Universitat Politècnica de Catalunya
Departament de Llenguatges i Sistemes Informàtics

PhD Thesis

Geometric Constraint Solving in 2D

November, 1998

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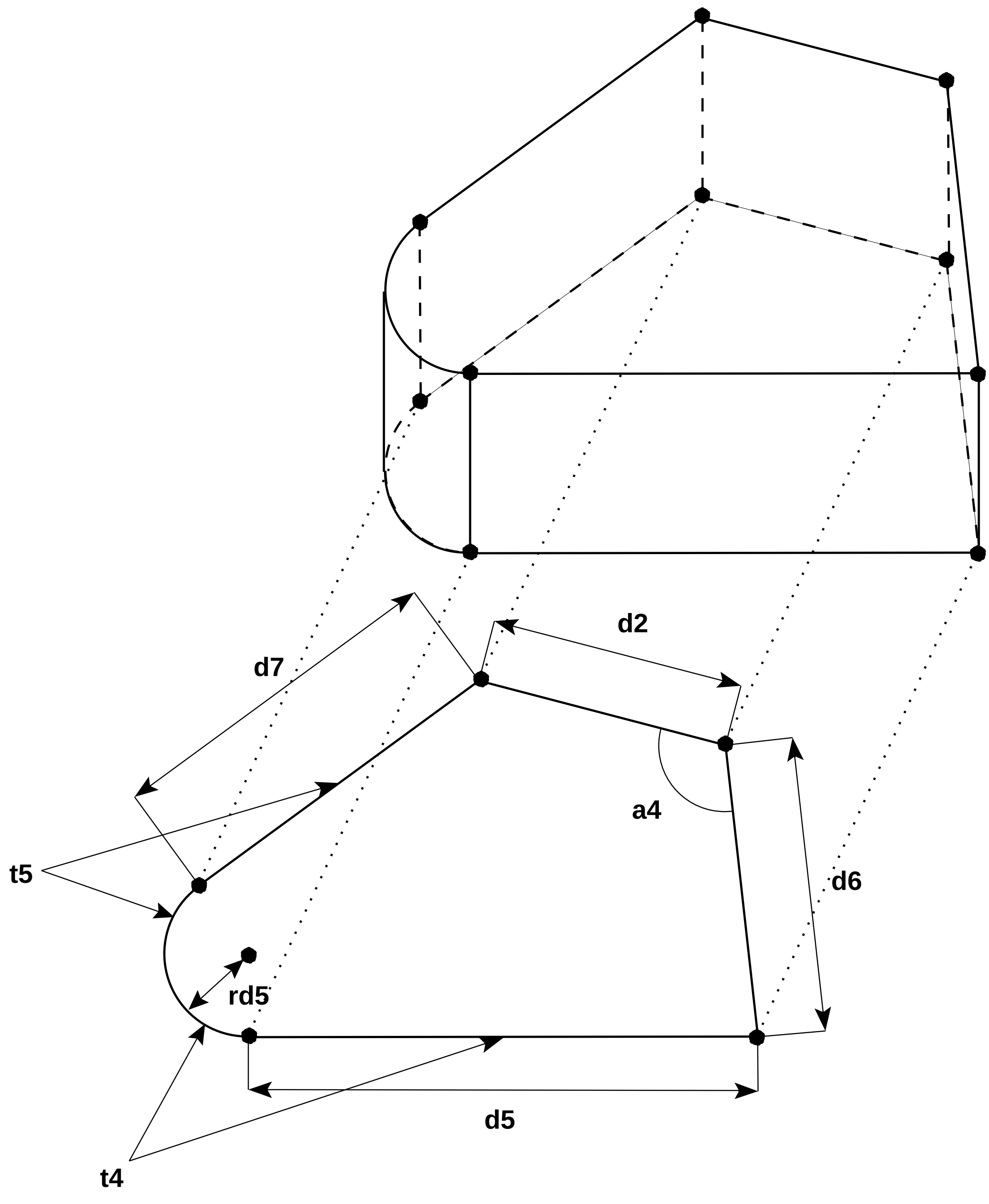
Contents

Motivation



Feature-based CAD systems

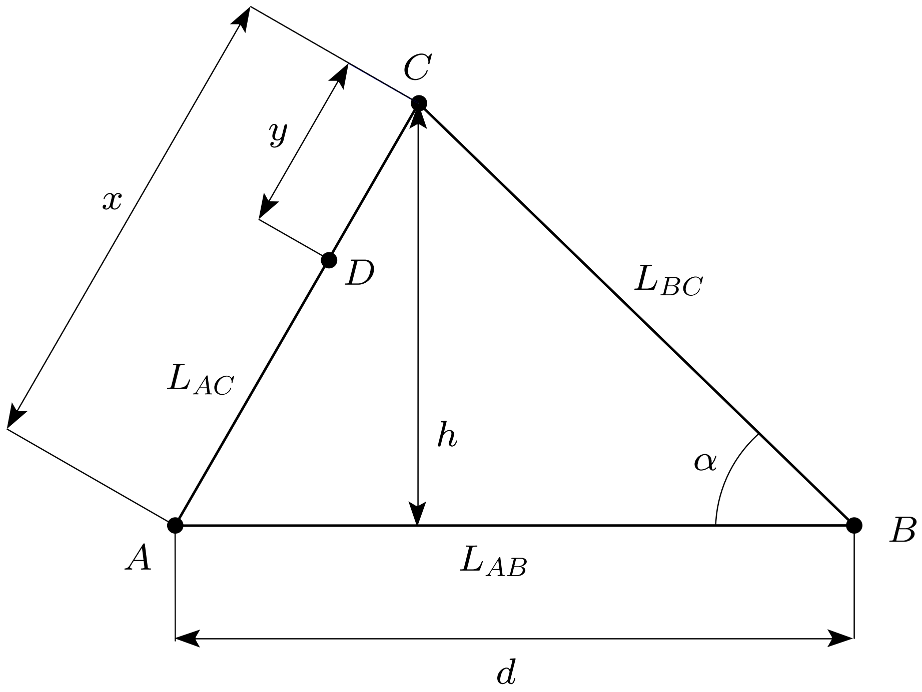
$+$

Editable representation (*Erep*)

$\Rightarrow$

2-dimensional constraint-based editor

Geometric Constraint Problem



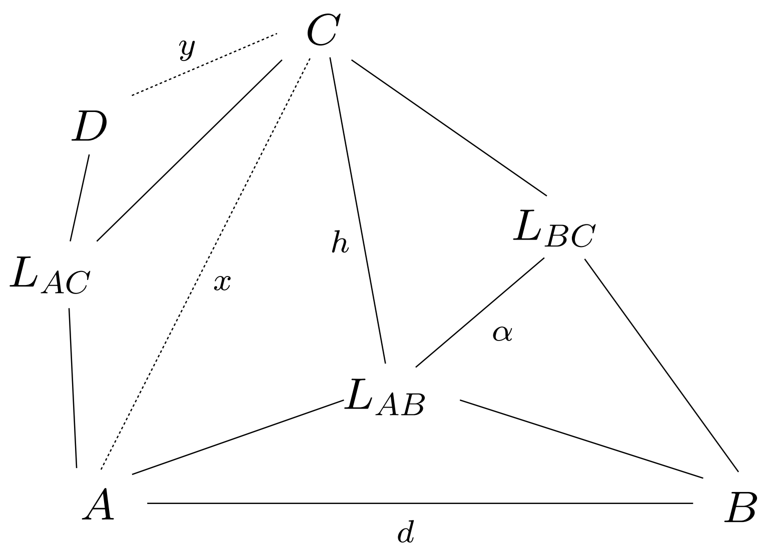
$$y = x \cdot v$$
$$v = 0.5$$

A *geometric constraint problem* (GCP) consists of

- A set of geometric elements $\{A, \cdots, L_{AB}, \cdots\}$.
- A set of values $\{d, \alpha, h\}$
- A set of dimensional variables $\{x, y\}$.
- A set of external variables $\{v\}$.
- A set of valuated and symbolic constraints.
- A set of equations.

Representation: Geometric Constraint Graph

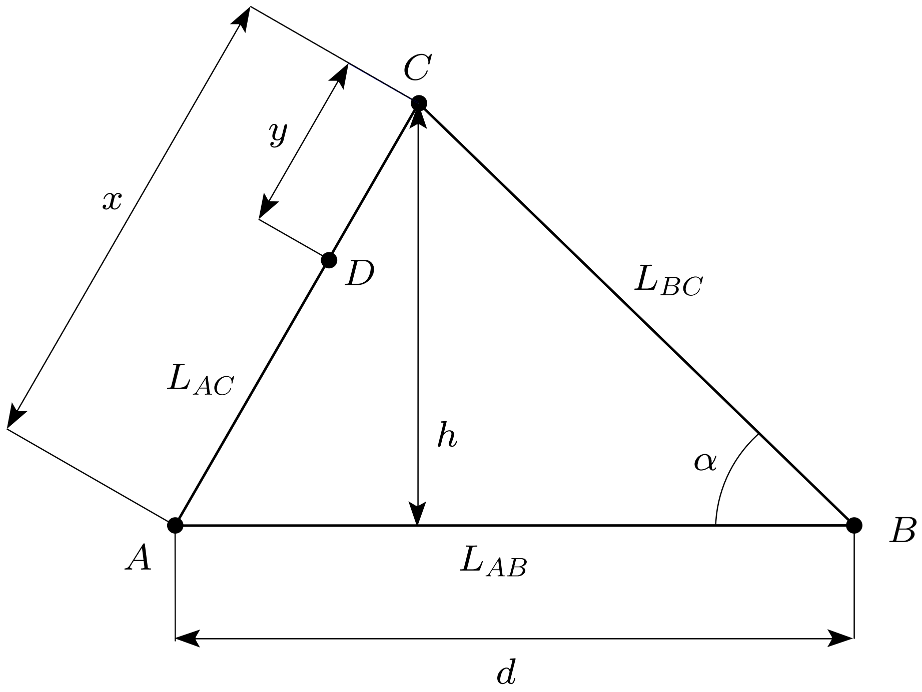
The geometric constraints of a GCP can be represented by a constraint graph $G = (E, V)$.



The *vertices* in V are two-dimensional geometric elements with two degrees of freedom.

The *edges* in E are constraints that reduce by one the degrees of freedom.

Representation: First-order Logic



$$
\begin{aligned}
y &= x \cdot v \\
v &= 0.5
\end{aligned}
$$

A geometric constraint problem can be represented by a formula in first-order logic.

$$
\begin{aligned}
&\varphi(A, B, C, D, L_{AB}, L_{AC}, L_{BC}, x, y, v) \\
&\quad \equiv \ \mathsf{d}(A, B) = d \wedge \mathsf{on}(A, L_{AB}) \wedge \mathsf{on}(B, L_{AB}) \wedge \\
&\qquad \mathsf{on}(A, L_{AC}) \wedge \mathsf{on}(C, L_{AC}) \wedge \mathsf{on}(D, L_{AC}) \wedge \\
&\qquad \mathsf{on}(B, L_{BC}) \wedge \mathsf{on}(C, L_{BC}) \wedge \\
&\qquad \mathsf{h}(C, L_{AB}) = h \wedge \mathsf{a}(L_{AB}, L_{BC}) = \alpha \wedge \\
&\qquad \mathsf{d}(A, C) = x \wedge \mathsf{d}(C, D) = y \wedge \\
&\qquad y = x \cdot v \wedge v = 0.5
\end{aligned}
$$

Geometric Constraint Solving

Geometric constraint solving (GCS) consists in proving the truth of the formula

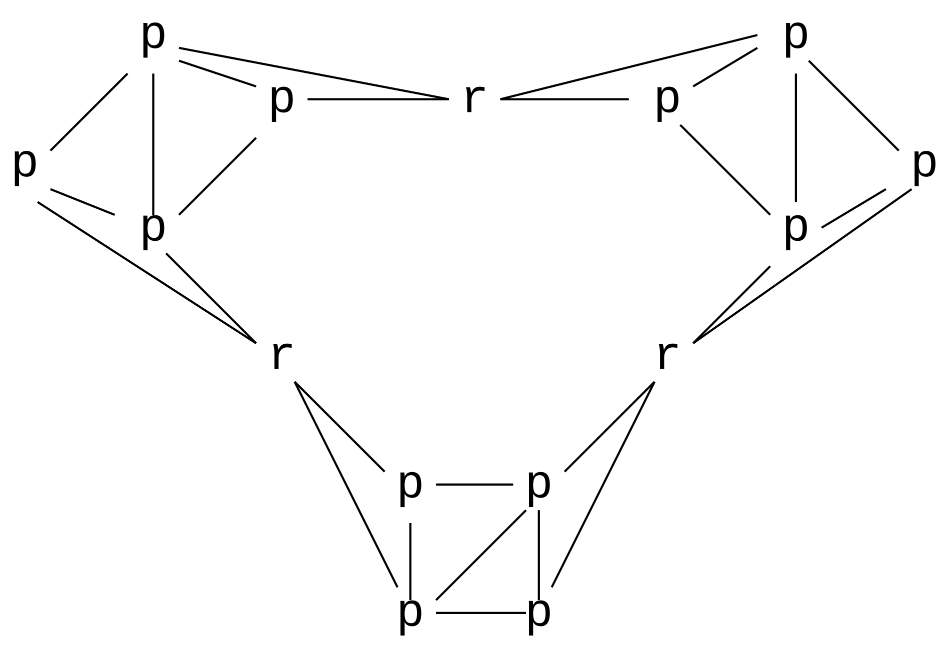
$$\exists A \, \exists B \, \exists C \, \exists D \, \exists L_{AB} \, \exists L_{AC} \, \exists L_{BC} \, \exists x \, \exists y \, \exists v \\ \varphi(A, B, C, D, L_{AB}, L_{AC}, L_{BC}, x, y, v)$$

by finding the position of the geometric elements and the values of tags and external variables that satisfy the constraints.

Over-constrained Geometric Constraint Problem

Theorem 1 (Laman, 1970)
Let $G = (P, D)$ a geometric constraint graph where the vertices in P are points in the two dimensional Euclidean space and the edges $D \subseteq P \times P$ are distance constraints. G is generically well constrained if and only if for all $G' = (P', D')$, subgraph of G induced by $P' \subseteq P$,

1. $|D'| \leq 2\,|P'| - 3$, and
2. $|D| = 2\,|P| - 3$.



Structurally over-constrained Geometric Constraint Problem

Definition 1 A geometric constraint graph is structurally over-constrained if and only if exists an induced subgraph with n vertices and m edges such that $m > 2 \cdot n - 3$.

Definition 2 A geometric constraint graph is structurally well-constrained if and only if it is not structurally over-constrained and $|E| = 2 \cdot |V| - 3$.

Definition 3 A geometric constraint graph is structurally under-constrained if and only if it is not structurally over-constrained and $|E| \leq 2 \cdot |V| - 3$.

Approaches to Geometric Constraint Solving

- Solving systems of equations
  - Numerical Constraint Solvers
  - Symbolic Constraint Solvers
  - Propagation Methods
  - Structural analysis
- Constructive Constraint Solvers
  - Graph based
  - Rule based
- Degrees of freedom analysis
- Geometric theorem proving

Constructive technique: Ruler-and-compass constructibility

A point P is *constructible* if there exists a finite sequence $P_0, P_1, \ldots, P_n = P$ of points in the plane with the following property. Let $S_j = \{P_0, P_1, \ldots, P_j\}$, for $1 \leq j \leq n$.

For each $2 \leq j \leq n$ is either

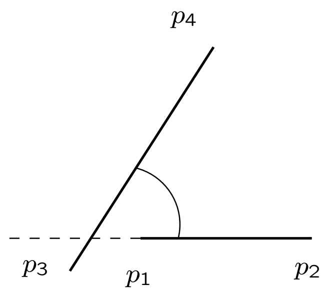
1. the intersection of two distinct straight lines, each joining two points of S_{j-1}, or

2. a point of intersection of a straight line joining two points of S_{j-1} and a circle with centre a point of S_{j-1} and radius the distance between two points of S_{j-1}, or

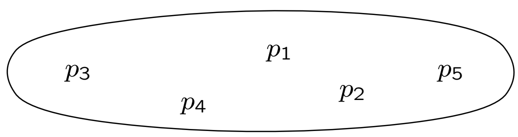
3. a point of intersection of two distinct circles, each with centre a point of S_{j-1} and radius the distance between two points of S_{j-1}.

Constructive technique: Constraints sets

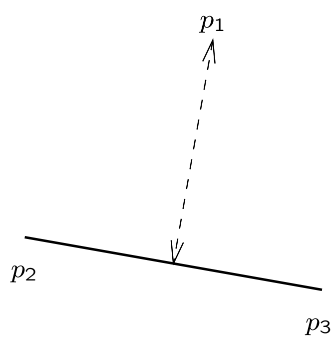
- A **CA** set is a pair of oriented segments which are mutually constrained in angle.



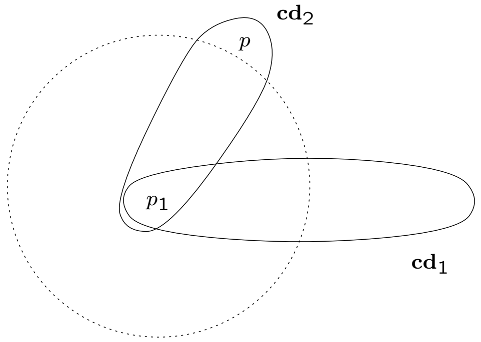
- A **CD** set is a set of points with mutually constrained distances.



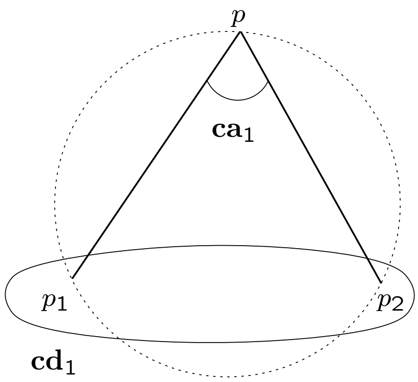
- A **CH** set is a point and a segment constrained by the perpendicular distance from the point to the segment.



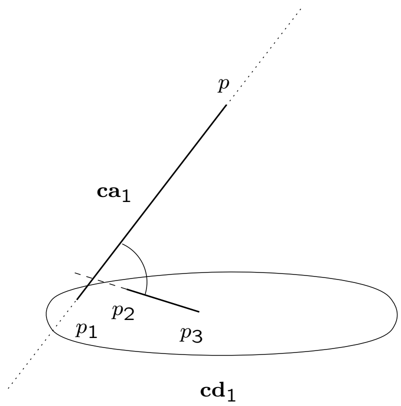
Constructive technique:
Geometric locus



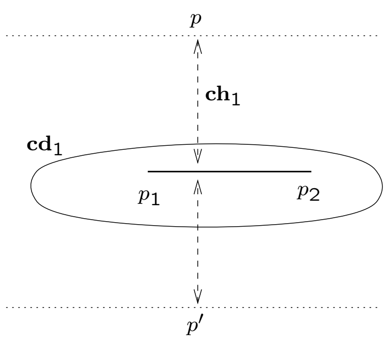
CR



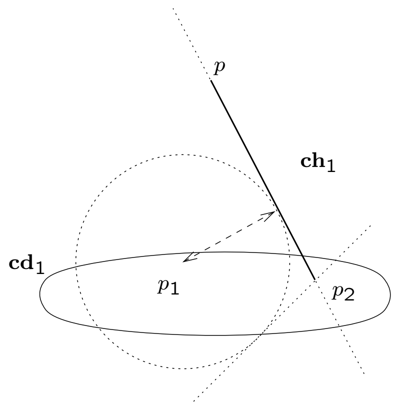
CC



RA

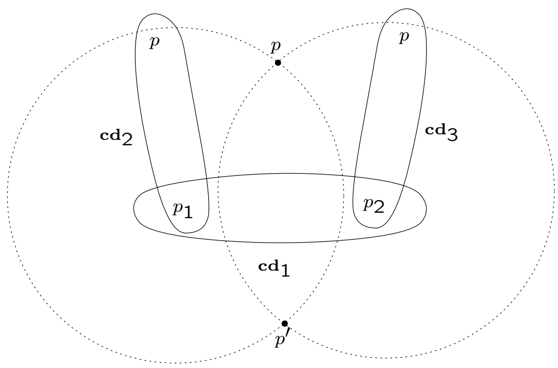


RP

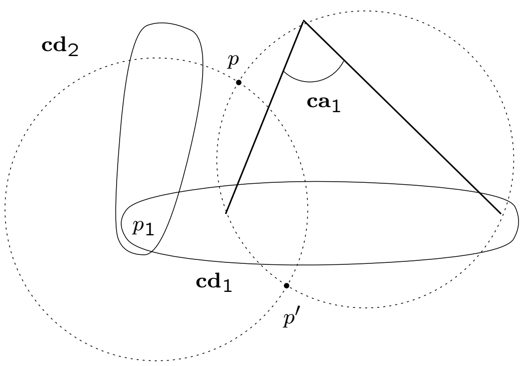


RT

Constructive technique: Set of rules



CR-CR



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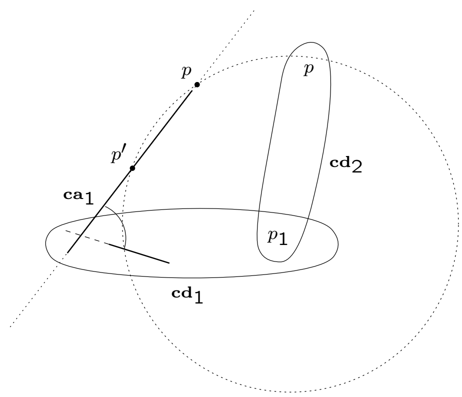
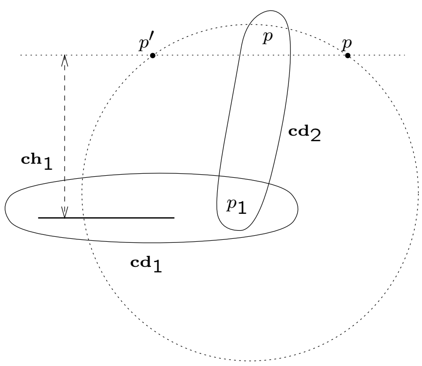
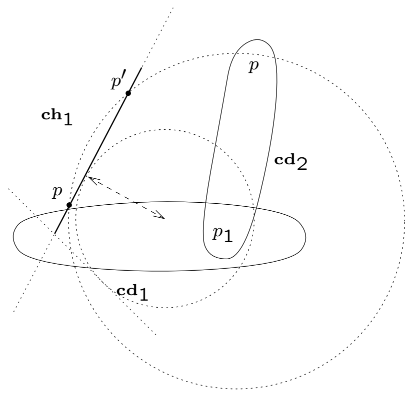


CR-RA



CR-RP

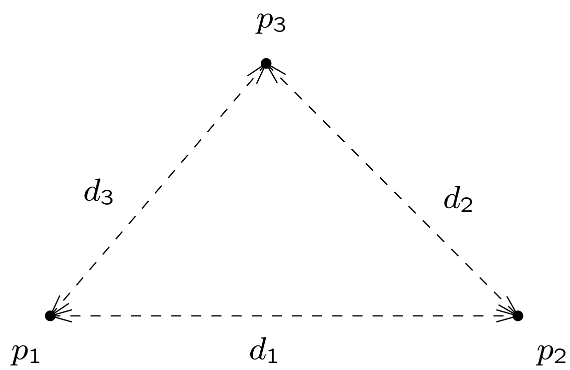


CR-RT

Constructive technique: Analysis and Construction phases

$\varphi(p_1, p_2, p_3) \equiv$ Geometric Constraint Problem

$$\mathsf{d}(p_1, p_2) = d_1 \wedge$$
$$\mathsf{d}(p_2, p_3) = d_2 \wedge$$
$$\mathsf{d}(p_3, p_1) = d_3$$



analysis

$\psi(p_1, p_2, p_3) \equiv$ Constructive Formula

$$p_1 = (0, 0) \wedge$$
$$p_2 = (d_1, 0) \wedge$$
$$p_3 = inter(circle(p_1, d_3), circle(p_2, d_2))$$

construction

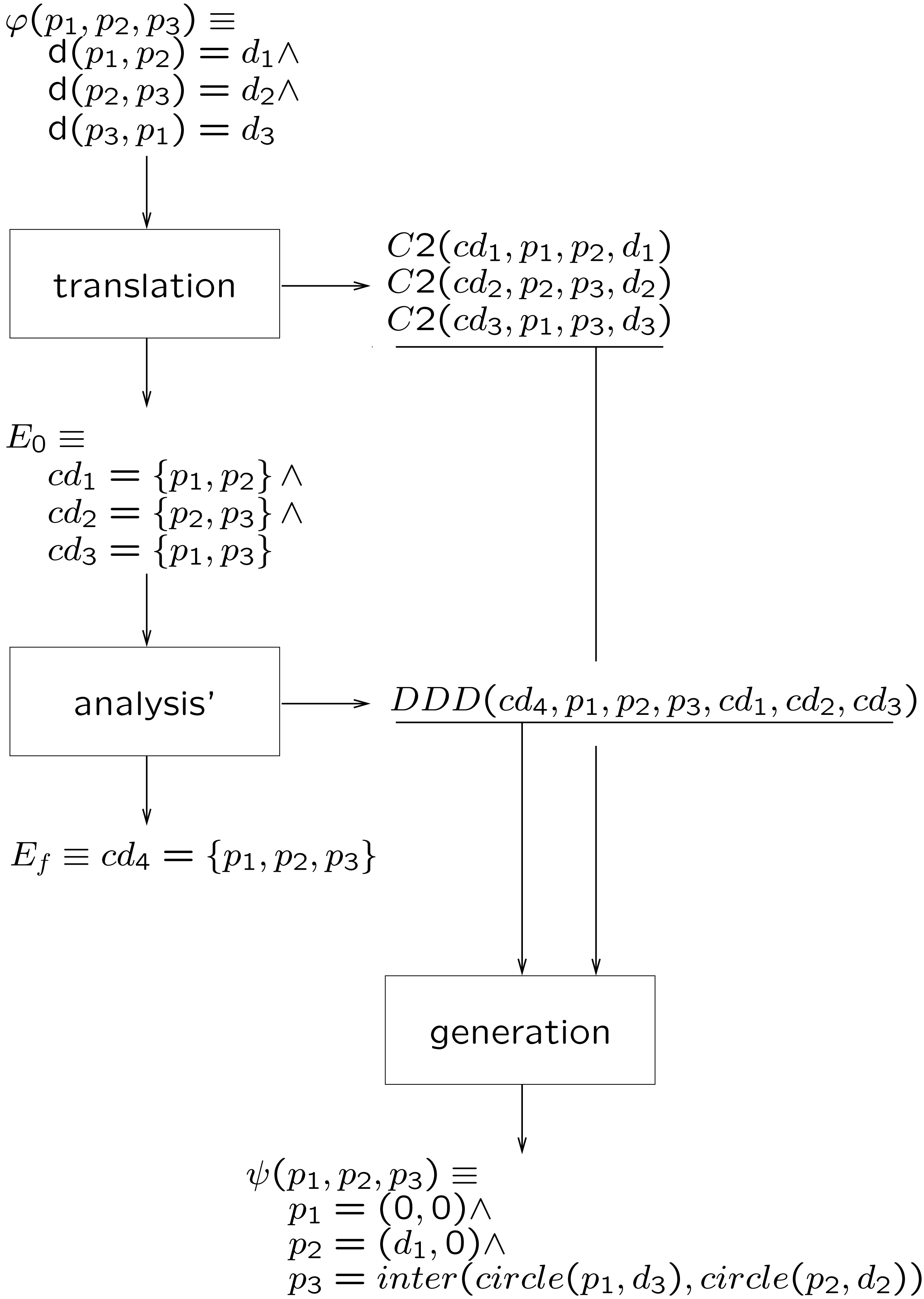
$\upsilon \equiv$ Values

$$d_1 = 200 \wedge$$
$$d_2 = 180 \wedge$$
$$d_3 = 220$$

$\psi_\upsilon(p_1, p_2, p_3) \equiv$ Valuated Formula

$$p_1 = (0, 0) \wedge$$
$$p_2 = (200, 0) \wedge$$
$$p_3 = (140, 169.706)$$

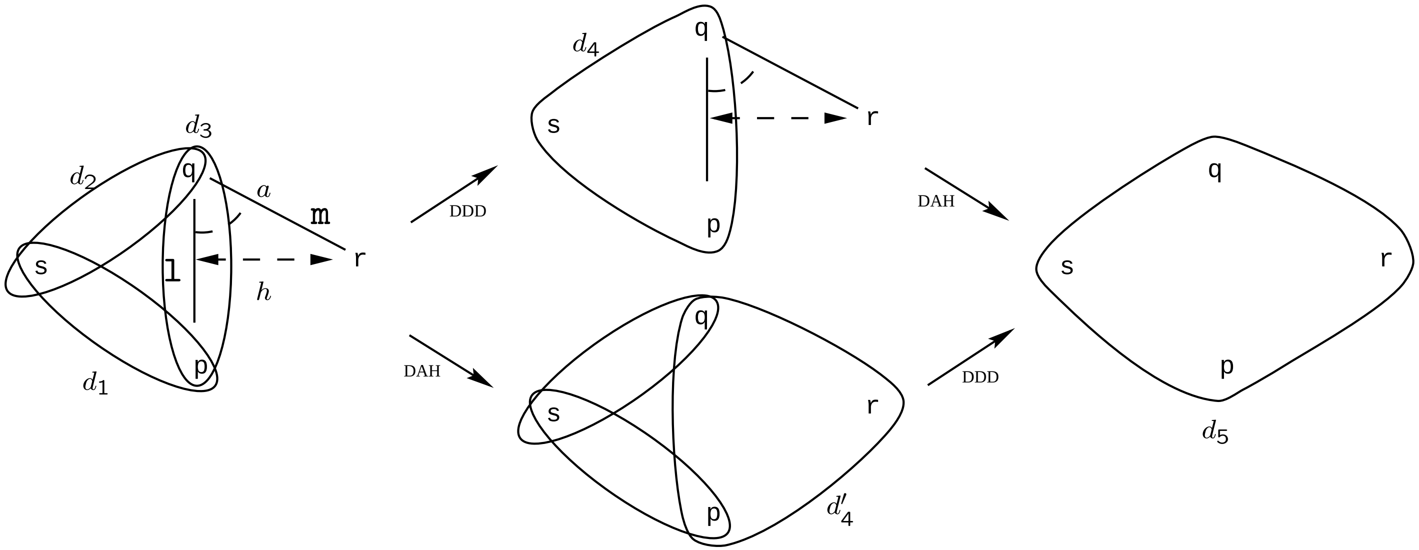
Constructive technique: Structure of the Analysis phase



Constructive technique: Correctness (1)

- Let R be a set of tuples (D, X) where
- D is a set of **CD** sets, and
- X is a set of **CA** sets and **CH** sets.
- Let $\longrightarrow_{\rho} = \longrightarrow_{\mathsf{DDD}} \cup \longrightarrow_{\mathsf{DDX}} \cup \longrightarrow_{\mathsf{DXX}}$ be a *reduction relation*.
- We define the *abstract reduction system* $\mathcal{R} = \langle R, \longrightarrow_{\rho} \rangle$.
- We proof *termination* and *confluence* that implies *canonicity* and *unique normal form property*.

Constructive technique: Correctness (2)



Constructive technique: Reduction rules (1)

$$(D, X) \longrightarrow_{\mathsf{DDD}} ((D - \{d_1, d_2, d_3\}) \cup \{d_1 \cup d_2 \cup d_3\}, X)$$

$$\text{if} \begin{cases} \{d_1, d_2, d_3\} \subseteq D \wedge \\ d_1 \cap d_2 = \{p1\} \wedge \\ d_2 \cap d_3 = \{p2\} \wedge \\ d_1 \cap d_3 = \{p3\} \wedge \\ p_1 \neq p_2 \neq p_3 \end{cases}$$

Constructive technique: Reduction rules (2)

$$(D, X) \quad \longrightarrow_{\mathsf{DDX}} \quad ((D - \{d_1, d_2\}) \cup \{d_1 \cup d_2 \cup \mathsf{punts}(x_1)\}, X - \{x_1\})$$

$$\text{if} \begin{cases} \{d_1, d_2\} \subseteq D \wedge \\ d_1 \cap d_2 = \{p_1\} \wedge \\ p \in d_2 \wedge \\ x_1 \in X \wedge \\ \mathsf{punts}(x_1) - d_1 = \{p\} \wedge \\ p \neq p_1 \end{cases}$$

Constructive technique: Reduction rules (3)

$$(D, X) \longrightarrow_{\mathsf{DXX}} ((D - \{d_1\}) \cup \{d_1 \cup \mathsf{punts}(x_1) \cup \mathsf{punts}(x_2)\}, X - \{x_1, x_2\})$$

$$\text{if } \begin{cases} d_1 \in D \wedge \\ \{x_1, x_2\} \subseteq X \wedge \\ \mathsf{punts}(x_1) - d_1 = \{p\} \wedge \\ \mathsf{punts}(x_2) - d_1 = \{p\} \end{cases}$$

Hybrid technique: Introduction

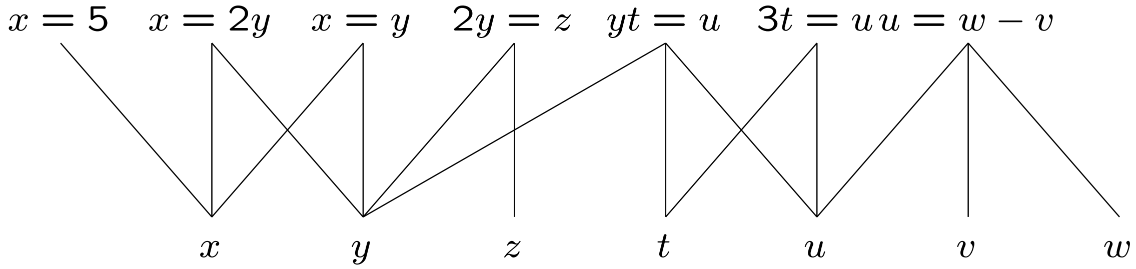
Goal Solve symbolic constraints keeping the two phases of the constructive technique for valuated constraints.

Idea Federate a constructive solver and an equational solver.

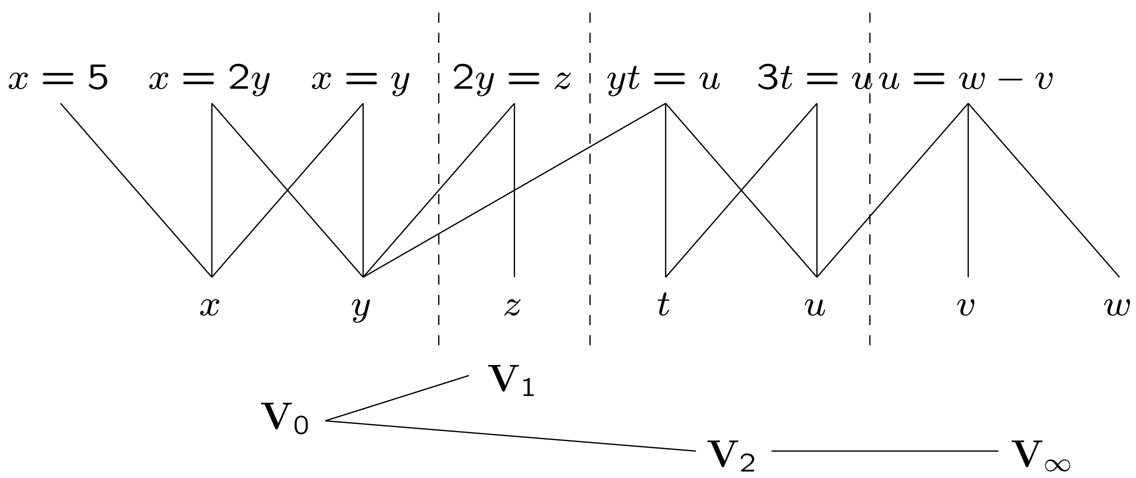
Required A technique of analysis of systems of equations.

Hybrid technique: Analysis of systems of equations

1. Represent the structure of the systems of equations by a *bipartite graph* (bigraph).

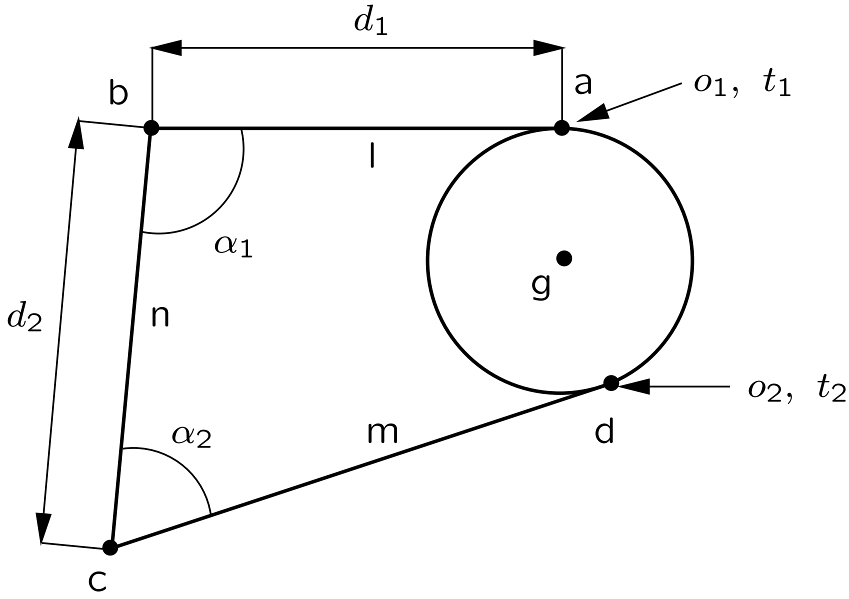


2. Compute the Dulmage-Mendelsohn decomposition of the bigraph. V_0 is the over-determined part, V_∞ is the under-determined part. $V_1, \cdots, V_n$ are the consistent part.



Hybrid technique: Motivation (1)

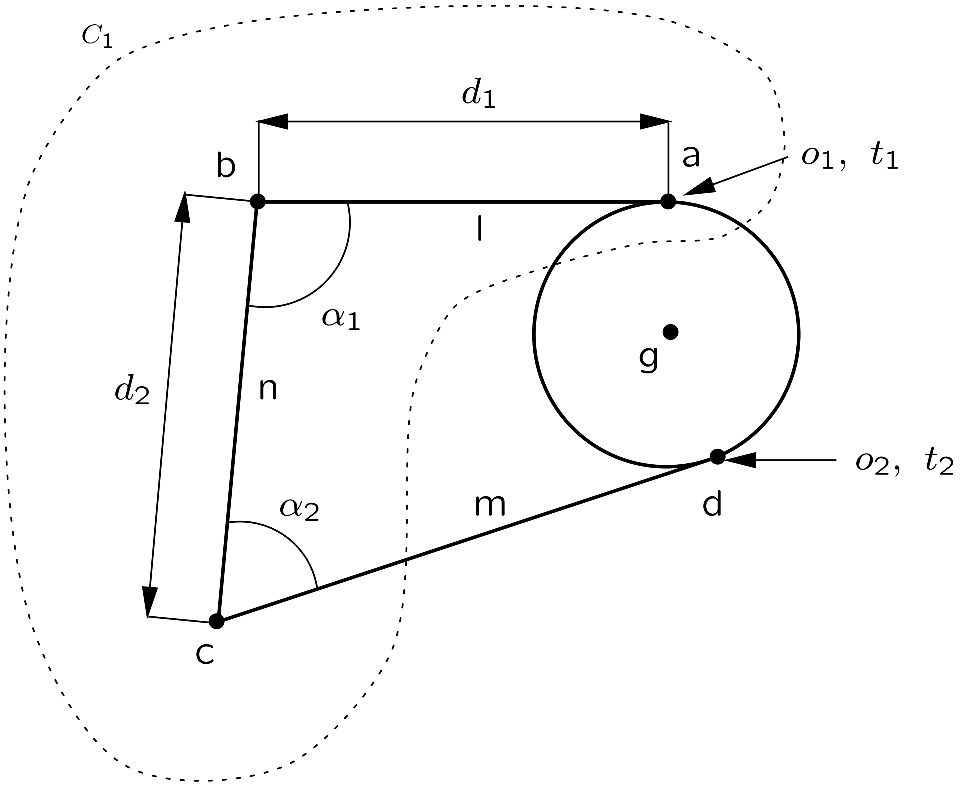
A geometric constraint problem that can not be solved without considering geometric variables.



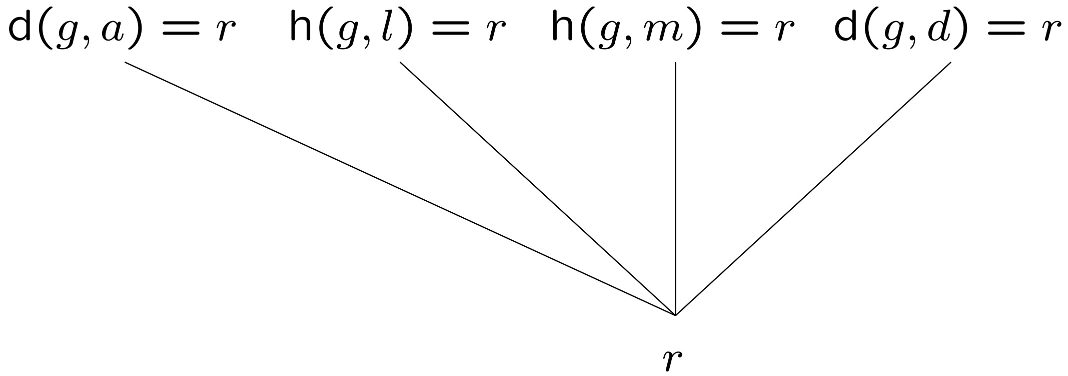
$$
\begin{aligned}
&\varphi(a,b,c,d,g,l,m,n,r)\\
&\quad\equiv\ \mathsf{d}(a,b)=d_1 \wedge \mathsf{d}(b,c)=d_2 \wedge\\
&\qquad \mathsf{on}(a,l)\wedge \mathsf{on}(b,l)\wedge \mathsf{on}(b,n)\wedge\\
&\qquad \mathsf{on}(c,n)\wedge \mathsf{on}(c,m)\wedge \mathsf{on}(d,m)\wedge\\
&\qquad \mathsf{a}(l,n)=\alpha_1 \wedge \mathsf{a}(l,m)=\alpha_2 \wedge\\
&\qquad \mathsf{d}(g,c)=r \wedge \mathsf{h}(g,l)=r \wedge\\
&\qquad \mathsf{h}(g,m)=r \wedge \mathsf{d}(g,d)=r
\end{aligned}
$$

Hybrid technique: Motivation (2)

Final state of the geometric analyzer.

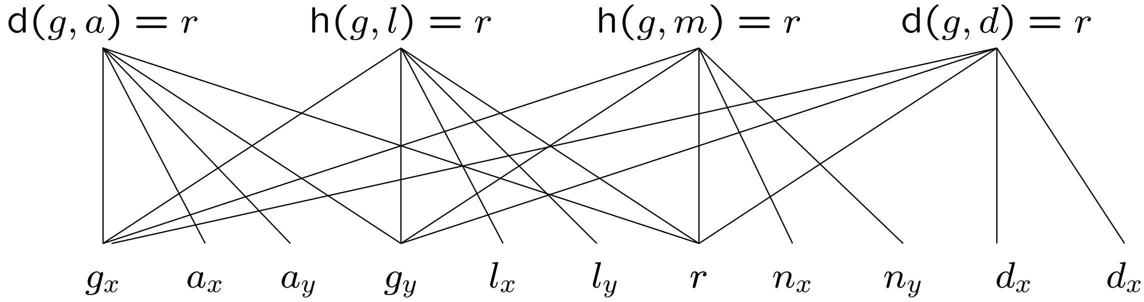


Final state of the equational analyzer.

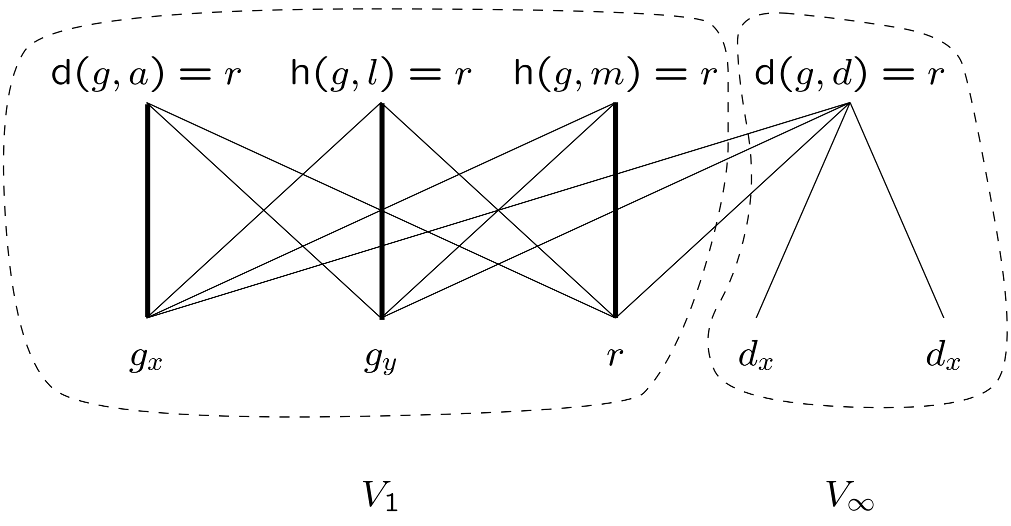


Hybrid technique: Technique (1)

1. Represent geometric variables in the bigraph (B).



2. Compute $R(B, C_1)$, the *restriction* of bigraph B by CD set C_1. The equations are analyzed with respect to a coordinate system (CD set).

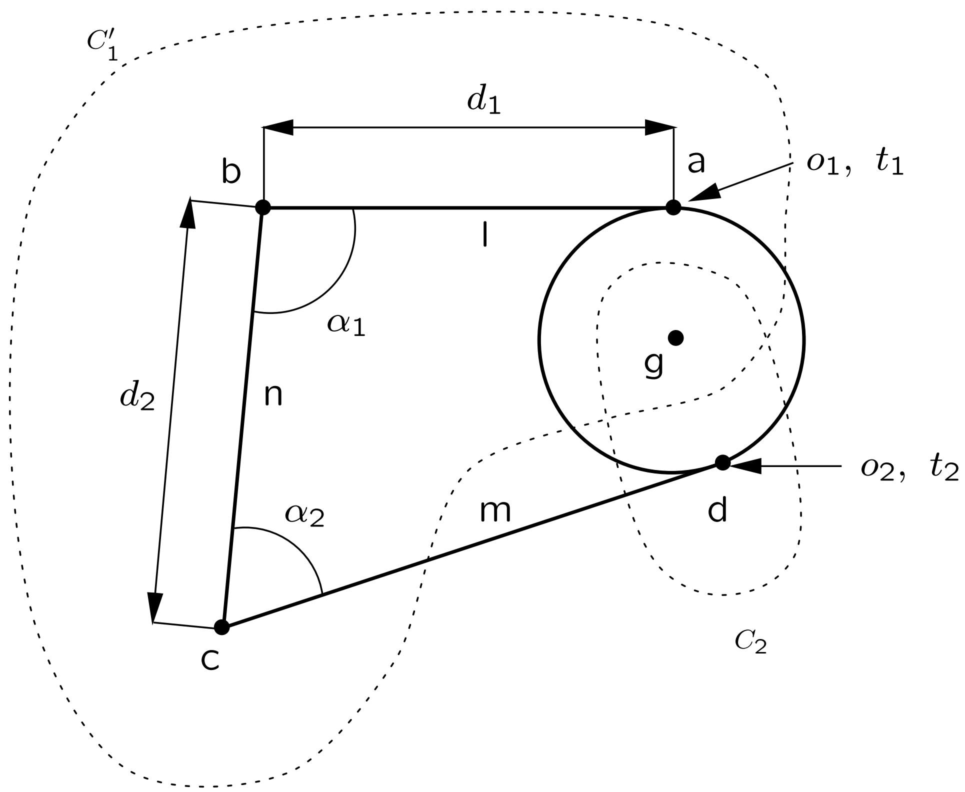


Hybrid technique: Technique (2)

3. For each solved dimensional variable, add a new constraint set to the state of the geometric analyzer.

4. For each pair of solved geometric variables (v_x, v_y), add the geometric element v to the projection CD set C_1.

5. Remove solved variables and equations from the bigraph B.



Hybrid technique: Correctness (1)

- Let S be a set of tuples (D, X, B) where
- D is a set of **CD** sets,
- X is a set of **CA** sets and **CH** sets, and
- B is a bigraf representing symbolic geometric constraints and equations.
- Let $\longrightarrow_{\rho'}$ be the constructive *reduction relation*.
- Let $\longrightarrow_{\kappa}$ be the equational analysis *reduction relation*.
- We define the *abstract reduction system* $\mathcal{S} = \langle S, \longrightarrow_{\rho'} \cup \longrightarrow_{\kappa} \rangle$.
- We proof *termination* and *confluence* that implies *canonicity* and *unique normal form property*.

Conclusions

- A correct ruler-and-compass constructive method.

- A clean phase structure.

- A correct hybrid method combining a constructive method and an equation analysis method.

- A prototype implementation.

Future work

- Study the domain of constructive methods.
- Extending the domain of our constructive method.
- Selection of the solution.
- Determine the range of values of a constraint.