The Dual Simplex Method

Combinatorial Problem Solving (CPS)

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Basic Idea

- Abuse of notation:
  Henceforth by “optimal” we mean “satisfying the optimality conditions”

- The algorithm as explained so far is known as primal simplex:
  starting with feasible basis,
  look for optimal basis while keeping feasibility

- There is an alternative algorithm known as dual simplex:
  starting with optimal basis,
  look for feasible basis while keeping optimality
Basic Idea

\[
\begin{align*}
\text{min} & \quad -x - y \\ 2x + y & \geq 3 \\ 2x + y & \leq 6 \\ x + 2y & \leq 6 \\ x & \geq 0 \\ y & \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad -x - y \\ 2x + y - s_1 & = 3 \\ 2x + y + s_2 & = 6 \\ x + 2y + s_3 & = 6 \\ x, y, s_1, s_2, s_3 & \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad -6 + y + s_3 \\ x & = 6 - 2y - s_3 \\ s_1 & = 9 - 3y - 2s_3 \\ s_2 & = -6 + 3y + 2s_3 \\
\end{align*}
\]

Basis \((x, s_1, s_2)\) is optimal
\(\text{(satisfies the optimality criterion)}\)

but is not feasible!
Basic Idea

\[ \begin{align*}
2x + y &\leq 6 \\
2x + y &\geq 3 \\
x + 2y &\leq 6 \\
x &\geq 0 \\
y &\geq 0 \\
(6, 0)
\end{align*} \]
Basic Idea

- Let us make the violating variables non-negative ...

  - Increase $s_2$ by making it non-basic

- ... while preserving optimality

  - If $y$ replaces $s_2$ in the basis, then $y = \frac{1}{3}(s_2 + 6 - 2s_3)$, $-x - y = -4 + \frac{1}{3}(s_2 + s_3)$

  - If $s_3$ replaces $s_2$ in the basis, then $s_3 = \frac{1}{2}(s_2 + 6 - 3y)$, $-x - y = -3 + \frac{1}{2}(s_2 - y)$

  - To preserve optimality, $y$ must replace $s_2$
Basic Idea

\[
\begin{align*}
\min & \quad -6 + y + s_3 \\
x & = 6 - 2y - s_3 \\
s_1 & = 9 - 3y - 2s_3 \\
s_2 & = -6 + 3y + 2s_3
\end{align*}
\]

\[
\begin{align*}
\min & \quad -4 + \frac{1}{3}s_2 + \frac{1}{3}s_3 \\
x & = 2 - \frac{2}{3}s_2 + \frac{1}{3}s_3 \\
y & = 2 + \frac{1}{3}s_2 - \frac{2}{3}s_3 \\
s_1 & = 3 - s_2
\end{align*}
\]

- Current basis is feasible and optimal!
Basic Idea

\[ \begin{align*}
2x + y &\leq 6 \\
x + 2y &\leq 6 \\
x &\geq 0 \\
y &\geq 0
\end{align*} \]
1. **Initialization**: Pick an optimal basis.

2. **Dual Pricing**: If all basic values are $\geq 0$, then return **OPTIMAL**. Else pick a basic variable with value $< 0$.

3. **Dual Ratio test**: Find non-basic variable for swapping while preserving optimality, i.e., sign constraints on reduced costs. If it does not exist, then return **INFEASIBLE**. Else swap chosen non-basic variable with violating basic variable.

4. **Update**: Update the tableau and go to 2.
Duality

- To understand better how the dual simplex works: theory of duality
- We can get lower bounds on LP optimum value by linearly combining constraints with convenient multipliers

\[
\begin{align*}
\begin{cases}
\min & -x - y \\
2x + y & \geq 3 \\
2x + y & \leq 6 \\
x + 2y & \leq 6 \\
x & \geq 0 \\
y & \geq 0
\end{cases}
\Rightarrow
\begin{cases}
\min & -x - y \\
2x + y & \geq 3 \\
-2x - y & \geq -6 \\
x - 2y & \geq -6 \\
x & \geq 0 \\
y & \geq 0
\end{cases}
\end{align*}
\]

\[
1 \cdot (x - 2y \geq -6) \\
1 \cdot (y \geq 0)
\]

\[
\begin{align*}
-x - 2y & \geq -6 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

\[
-x - y \geq -6
\]
Duality

- There may be different choices of multipliers
- Each choice may give different lower bounds

\[
\begin{align*}
\text{min } -x - y & \\
2x + y & \geq 3 \\
-2x - y & \geq -6 \\
-x - 2y & \geq -6 \\
x & \geq 0 \\
y & \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
1 \cdot ( & \quad 2x + y \quad \geq \quad 3 ) \\
2 \cdot ( & \quad -2x - y \quad \geq \quad -6 ) \\
1 \cdot ( & \quad x \quad \geq \quad 0 ) \\
\end{align*}
\]

\[
\begin{align*}
2x + y & \geq 3 \\
-4x - 2y & \geq -12 \\
x & \geq 0 \\
\end{align*}
\]

\[
-x - y \geq -9
\]
Duality

- In general:

\[
\begin{align*}
\min & -x - y \\
2x + y & \geq 3 \\
-2x - y & \geq -6 \\
-x - 2y & \geq -6 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

\[
\mu_1 \cdot (2x + y) \geq 3 \\
\mu_2 \cdot (-2x - y) \geq -6 \\
\mu_3 \cdot (-x - 2y) \geq -6 \\
\mu_4 \cdot (x) \geq 0 \\
\mu_5 \cdot (y) \geq 0
\]

\[
2\mu_1 x + \mu_1 y \geq 3\mu_1 \\
-2\mu_2 x - \mu_2 y \geq -6\mu_2 \\
-\mu_3 x - 2\mu_3 y \geq -6\mu_3 \\
\mu_4 x \geq 0 \\
\mu_5 y \geq 0
\]

\[
(2\mu_1 - 2\mu_2 - \mu_3 + \mu_4) x + (\mu_1 - \mu_2 - 2\mu_3 + \mu_5) y \geq 3\mu_1 - 6\mu_2 - 6\mu_3
\]

- If \( \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \mu_4 \geq 0, \mu_5 \geq 0, \)

\[2\mu_1 - 2\mu_2 - \mu_3 + \mu_4 = -1 \quad \text{and} \quad \mu_1 - \mu_2 - 2\mu_3 + \mu_5 = -1\]

then \( 3\mu_1 - 6\mu_2 - 6\mu_3 \) is a lower bound
Duality

In general:

\[
\begin{align*}
\min \quad & -x - y \\
2x + y & \geq 3 \\
-2x - y & \geq -6 \\
-x - 2y & \geq -6 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

\[
\begin{align*}
\mu_1 \cdot (2x + y) & \geq 3 \\
\mu_2 \cdot (-2x - y) & \geq -6 \\
\mu_3 \cdot (-x - 2y) & \geq -6
\end{align*}
\]

\[
\begin{align*}
2\mu_1 x + \mu_1 y & \geq 3\mu_1 \\
-2\mu_2 x - \mu_2 y & \geq -6\mu_2 \\
-\mu_3 x - 2\mu_3 y & \geq -6\mu_3
\end{align*}
\]

\[
(2\mu_1 - 2\mu_2 - \mu_3) x + (\mu_1 - \mu_2 - 2\mu_3) y \geq 3\mu_1 - 6\mu_2 - 6\mu_3
\]

If \( \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \)

\[
2\mu_1 - 2\mu_2 - \mu_3 + \mu_4 \leq -1 \quad \text{and} \quad \mu_1 - \mu_2 - 2\mu_3 + \mu_5 \leq -1
\]

then \( 3\mu_1 - 6\mu_2 - 6\mu_3 \) is a lower bound
Duality

Best possible lower bound can be found by solving

\[
\begin{align*}
\text{max} & \quad 3\mu_1 - 6\mu_2 - 6\mu_3 \\
2\mu_1 - 2\mu_2 - \mu_3 & \leq -1 \\
\mu_1 - \mu_2 - 2\mu_3 & \leq -1 \\
\mu_1, \mu_2, \mu_3 & \geq 0
\end{align*}
\]
Duality

- Best possible lower bound can be found by solving

\[
\begin{align*}
\text{max} & \quad 3\mu_1 - 6\mu_2 - 6\mu_3 \\
& \quad 2\mu_1 - 2\mu_2 - \mu_3 \leq -1 \\
& \quad \mu_1 - \mu_2 - 2\mu_3 \leq -1 \\
& \quad \mu_1, \mu_2, \mu_3 \geq 0
\end{align*}
\]

- Best solution is given by \((\mu_1, \mu_2, \mu_3) = (0, \frac{1}{3}, \frac{1}{3})\)

\[
\begin{align*}
0 \cdot ( & \quad 2x + y \geq 3) \\
\frac{1}{3} \cdot ( & \quad -2x - y \geq -6) \\
\frac{1}{3} \cdot ( & \quad -x - 2y \geq -6)
\end{align*}
\]

Matches with optimum!

\[
-x - y \geq -4
\]
Dual Problem

- Given a LP (called **primal**)

\[
\begin{align*}
\text{min} & \quad c^T x \\
Ax & \geq b \\
x & \geq 0
\end{align*}
\]

its **dual** is the LP

\[
\begin{align*}
\text{max} & \quad b^T y \\
A^T y & \leq c \\
y & \geq 0
\end{align*}
\]

- Primal variables associated with columns of \( A \)
- Dual variables (**multipliers**) associated with rows of \( A \)
- Objective and right-hand side vectors swap their roles
Dual Problem

- **Prop.** The dual of the dual is the primal.

Proof:

\[
\begin{align*}
\max & \quad b^T y \\
A^T y & \leq c \\
y & \geq 0
\end{align*}
\implies
\begin{align*}
\min & \quad (-b)^T y \\
-A^T y & \geq -c \\
y & \geq 0
\end{align*}
\]

\[
\begin{align*}
\max & \quad -c^T x \\
(-A^T)^T x & \leq -b \\
x & \geq 0
\end{align*}
\implies
\begin{align*}
\min & \quad c^T x \\
Ax & \geq b \\
x & \geq 0
\end{align*}
\]

- We say the primal and the dual form a **primal-dual pair**
Dual Problem

Prop. \( \begin{align*}
\min c^T x \\
Ax = b \\
x \geq 0
\end{align*} \) and \( \begin{align*}
\max b^T y \\
A^T y \leq c
\end{align*} \) form a primal-dual pair

Proof:

\( \begin{align*}
\min c^T x \\
Ax = b \\
x \geq 0 \\
\Rightarrow \\
\min c^T x \\
Ax = b \\
x \geq 0
\end{align*} \)

\( \begin{align*}
\max b^T y_1 - b^T y_2 \\
A^T y_1 - A^T y_2 \leq c \\
y_1, y_2 \geq 0 \\
y := y_1 - y_2 \\
\Rightarrow \\
\max b^T y \\
A^T y \leq c
\end{align*} \)
Duality Theorems

- Th. (Weak Duality) Let \((P, D)\) be a primal-dual pair

\[
\begin{align*}
(P) & \quad \min \ c^T x \\
& \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\quad \text{and} \quad
\begin{align*}
(D) & \quad \max \ b^T y \\
& \quad A^T y \leq c
\end{align*}
\]

If \(x\) is feasible solution to \(P\) and \(y\) is feasible solution to \(D\) then \(b^T y \leq c^T x\)

Proof:

\[
b^T y = y^T b = y^T Ax \leq c^T x
\]

as \(c - A^T y \geq 0\), i.e., \(c^T - y^T A \geq 0\), and \(x \geq 0\) imply \(c^T x - y^T Ax \geq 0\).
Duality Theorems

- Feasible solutions to $D$ give lower bounds on $P$
- Feasible solutions to $P$ give upper bounds on $D$
- Can the two optimum values ever be equal? If so, are they always equal?
Duality Theorems

- Feasible solutions to $D$ give lower bounds on $P$
- Feasible solutions to $P$ give upper bounds on $D$
- Can the two optimum values ever be equal? If so, are they always equal?

Th. (Strong Duality) Let $(P, D)$ be a primal-dual pair

\[
\begin{align*}
(P) \quad \min & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
(D) \quad \max & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c
\end{align*}
\]

If any of $P$ or $D$ has a feasible solution and a finite optimum then the same holds for the other problem and the two optimum values are equal.
Duality Theorems

Proof (Th. of Strong Duality):

By symmetry it is sufficient to prove only one direction. Wlog. let us assume $P$ is feasible with finite optimum.

After executing the Simplex algorithm to $P$ we find $B$ optimal feasible basis. Then:

- $c_B^T B^{-1} a_j \leq c_j$ for all $j \in R$ (optimality conds hold)
- $c_B^T B^{-1} a_j = c_j$ for all $j \in B$

So $\pi^T := c_B^T B^{-1}$ is dual feasible: $\pi^T A \leq c^T$, i.e. $A^T \pi \leq c$.
Moreover, $c_B^T \beta = c_B^T B^{-1} b = \pi^T b = b^T \pi$

By the theorem of weak duality, $\pi$ is optimum for $D$

If $B$ is an optimal feasible basis for $P$, then simplex multipliers $\pi^T := c_B^T B^{-1}$ are optimal feasible solution for $D$
Duality Theorems

**Prop.** Let $(P, D)$ be a primal-dual pair

\[
\begin{align*}
(P) & \quad \min c^T x \\
Ax &= b \\
x &\geq 0
\end{align*}
\quad \text{and} \quad
\begin{align*}
(D) & \quad \max b^T y \\
A^T y &\leq c
\end{align*}
\]

If $P$ (resp., $D$) has a feasible solution but the objective value is not bounded, then $D$ (resp., $P$) is infeasible

**Proof:**

By contradiction. If $y$ were a feasible solution to $D$, by the weak duality theorem objective of $P$ would be lower bounded!
Duality Theorems

- **Prop.** Let \((P, D)\) be a primal-dual pair

\[
\begin{align*}
\text{(P)} & \quad \min \ c^T x \\
Ax &= b \\
x &\geq 0
\end{align*}
\]

and

\[
\begin{align*}
\text{(D)} & \quad \max \ b^T y \\
A^T y &\leq c
\end{align*}
\]

If \(P\) (resp., \(D\)) has a feasible solution but the objective value is not bounded, then \(D\) (resp., \(P\)) is infeasible

- And the converse?
Duality Theorems

Prop. Let \((P, D)\) be a primal-dual pair

\[
\begin{align*}
(P) \quad & \min c^T x \\
& Ax = b \\
& x \geq 0
\end{align*}
\]

and

\[
\begin{align*}
(D) \quad & \max b^T y \\
& A^T y \leq c
\end{align*}
\]

If \(P\) (resp., \(D\)) has a feasible solution but the objective value is not bounded, then \(D\) (resp., \(P\)) is infeasible

And the converse?

\[
\begin{align*}
\min \quad & 3x_1 + 5x_2 \\
& x_1 + 2x_2 = 3 \\
& 2x_1 + 4x_2 = 1 \\
& x_1, x_2 \text{ free}
\end{align*}
\]

\[
\begin{align*}
\max \quad & 3y_1 + y_2 \\
& y_1 + 2y_2 = 3 \\
& 2y_1 + 4y_2 = 5 \\
& y_1, y_2 \text{ free}
\end{align*}
\]
# Duality Theorems

<table>
<thead>
<tr>
<th>Primal unbounded</th>
<th>$\implies$</th>
<th>Dual infeasible</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dual unbounded</td>
<td>$\implies$</td>
<td>Primal infeasible</td>
</tr>
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<td>Dual { infeasible, unbounded }</td>
</tr>
<tr>
<td>Dual infeasible</td>
<td>$\implies$</td>
<td>Primal { infeasible, unbounded }</td>
</tr>
</tbody>
</table>

- Consider a primal-dual pair of the form

\[
\begin{align*}
\min & \quad c^T x \\
\text{max} & \quad b^T y \\
Ax &= b \\
x &\geq 0 \\
A^T y + w &= c \\
w &\geq 0
\end{align*}
\]

- Karush-Kuhn-Tucker (KKT) optimality conditions are

- They are necessary and sufficient conditions for optimality of the pair of primal-dual solutions \((x, (y, w))\)

- Used, e.g., as a test of quality in LP solvers

\[ \begin{align*}
\min & \quad c^T x \\
(P) & \quad Ax = b \quad x \geq 0 \\
\max & \quad b^T y \\
(D) & \quad A^T y + w = c \quad w \geq 0
\end{align*} \]

\( (KKT) \)
- \( Ax = b \)
- \( A^T y + w = c \)
- \( x, w \geq 0 \)
- \( x^T w = 0 \)

\[ \text{Th.} \quad (x, (y, w)) \text{ is solution to } KKT \text{ iff } x \text{ optimal solution to } P \text{ and } (y, w) \text{ optimal solution to } D \]

\text{Proof:}

\[ \Rightarrow \quad \text{By } 0 = x^T w = x^T (c - A^T y) = c^T x - b^T y, \text{ and Weak Duality} \]

\[ \Leftarrow \quad x \text{ is feasible solution to } P, \ (y, w) \text{ is feasible solution to } D. \]

By Strong Duality \( x^T w = x^T (c - A^T y) = c^T x - b^T y = 0 \)

as both solutions are optimal
Relating Bases

\[
\min \ z = c^T x \\
(P) \ Ax = b \\
x \geq 0
\]

\[
\max \ Z = b^T y \\
(D) \ A^T y \leq c 
\]

\[\iff\]

\[
\max \ Z = b^T y \\
A^T y + w = c \\
w \geq 0
\]

- Let \( B \) be basis of \( P \).
- Reorder rows in \( D \) so that \( B \)-basic variables are first \( m \).
- Reorder columns in \( D \) so that the matrix is

\[
\left( \begin{array}{c|c|c}
B^T & I & 0 \\
R^T & 0 & I
\end{array} \right) \left( \begin{array}{c}
y \\
w_B \\
w_R
\end{array} \right)
\]

Submatrix of vars \( y \) and vars \( w_R \):

\[
\hat{B} = \left( \begin{array}{c|c}
B^T & 0 \\
R^T & I
\end{array} \right)
\]

- Note \( \hat{B} \) is a square matrix
Relating Bases

\( \hat{B} = (y, w_R) \) is a basis of \( D \):

\[
\hat{B} = \begin{pmatrix}
B^T & 0 \\
R^T & I
\end{pmatrix}
\]

\[
\hat{B}^{-1} = \begin{pmatrix}
B^{-T} & 0 \\
-R^T B^{-T} & I
\end{pmatrix}
\]

- Recall each variable \( w_j \) in \( D \) is associated to a variable \( x_j \) in \( P \).
- \( w_j \) is \( \hat{B} \)-basic iff \( x_j \) is not \( B \)-basic.
Dual Feasibility = Primal Optimality

- Let’s apply simplex algorithm to dual problem
- Let’s see correspondence of dual feasibility in primal LP

\[ \hat{B}^{-1}c = \begin{pmatrix} B^{-T} \\ -R^TB^{-T} \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} \begin{pmatrix} c_B \\ c_R \end{pmatrix} = \begin{pmatrix} B^{-T}c_B \\ -R^TB^{-T}c_B + c_R \end{pmatrix} \]

- There is no restriction on the sign of \( y_1, \ldots, y_m \)
- Variables \( w_j \) have to be non-negative. But

\[ -R^TB^{-T}c_B + c_R \geq 0 \text{ iff } c_R^T - c_BB^{-1}R \geq 0 \text{ iff } d_R^T \geq 0 \]

- \( \hat{B} \) is dual feasible iff \( d_j \geq 0 \) for all \( j \in R \)
- Dual feasibility is primal optimality!
Dual Optimality = Primal Feasibility

- $\hat{B}$-basic dual vars: $(y \mid w_R)$ with costs $(b^T \mid 0)$
- Non $\hat{B}$-basic dual vars: $w_B$ with costs $(0)$
- Optimality condition: reduced costs $\leq 0$ (maximization!)

$$0 \geq (0) - (b^T \mid 0) \begin{pmatrix} B^{-T} & 0 \\ -R^T B^{-T} & I \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} = (0) - (b^T B^{-T} \mid 0) \begin{pmatrix} I \\ 0 \end{pmatrix} = ( -\beta^T ) \text{ iff } \beta \geq 0$$

- For all $1 \leq p \leq m$, $w_{kp}$ is not (dual) improving iff $\beta_p \geq 0$
- Dual optimality is primal feasibility!
Improving a Non-Optimal Solution

Let $p$ ($1 \leq p \leq m$) be such that $\beta_p < 0 \iff b^T B^{-T} e_p < 0$

Let $\rho_p^T$ be the $p$-th row of $B^{-1}$, i.e., $\rho_p = B^{-T} e_p$.

So $b^T \rho_p = \beta_p$. If $w_{kp}$ takes value $t \geq 0$:

$$
\begin{pmatrix}
y(t) \\
w_R(t)
\end{pmatrix} = \hat{B}^{-1} c - \hat{B}^{-1} t e_p =
$$

$$
\begin{pmatrix}
B^{-T} c_B \\
d_R
\end{pmatrix} - \begin{pmatrix}
B^{-T} \\ -R^T B^{-T}
\end{pmatrix}
\begin{pmatrix}
0 \\
I
\end{pmatrix}
\begin{pmatrix}
te_p \\
0
\end{pmatrix} =
$$

$$
\begin{pmatrix}
B^{-T} c_B - t \rho_p \\
\frac{d_R}{d_R + t R^T \rho_p}
\end{pmatrix}
$$

Dual objective value improvement is

$$
\Delta Z = b^T y(t) - b^T y(0) = -t b^T \rho_p = -t \beta_p
$$
Improving a Non-Optimal Solution

- Only $w$ variables need to be $\geq 0$: for $j \in \mathcal{R}$

\[
\begin{align*}
  w_j(t) &= d_j + t a_j^T \rho_p = d_j + t \rho_p^T a_j = \\
  &= d_j + t e_p^T B^{-1} a_j = d_j + t e_p^T \alpha_j = d_j + t \alpha_j^p
\end{align*}
\]

\[w_j(t) \geq 0 \iff d_j + t \alpha_j^p \geq 0\]

- If $\alpha_j^p \geq 0$ the constraint is satisfied for all $t \geq 0$
- If $\alpha_j^p < 0$ we need $\frac{d_j}{-\alpha_j^p} \geq t$

- Best improvement achieved with

\[\Theta_D := \min\left\{ \frac{d_j}{-\alpha_j^p} \mid \alpha_j^p < 0 \right\}\]

- Variable $w_q$ is blocking when $\Theta_D = \frac{d_q}{-\alpha_q^p}$
Improving a Non-Optimal Solution

1. If $\Theta_D = +\infty$ (there is no $j$ such that $j \in \mathcal{R}$ and $\alpha_j^p < 0$):

   Value of dual objective can be increased infinitely.

   Dual LP is unbounded.

   Primal LP is infeasible.

2. If $\Theta_D < +\infty$ and $w_q$ is blocking:

   When setting $w_{kp} = \Theta_D$ sign of basic slack vars of dual (reduced costs of non-basic vars of primal) is respected

   In particular, $w_q(\Theta_D) = d_q + \Theta_D \alpha_q^p = d_q + (\frac{d_q}{\alpha_q^p})\alpha_q^p = 0$

   We can make a basis change:
   - In dual: $w_{kp}$ enters $\hat{B}$ and $w_q$ leaves
   - In primal: $x_{kp}$ leaves $B$ and $x_q$ enters
Update

- We forget about dual LP and work only with primal LP

- New basic indices: \( \bar{B} = B - \{k_p\} \cup \{q\} \)

- New dual objective value: \( \bar{Z} = Z - \Theta_D \beta_p \)

- New dual basic sol: \( \bar{y} = y - \Theta_D \rho_p \)
  \( \bar{d}_j = d_j + \Theta_D \alpha_j^p \) if \( j \in \mathcal{R} \), \( \bar{d}_{kp} = \Theta_D \)

- New primal basic sol: \( \bar{\beta}_p = \Theta_P \), \( \bar{\beta}_i = \beta_i - \Theta_P \alpha_i^q \) if \( i \neq p \)
  where \( \Theta_P = \frac{\beta_p}{\alpha_q^p} \)

- New basis inverse: \( \bar{B}^{-1} = EB^{-1} \)
  where \( E = (e_1, \ldots, e_{p-1}, \eta, e_{p+1}, \ldots, e_m) \) and
  \( \eta^T = \left( \left( \frac{-\alpha_1^p}{\alpha_q^p} \right), \ldots, \left( \frac{-\alpha_{p-1}^p}{\alpha_q^p} \right), \frac{1}{\alpha_q^p} \left( \frac{-\alpha_{p+1}^p}{\alpha_q^p} \right), \ldots, \left( \frac{-\alpha_m^p}{\alpha_q^p} \right) \right)^T \)
Algorithmic Description

1. **Initialization:** Find an initial dual feasible basis \( \mathcal{B} \)
   
   Compute \( B^{-1}, \beta = B^{-1}b \), 
   
   \[ y^T = c_B^T B^{-1}, \ d^T_R = c_R^T - y^T R, \ Z = b^T y \]

2. **Dual Pricing:**
   
   If for all \( i \in \mathcal{B}, \beta_i \geq 0 \) then return **OPTIMAL**
   
   Else let \( p \) be such that \( \beta_p < 0 \).
   
   Compute \( \rho_p^T = c_p^T B^{-1} \) and \( \alpha_j^p = \rho_p^T a_j \) for \( j \in \mathcal{R} \)

3. **Dual Ratio test:** Compute \( \mathcal{J} = \{ j \mid j \in \mathcal{R}, \alpha_j^p < 0 \} \).
   
   If \( \mathcal{J} = \emptyset \) then return **INFEASIBLE**
   
   Else compute \( \Theta_D = \min_{j \in \mathcal{J}} \left( \frac{d_j}{-\alpha_j^p} \right) \) and \( q \) st. \( \Theta_D = \frac{d_q}{-\alpha_q^p} \)
Algorithmic Description

4. Update:

\[
\bar{B} = B - \{k_p\} \cup \{q\} \\
\bar{Z} = Z - \Theta_D \beta_p
\]

Dual solution

\[
\bar{y} = y - \Theta_D \rho_p \\
\bar{d}_j = d_j + \Theta_D \alpha^p_j \text{ if } j \in \mathcal{R}, \quad \bar{d}_{k_p} = \Theta_D
\]

Primal solution

Compute \( \alpha_q = B^{-1} a_q \) and \( \Theta_P = \frac{\beta_p}{\alpha_q} \)

\[
\bar{\beta}_p = \Theta_P, \quad \bar{\beta}_i = \beta_i - \Theta_P \alpha^i_q \text{ if } i \neq p
\]

\( \bar{B}^{-1} = E B^{-1} \)

Go to 2.
## Primal vs. Dual Simplex

<table>
<thead>
<tr>
<th>PRIMAL</th>
<th>DUAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio test: $O(m)$ divs</td>
<td>Ratio test: $O(n - m)$ divs</td>
</tr>
<tr>
<td>Can handle bounds efficiently</td>
<td>Can handle bounds efficiently (not explained here)</td>
</tr>
<tr>
<td>Many years of research and implementation</td>
<td>Developments in the 90’s made it an alternative</td>
</tr>
<tr>
<td>There are classes of LP’s for which it is the best</td>
<td>Nowadays on average it gives better performance</td>
</tr>
<tr>
<td>Not suitable for solving LP’s with integer variables</td>
<td><strong>Suitable</strong> for solving LP’s with <strong>integer</strong> variables</td>
</tr>
</tbody>
</table>