$\land \left( angleLL(l_3, l_4, a_1) \right)$$
Parameters assignments

- A *parameters assignment* $\alpha$ assigns values to parameters symbols.

$$
\begin{align*}
\alpha(a_1) &= -1.222 \\
\alpha(a_2) &= 1.0472 \\
\alpha(h_1) &= 160.0 \\
\alpha(d_1) &= 290.0 \\
\alpha(d_2) &= 130.0
\end{align*}
$$
Instance problems

- $\alpha.A = \langle G, \alpha.C, P \rangle$ is an instance problem.

$$\alpha.C = \{ \text{onPL}(p_1, l_1),$$
$$\text{onPL}(p_1, l_2),$$
$$\text{onPL}(p_2, l_1),$$
$$\text{onPL}(p_2, l_3),$$
$$\text{onPL}(p_3, l_3),$$
$$\text{onPL}(p_3, l_4),$$
$$\text{onPL}(p_4, l_2),$$
$$\text{onPL}(p_4, l_4),$$
$$\text{distPP}(p_2, p_3, 290.0),$$
$$\text{distPP}(p_3, p_4, 130.0),$$
$$\text{distPL}(p_1, l_3, 160.0),$$
$$\text{angleLL}(l_3, l_1, 1.0472),$$
$$\text{angleLL}(l_3, l_4, -1.222) \}$$
Geometry assignments

- A *geometry assignment* \( \kappa \) assigns coordinates to geometric elements.

\[
\begin{align*}
\kappa(p_1) & = (92.38, 160) \\
\kappa(p_2) & = (0, 0) \\
\kappa(p_3) & = (290, 0) \\
\kappa(p_4) & = (245.54, 122.16) \\
\kappa(l_1) & = (-0.87, 0.5, 0) \\
\kappa(l_2) & = (-0.24, -0.97, 177.48) \\
\kappa(l_3) & = (0, -1, 0) \\
\kappa(l_4) & = (0.94, 0.34, -272.51)
\end{align*}
\]
Which are the solutions of an abstract problem?

- A *realization* of an instance problem $\alpha.A$ is a geometry assignment $\kappa$ for which the formula $\Psi(\kappa.\alpha.A)$ holds.

- $V(\alpha.A)$ is the set of realizations of the instance problem $\alpha.A$.

\[
V(\alpha.A) = \{ \kappa \mid \Psi(\kappa.\alpha.A) \}
\]
Abstract plans

- An abstract plan \( S = \langle G, P, L, I \rangle \) is a sequence of geometric operations \( L \) that computes the coordinates of the geometric elements in \( G \).

\[
L = \{ p_2 = \text{pointXY}(O_x, O_y) \\
p_3 = \text{pointXY}(d_1, O_y) \\
c_1 = \text{circleCR}(p_3, d_2) \\
l_3 = \text{linePP}(p_2, p_3) \\
l_4 = \text{lineAP}(l_3, a_1, p_3) \\
p_4 = \text{interLC}(l_4, c_1, s_1) \\
l_1 = \text{lineAP}(l_3, a_2, p_2) \\
l_8 = \text{lineLD}(l_3, h_1, s_2) \\
p_1 = \text{interLL}(l_1, l_8) \\
l_2 = \text{linePP}(p_1, p_4) \}
\]
What does an abstract plan mean?

- Characteristic formula $\Phi$.

$$
\Phi(S) \equiv (p_2 = pointXY(O_x, O_y) \\
\land p_3 = pointXY(d_1, O_y) \\
\land c_1 = circleCR(p_3, d_2) \\
\land l_3 = linePP(p_2, p_3) \\
\land l_4 = lineAP(l_3, a_1, p_3) \\
\land p_4 = interLC(l_4, c_1, s_1) \\
\land l_1 = lineAP(l_3, a_2, p_2) \\
\land l_8 = lineLD(l_3, h_1, s_2) \\
\land p_1 = interLL(l_1, l_8) \\
\land l_2 = linePP(p_1, p_4))
$$
Index assignments

- An *index assignment* \( \iota \) assigns values to the sign parameters in the index.

\[
\iota(s_1) = +1 \\
\iota(s_2) = +1
\]
Instance plans and indexed plans

- $\alpha.S = \langle G, P, \alpha.L, I \rangle$ is an instance plan.
- $\iota.S = \langle G, P, \iota.L, I \rangle$ is an indexed plan.
- An example of $\iota.\alpha.S$.

$$\begin{align*}
\iota.\alpha.L &= \{ p_2 = \text{pointXY}(O_x, O_y) \\
&\quad \land p_3 = \text{pointXY}(290.0, O_y) \\
&\quad \land c_1 = \text{circleCR}(p_3, 130.0) \\
&\quad \land l_3 = \text{linePP}(p_2, p_3) \\
&\quad \land l_4 = \text{lineAP}(l_3, -1.222, p_3) \\
&\quad \land p_4 = \text{interLC}(l_4, c_1, +1) \\
&\quad \land l_1 = \text{lineAP}(l_3, 1.0472, p_2) \\
&\quad \land l_8 = \text{lineLD}(l_3, 160.0, +1) \\
&\quad \land p_1 = \text{interLL}(l_1, l_8) \\
&\quad \land l_2 = \text{linePP}(p_1, p_4) \}\end{align*}$$
Which are the solutions of an abstract plan?

- An *indexed anchor* of an instance plan $\alpha_.S$ is a geometry assignment $\kappa$ for which there is an index assignment $\iota$ such that the formula $\Phi(\kappa.\iota.\alpha_.S)$ holds.

- $V(\alpha_.S)$ is the set of indexed anchors of the instance plan $\alpha_.S$.

\[
V(\alpha_.S) = \{ \kappa \mid \exists \iota \Phi(\iota.\kappa.\alpha_.S) \}
\]
Analyzers

- An analyzer computes an abstract plan $S = \langle G, P, L, I \rangle$ from an abstract problem $A = \langle G, C, P \rangle$.

- Correct analyzers compute construction plans that generate realizations when carried out.

  $$V(\alpha.S) \subseteq V(\alpha.A)$$

- Complete analyzers compute construction plans that generate exactly the set of realizations when carried out.

  $$V(\alpha.S) = V(\alpha.A)$$
Constructors

- A constructor carries out the geometric operations in an abstract plan \( S = \langle G, P, L, I \rangle \).

\[ \text{Abstract plan } S \]

\[ \text{Parameters ass. } \alpha \]

\[ \text{Index assign. } \iota \]

\[ \text{Constructor} \]

\[ \text{Realization } \kappa \]

- The geometric assignment \( \kappa \) is a realization provided than the analyzer which has computed \( S \) is correct.

- \( \kappa \) is such that \( \Phi(\kappa, \iota, \alpha, S) \) holds.
Index selectors

- An *index selector* selects a unique solution among a possibly exponential number of solutions described in the construction plan.

- The anchor-based index selector computes an index assignment $\iota$ such that $\Phi(\iota,\kappa,\alpha_{\kappa},S)$ holds. $\kappa$ is a geometry assignment and $\alpha_{\kappa}$ is a parameters assignment obtained from $\kappa$. 
Conclusions

We have presented the definition of an architecture for constructive geometric constraint solvers.

- The architecture is precisely and concisely defined.

- It is independent of any particular implementation of the functional units.

- It is well suited for interactive applications.

- The functional units are reusable to solve problems which are not geometric constraint solving problems but are related.
Future work

Development of a common platform for researchers in geometric constraint solving.

This requires:

- The definition of data interchange standards for the data entities.

- The definition of a common set of geometric elements, constraints and geometric operations and its precise semantics.