

Splitting on Demand in SAT Modulo Theories

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Abstract. *Lazy* algorithms for *Satisfiability Modulo Theories* (SMT) combine a generic DPLL-based SAT *engine* with a *theory solver* for the given theory T that can decide the T -consistency of *conjunctions* of ground literals. For many theories of interest, theory solvers need to reason by performing internal case splits. Here we argue that it is more convenient to delegate these case splits to the DPLL engine instead. This can be done on demand for solvers that can encode their internal case splits into one or more clauses, possibly including new constants and literals. It results in drastically simpler theory solvers, and can lead, we believe, to more efficient overall systems. We present this in an improved version of DPLL(T), a general SMT architecture for the lazy approach, and formalize and prove it correct in an extension of *Abstract DPLL Modulo Theories*, a framework for modeling and reasoning about lazy algorithms for SMT. A remarkable additional feature of the architecture, also discussed in the paper, is that it naturally includes an efficient Nelson-Oppen-like combination of multiple theories and their solvers.

1 Introduction

The performance of propositional SAT solvers based on the Davis-Putnam-Logemann-Loveland (DPLL) procedure [8, 7] has importantly improved during the last years, and DPLL-based solvers are becoming the tool of choice for attacking more and more practical problems. The DPLL procedure has also been adapted for handling problems in more expressive logics, and, in particular, for the *SAT Modulo Theories* (SMT) problem: deciding the satisfiability of ground first-order formulas with respect to background theories such as the integer or real numbers, or arrays. SMT problems frequently arise in formal hardware and software verification applications, where typical formulas consist of very large sets of clauses like:

$$p \vee \neg q \vee a=f(b-c) \vee read(s, f(b-c))=d \vee a-g(c) \leq 7$$

with propositional atoms as well as atoms over (combined) theories like the integers, arrays, or Equality with Uninterpreted Functions (EUF). SMT has become a very active area of research, and efficient SMT solvers exist that can handle (combinations of) many such theories T . Currently most SMT solvers follow the so-called *lazy* approach to SMT, combining (i) *theory solvers* that can

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handle *conjunctions* of literals over the given theory T , with (ii) DPLL *engines* for dealing with the boolean structure of the formulas.

DPLL(T) is a general SMT architecture for the lazy approach [9]. It consists of a DPLL(X) engine, whose parameter X can be instantiated with a T -solver $Solver_T$, thus producing a DPLL(T) system. The DPLL(X) engine always considers the problem as a purely propositional one. For example, if the theory T is EUF, at some point DPLL(X) might consider a partial assignment containing, among many others, the four literals $a = b$, $f(a) = c$, $f(b) = d$, and $c \neq d$ without noticing its T -inconsistency, because it just considers such literals as propositional (syntactic) objects. But $Solver_T$ continuously analyzes the partial model that DPLL(X) is building (a conjunction of literals). It can warn DPLL(X) about this T -inconsistency, and generate a clause, called a *theory lemma*, $a \neq b \vee f(a) \neq c \vee f(b) \neq d \vee c = d$, that can be used by DPLL(X) for backjumping. $Solver_T$ sometimes also does *theory propagation*: as soon as, e.g., $a = b$, $f(a) = c$, and $f(b) = d$ become true, it can notify DPLL(X) about T -consequences like $c = d$ that occur in the input formula. The modular DPLL(T) architecture is flexible, and, compared with other SMT techniques, DPLL(T) is also very efficient and has good scaling properties: the BarcelogicTools implementation of DPLL(T) won all the four divisions it entered at the 2005 SMT Competition [1].

Here we propose an improved version of the DPLL(T) architecture, to rationalize and simplify the construction of lazy SMT systems where $Solver_T$ does reasoning by cases. We present it formally by means of a corresponding extension of *Abstract DPLL Modulo Theories*, a uniform, declarative framework introduced in [11] for modeling and reasoning about lazy SMT procedures.

Example 1. Consider the equation $read(write(A, i, v), j) = read(A, j)$ in the theory of arrays. It holds in two situations: when the indices i and j are distinct, or when they are equal but the $write(A, i, v)$ changes nothing, i.e., the value of array A at position i is already v . Deciding the T -consistency of a large conjunction of equations and disequations over arrays essentially requires $Solver_T$ to do an analysis of many Boolean combinations of such cases. In the extension of DPLL(T) we propose here, $Solver_T$ can delegate all such case splittings to the DPLL(X) engine, e.g., it can *demand* DPLL(X) to split on atoms like $i = j$, by sending it a *theory lemma* (i.e., a ground clause valid in the theory) that encodes the split—for instance, a clause like $read(write(A, i, v), j) \neq read(A, j) \vee i \neq j \vee read(A, i) = v$. \square

The main novelty, and complication, versus the previous version of DPLL(T) is that the lemma may contain atoms that *do not occur in the input formula*. Sometimes even new constant symbols may be introduced. For example, in (fragments of) set theory [6], a set disequality $s \neq s'$ may be handled by the theory solver by reducing it to the disjunction $(a \in s \wedge a \notin s') \vee (a \notin s \wedge a \in s')$, where a is a fresh Skolem constant. Centralizing all case splitting into the engine avoids the duplication of this functionality and drastically simplifies the implementation of $Solver_T$, which now needs no case-splitting infrastructure. Roughly, the solver's

only requirement becomes that it must be able to detect T -inconsistencies once all case splits it has requested have been done. This sort of architecture is likely to also produce performance gains, since the engine is already specialized in reasoning efficiently over disjunctions, and because its heuristics are not sabotaged, so to speak, by the internal splitting done by the theory solver.

The second main contribution of this paper is that the extension can easily and efficiently accommodate theories T defined as the union of several component theories T_1, \dots, T_n . This gives rise to a DPLL(T_1, \dots, T_n) architecture.

Example 2. Let T be the union of two disjoint theories T_1 and T_2 where T_1 is EUF and T_2 is (some fragment of) arithmetic, two of the most common theories in SMT. Let F be the conjunction $a=b \wedge f(a)-c \leq 3 \wedge f(b)-c \geq 4$ over the combined signature of T_1 and T_2 . Introducing new constants c_1 and c_2 , F can be *purified*, into an equisatisfiable conjunction of the T_1 -pure formula F_1 and the T_2 -pure formula F_2 below:

$$a=b \wedge c_1=f(a) \wedge c_2=f(b) \qquad c_1-c \leq 3 \wedge c_2-c \geq 4.$$

In general, an *arrangement* A for such pure conjunctions $F_1 \dots F_n$ is a conjunction saying, for every two constants *shared* between at least two different F_i 's, whether the constants are equal or distinct. A general combination result underpinning the Nelson-Oppen method [10] states that for *stably infinite* and *signature disjoint* T_i 's, F is T -consistent if, and only if, for some arrangement A each $F_i \wedge A$ is T_i -consistent (see, e.g., [13] for precise definitions and details). This can be decided by the respective T_i -solvers. In this example, F is T -inconsistent since $F_1 \wedge c_1 \neq c_2$ is T_1 -inconsistent and $F_2 \wedge c_1=c_2$ is T_2 -inconsistent.

In practice, it is useful if each T_i -solver is able to *generate* all clauses $c_1 = c'_1 \vee \dots \vee c_k = c'_k$ over the shared constants that are T_i -entailed by the conjunction F_i . For *convex* T_i , these entailed clauses are in fact always unit. It is not difficult to see that, if these two properties hold for all T_i , we only have to consider one arrangement: the one where every two constants not equated in a propagated equality are distinct. But usually the situation is less ideal. If some T_i is non-convex, it is necessary to do case splitting over the T_i -entailed non-unit clauses, and if some T_i -solver has limited (or too expensive) generation capabilities, the possible arrangements need to be (partially) guessed and tried.

By centralizing case splitting into the DPLL(X) and extending it to equalities over shared constants we can use the engine to efficiently enumerate the arrangements on demand, that is, as requested by the individual theory solvers. In the resulting DPLL(T_1, \dots, T_n) architecture, again the engine must handle literals not in the input clause—in this case, (dis)equalities between shared variables. \square

Section 2 of this paper introduces and formally proves the correctness of the Extended Abstract DPLL Modulo Theories framework that can accommodate and formalize these approaches. Section 3 illustrates how to use the framework to avoid internal case splits in a very general class of theory solvers. Section 4 is on the application of the framework to DPLL(T_1, \dots, T_n). Finally Section 5 concludes.

Related work. Some of the ideas formalized in this paper on centralizing case splits in the Boolean engine are implemented in the system CVC [4]. But apart from a brief note in Clark Barrett’s PhD Thesis [3], we are not aware of any other description of them in the literature. Bozzano *et al.* propose in [5] to use the Boolean engine in multi-theory SMT systems to do case splitting over the (potentially large) space of all possible arrangements. Besides allowing case splits on a more expressive language (i.e., not restricted to just the shared equalities), our approach further develops the ideas of [5] by proposing a formal framework for the flexible combination of guessing (by the engine) and deduction (by the solvers) of partial arrangements.

2 Extended Abstract DPLL Modulo Theories

Abstract DPLL Modulo Theories [11] is a framework for modeling and reasoning about DPLL-based SAT and SMT systems in terms of simple transition rules and rule application strategies. The framework facilitates the understanding and the comparison of different approaches as well as the proving of their correctness. In this section, we briefly describe the framework (see [11] for a more thorough description of the framework) and then extend it so that it can be used to formalize our new version of DPLL(T) and, more generally, SMT approaches where new atoms and new symbols are introduced.

2.1 Abstract DPLL Modulo Theories

As usual in SMT, given a background theory T (a set of closed first-order formulas), we will only consider the SMT problem for *ground* (and hence quantifier-free) CNF formulas F . Such formulas may contain *free* constants, i.e., constant symbols not in the signature of T , which, as far as satisfiability is concerned, can be equivalently seen as existential variables. Other than free constants, all other predicate and function symbols in the formulas will instead come from the signature of T . From now on, we will assume that all formulas satisfy these restrictions.

The formalism we describe is based on a set of *states* together with a binary relation \Longrightarrow (called the *transition relation*) over these states, defined by means of *transition rules*. Starting with a state containing an input formula F , one can use the rules to generate a finite sequence of states, where the final state indicates whether or not F is T -consistent.

A *state* is either the distinguished state *FailState* (denoting T -unsatisfiability) or a pair of the form $M \parallel F$, where M is a sequence of literals, with \emptyset denoting the empty sequence, and F is a formula in conjunctive normal form (CNF), i.e., a finite set of disjunctions of literals. We additionally require that M never contains both a literal and its negation and that each literal in M is annotated as either a *decision* literal (indicated by l^d) or not. Frequently, we will refer to M as a *partial assignment* or consider M just as a set or conjunction of literals, ignoring both the annotations and the order of its elements.

In what follows, a possibly subscripted or primed lowercase l *always* denotes a literal. Similarly C and D always denote clauses (disjunctions of literals), F and G denote conjunctions of clauses, and M and N denote partial assignments.

We write $M \models F$ to indicate that M propositionally satisfies F . If C is a clause $l_1 \vee \dots \vee l_n$, we sometimes write $\neg C$ to denote the formula $\neg l_1 \wedge \dots \wedge \neg l_n$. We say that C is *conflicting* in a state $M \parallel F, C$ if $M \models \neg C$.

A formula F is called *T-(in)consistent* if $F \wedge T$ is (un)satisfiable in the first-order sense. We say that M is a *T-model* of F if $M \models F$ and M , seen as a conjunction of literals, is *T-consistent*. It is not difficult to see that F is *T-consistent* if, and only if, it has a *T-model*. If F and G are formulas, then F *entails G in T*, written $F \models_T G$, if $F \wedge \neg G$ is *T-inconsistent*. If $F \models_T G$ and $G \models_T F$, we say that F and G are *T-equivalent*. A *theory lemma* is a clause C such that $\emptyset \models_T C$.

We start with the transition system first presented in [11].

Definition 1. *Abstract DPLL Modulo Theories consists of the following rules:*

UnitPropagate :

$$M \parallel F, C \vee l \implies M l \parallel F, C \vee l \quad \text{if} \quad \begin{cases} M \models \neg C \\ l \text{ is undefined in } M \end{cases}$$

Decide :

$$M \parallel F \implies M l^d \parallel F \quad \text{if} \quad \begin{cases} l \text{ or } \neg l \text{ occurs in a clause of } F \\ l \text{ is undefined in } M \end{cases}$$

Fail :

$$M \parallel F, C \implies \text{FailState} \quad \text{if} \quad \begin{cases} M \models \neg C \\ M \text{ contains no decision literals} \end{cases}$$

Restart :

$$M \parallel F \implies \emptyset \parallel F$$

T-Learn :

$$M \parallel F \implies M \parallel F, C \quad \text{if} \quad \begin{cases} \text{each atom of } C \text{ occurs in } F \text{ or in } M \\ F \models_T C \end{cases}$$

T-Forget :

$$M \parallel F, C \implies M \parallel F \quad \text{if} \quad \{ F \models_T C \}$$

T-Backjump :

$$M l^d N \parallel F, C \implies M l' \parallel F, C \quad \text{if} \quad \begin{cases} M l^d N \models \neg C, \text{ and there is} \\ \text{some clause } C' \vee l' \text{ such that:} \\ F, C \models_T C' \vee l' \text{ and } M \models \neg C', \\ l' \text{ is undefined in } M, \text{ and} \\ l' \text{ or } \neg l' \text{ occurs in } F \text{ or in } M l^d N \end{cases}$$

T-Propagate :

$$M \parallel F \implies M l \parallel F \quad \text{if} \quad \begin{cases} M \models_T l \\ l \text{ or } \neg l \text{ occurs in } F \\ l \text{ is undefined in } M \end{cases}$$

The Basic DPLL Modulo Theories system consists of the rules *Decide*, *Fail*, *UnitPropagate*, *T-Propagate* and *T-Backjump*. We denote the transition relation defined by these rules by $\Longrightarrow_{\text{B}}$. We denote the transition relation defined by all the rules by $\Longrightarrow_{\text{FT}}$.

For a transition relation \Longrightarrow , we denote by \Longrightarrow^* the reflexive-transitive closure of \Longrightarrow . We call any sequence of the form $S_0 \Longrightarrow S_1, S_1 \Longrightarrow S_2, \dots$ a *derivation*, and denote it by $S_0 \Longrightarrow S_1 \Longrightarrow S_2 \Longrightarrow \dots$. We call any subsequence of a derivation a *subderivation*. If $S \Longrightarrow S'$ we say that there is a *transition* from S to S' . A state S is *final* with respect to \Longrightarrow if there are no transitions from S .

The relevant derivations in the Abstract DPLL Modulo Theories system are those that start with a state of the form $\emptyset \parallel F$, where F is a formula to be checked for T -consistency, and end in a state that is final with respect to $\Longrightarrow_{\text{B}}$.

2.2 The Extended Abstract DPLL Modulo Theories System

Any realization of the Abstract DPLL Modulo Theories framework, in addition to implementing the rules and a terminating strategy, must be able to determine the T -consistency of M when a final state $M \parallel F$ is reached. For this purpose, one typically assumes the existence of $Solver_T$, which can do precisely that.

However, for some important theories, determining the T -consistency of a conjunction of literals requires additional *internal* case splitting. In order to simplify $Solver_T$ and centralize the case splitting in the DPLL engine, it is desirable to relax the requirement on $Solver_T$ by allowing it to demand that the DPLL engine do additional case splits before determining the T -consistency of the partial assignment. For flexibility—and because it is needed by actual theories of interest—the theory solver should be able to demand case splits not only on literals that do not appear in M or F , but also on literals that may introduce new constant symbols not appearing in M or F .

It is not hard to see, however, that allowing this kind of flexibility poses a potential problem for termination of the procedure. We can overcome this difficulty as follows. As a purely theoretical construction, we assume that, for each given (finite) input formula F , a finite set of literals $\mathcal{L}(F)$ exists that contains all literals on which the theory solver may demand case splits when starting with a conjunction of literals from F . For example, for the theory of arrays $\mathcal{L}(F)$ could contain atoms of the form $i = j$, where i and j are array indices occurring in F . This technical requirement poses no limitations on any of the practically useful theory solver procedures we are aware of (see Section 3). Also, for the proofs here there is no need to construct the set $\mathcal{L}(F)$. It is enough to know that it exists. Formally, we require the following.

Definition 2. \mathcal{L} is a suitable literal-generating function if for every finite set of literals L :

1. \mathcal{L} maps L to a new finite set of literals L' such that $L \subseteq L'$.
2. For each atomic formula α , $\alpha \in \mathcal{L}(L)$ iff $\neg\alpha \in \mathcal{L}(L)$.
3. If L' is a set of literals and $L \subseteq L'$, then, $\mathcal{L}(L) \subseteq \mathcal{L}(L')$ (*monotonicity*).

4. $\mathcal{L}(\mathcal{L}(L)) = \mathcal{L}(L)$ (*idempotence*).

For convenience, given a formula F , we denote by $\mathcal{L}(F)$ the result of applying \mathcal{L} to the set of all literals appearing in F .

The introduction of new constant symbols poses potential problems not only for termination, but also for soundness. One property of the transition relation \Longrightarrow_{FT} is that whenever $\emptyset \parallel F \Longrightarrow_{FT}^* M \parallel F'$, the formulas F and F' are T -equivalent. This will no longer be true if we allow the introduction of new constant symbols. However, it is sufficient to simply ensure T -equisatisfiability of F and F' . To this end, we introduce the following definition.

Definition 3. *Given a formula F and a formula G , we define $\gamma_F(G)$ as follows:*

1. Let G' be the formula obtained by replacing each constant symbol in G that does not appear in F with a fresh variable.
2. Let \bar{v} be the set of all fresh variables introduced in the previous step.
3. Then, $\gamma_F(G) = \exists \bar{v}. G'$.

Now we can give a new transition rule called **Extended T -Learn** which replaces T -Learn and allows for the desired additional flexibility.

Definition 4. *The Extended DPLL Modulo Theories system, denoted as \Longrightarrow_{XT} , consists of the rules of Basic DPLL Modulo Theories, together with the rules Restart, T -Forget and the rule Extended T -Learn below:*

Extended T -Learn

$$M \parallel F \quad \Longrightarrow \quad M \parallel F, C \quad \text{if} \quad \begin{cases} \text{each atom of } C \text{ occurs in } F \text{ or in } \mathcal{L}(M) \\ F \models_T \gamma_F(C) \end{cases}$$

The key observation is that an implementation using **Extended T -Learn** has more flexibility when a state $M \parallel F$ is reached which is final with respect to \Longrightarrow_B . Whereas before it would have been necessary to determine the T -consistency of M when such a state was reached, the **Extended T -Learn** rule allows the possibility of *delaying* a response by demanding that additional case splits (on new literals appearing in the clause C) be done first. As we will show below, the properties of \mathcal{L} ensure that the response cannot be delayed indefinitely.

2.3 Correctness of Extended Abstract DPLL Modulo Theories

A decision procedure for SMT can be obtained by generating a derivation using \Longrightarrow_{XT} with a particular strategy. As with \Longrightarrow_{FT} , the aim of a derivation is to compute a state S such that: (i) S is final with respect to the rules of Basic DPLL Modulo Theories and (ii) if S is of the form $M \parallel F$ then M is T -consistent. We start by stating some invariants.¹

Lemma 1. *If $\emptyset \parallel F \Longrightarrow_{XT}^* M \parallel G$ then the following hold.*

¹ Because of space constraints we must refer the reader to [2] for the proofs of all the results in this section.

1. All the literals in M and all the literals in G are in $\mathcal{L}(F)$.
2. M contains no literal more than once and is indeed an assignment, i.e., it contains no pair of literals of the form p and $\neg p$.
3. $G \models_T F$ and for some H , $F \models_T \gamma_H(G)$.
4. If M is of the form $M_0 l_1 M_1 \dots l_n M_n$, where l_1, \dots, l_n are all the decision literals of M , then $G, l_1, \dots, l_i \models_T M_i$ for all i in $0 \dots n$.

Theorem 1 (Termination of \implies_{XT}). Every derivation Der of the form $\emptyset \parallel F = S_0 \implies_{\text{XT}} S_1 \implies_{\text{XT}} \dots$ is finite if the following two conditions hold:

1. Der has no infinite subderivations consisting of only *Extended T-Learn* and *T-Forget* steps.
2. For every subderivation of Der of the form $S_{i-1} \implies_{\text{XT}} S_i \implies_{\text{XT}} \dots \implies_{\text{XT}} S_j \implies_{\text{XT}} \dots \implies_{\text{XT}} S_k$ where the only three *Restart* steps are the ones producing S_i , S_j , and S_k , either:
 - there are more *Basic DPLL Modulo Theories* steps in $S_j \implies_{\text{XT}} \dots \implies_{\text{XT}} S_k$ than in $S_i \implies_{\text{XT}} \dots \implies_{\text{XT}} S_j$, or
 - in $S_j \implies_{\text{XT}} \dots \implies_{\text{XT}} S_k$ a new clause is learned that is not forgotten in Der .

Lemma 2. If $\emptyset \parallel F \implies_{\text{XT}}^* M \parallel F'$ and there is some conflicting clause in $M \parallel F'$, i.e., $M \models \neg C$ for some clause C in F' , then either *Fail* or *T-Backjump* applies to $M \parallel F'$.

Property 1. If $\emptyset \parallel F \implies_{\text{XT}}^* M \parallel F'$ and M is T -inconsistent, then either there is a conflicting clause in $M \parallel F'$, or else *Extended T-Learn* applies to $M \parallel F'$, generating a clause enabling some *Basic DPLL Modulo Theories* step.

Even if it is very easy to generate non-terminating derivations for \implies_{XT} , Theorem 1 defines a very general strategy for avoiding that. Lemma 2 and Property 1 show that, for a state of the form $M \parallel F$, if there is some literal of F undefined in M , or there is some conflicting clause, or M is T -inconsistent, then a rule of *Basic DPLL Modulo Theories* is always applicable, possibly after a single *Extended T-Learn* step. Together with Theorem 1 (Termination), this shows how to compute a state to which the following main theorem is applicable.

Theorem 2. Let Der be a derivation $\emptyset \parallel F \implies_{\text{XT}}^* S$, where S is (i) final with respect to *Basic DPLL Modulo Theories*, and (ii) if S is of the form $M \parallel F'$ then M is T -consistent. Then

1. S is *FailState* if, and only if, F' is T -inconsistent.
2. If S is of the form $M \parallel F'$ then M is a T -model of F .

For a given theory T , Theorems 1 and 2 show how to obtain a decision procedure for the T -consistency of formulas as long as we have a theory solver and can prove the existence of a suitable literal-generating function \mathcal{L} so that, for every state of the form $M \parallel F$ that is final with respect to \implies_{B} , it is always possible to: (i) determine that M is T -inconsistent; (ii) determine that M is T -consistent; or (iii) generate a new clause via *Extended T-Learn* that enables some *Basic DPLL Modulo Theories* step.

3 Avoiding case splitting within theory solvers

In this section we show how rule-based theory solvers can be used in the context of Extended DPLL Modulo Theories. We show how the *Extended T-Learn* rule allows such solvers to avoid any internal case splitting, and we explain why, for these types of solvers, the existence of \mathcal{L} is reasonable. Recall that theory solvers only need to deal with conjunctions of literals, here represented as sets of literals.

3.1 Rule-based Theory Solvers

A large class of theory solvers can be defined using inference rules which describe how to take a set of literals and transform it in some way to get new sets of literals (or \perp , indicating T -inconsistency). Consider a theory T . For our purposes, let us assume that an inference rule has one of the following two formats:

$$\frac{\Gamma, \Delta}{\perp} \qquad \frac{\Gamma, \Delta}{\Gamma, \Delta_1 \quad \Gamma, \Delta_2 \quad \cdots \quad \Gamma, \Delta_n}$$

We call rules of the first kind *refuting* rules and rules of the second kind *progress* rules. Γ, Δ and Δ_i are meta-variables representing sets of literals. Typically, Δ has side-conditions or is a schema, while Γ can represent any set of literals. Progress rules describe a local change based on a small number of literals (i.e. the ones in Δ), while all of the other literals are unchanged (i.e. the ones in Γ).

A refuting rule is *sound* iff any legal instance δ of Δ is T -inconsistent. A progress rule is *sound* if whenever $\Delta, \Delta_1, \dots, \Delta_n$ are instantiated with $\delta, \delta_1, \dots, \delta_n$ respectively, δ is T -consistent iff $\bigvee_{i=1}^n \delta_i$ is T -consistent. We say that a set Φ of literals is (*ir*)*reducible* with respect to a set of derivation rules R if (n)one of the rules in R applies to it, i.e., if (no)some subset of Φ is a legal instance of Δ in a rule of R . A *strategy* is a function that, given a reducible set of literals Φ , chooses a rule from R to apply.

Given a set of rules R and a strategy S , a *derivation tree* for a set of literals Φ is a finite tree with root Φ such that for each internal node E of the tree, E is reducible and its children are the conclusions of the rule selected by S for E . A *refutation tree* (for Φ) is a derivation tree all of whose leaves are \perp . A *derivation* is a sequence of derivation trees starting with the single-node tree containing Φ , where each tree is derived from the previous one by the application of a rule from R to one of its leaves. A *refutation* is a finite derivation ending with a refutation tree. A strategy S is *terminating* if every derivation using S is finite. A strategy S is *complete* if whenever Φ is T -inconsistent, S produces a refutation for Φ .

It is not hard to see that a set R of sound inference rules together with a terminating and complete strategy S provide a decision procedure for the T -consistency of sets of ground literals. In fact, all decision procedures typically associated with applications of Satisfiability Modulo Theories can be described in this way. We will now describe how such decision procedures can be incorporated into the Extended Abstract DPLL Modulo Theories formalism.

3.2 Integration with Rule-based Theory Solvers

Recall that the original DPLL Modulo Theories framework requires that for every state $M \parallel F$ that is final with respect to Basic DPLL Modulo Theories, the theory solver can determine the T -consistency of M . Given a set of sound inference rules and a terminating and complete strategy, M can be checked for T -consistency simply by generating the derivation starting with M and determining whether it results in a refutation tree or not.

Note that this process may require a large derivation tree with many branches. The purpose of the **Extended T -Learn** rule is to allow the theory solver to avoid having to do any splitting itself. This can be accomplished as follows. Given a state $M \parallel F$ which is final with respect to Basic DPLL Modulo Theories, the theory solver begins applying rules starting with M . However, this time, as soon as a splitting rule is encountered (a progress rule with $n > 1$), the theory solver halts and uses **Extended T -Learn** to return a clause representing the case split. The theory solver is then suspended until another final state $M' \parallel F'$ is reached.

The obvious remaining question is how to capture the case split with a learned clause. As we show in [2], one way to do this that will work for any rule-based theory solver is to encode the number of possible case splits using Boolean constants. In practice, however, it is usually possible and desirable to encode splitting rules more directly. For example, a progress rule of the form seen in the previous subsection (where $n > 1$) corresponds to the following formula schema: $\neg(\Delta) \vee \bigvee_{i=1}^n \Delta_i$. Any instance of this schema can be converted into CNF and the resulting clauses sent to the DPLL engine via **Extended T -Learn**. For this to work, one additional requirement is that the rules be *refining*. We say that an inference rule is *refining* if it is a refuting rule or if whenever $\Delta, \Delta_1, \dots, \Delta_n$ are instantiated with $\delta, \delta_1, \dots, \delta_n$ respectively, $\delta \models_T \gamma_\delta(\bigvee_{i=1}^n \delta_i)$. This is essentially a stronger version of soundness. It requires that any model of the premise can be refined into a model of some of the consequents. It is necessary in order to satisfy the side conditions of **Extended T -Learn**.

We must also check that an appropriate literal-generating function \mathcal{L} exists. Assume we are given a set of rules R and a terminating strategy S . First, define \mathcal{D} to be a function which, given a set Φ of literals returns all literals that may appear along any branch of the derivation tree with any subset of Φ at its root. And let \mathcal{N} be a function which, given a set Φ of literals, returns all literals that can be formed from the atomic formulas in Φ . Now, we define a series of functions \mathcal{L}_i as follows. Let \mathcal{L}_0 be the identity function and for $i > 0$, let $\mathcal{L}_i(\Phi) = \mathcal{N}(\mathcal{D}(\mathcal{L}_{i-1}(\Phi)))$. If for some $k > 0$, $\mathcal{L}_k = \mathcal{L}_{k+1}$, then we say that R is *literal-bounded under S* , and define $\mathcal{L} = \mathcal{L}_k$.

Property 2. If R is a set of sound refining rules for a theory T , S is a strategy for R that is terminating and complete, and R is literal-bounded under S , then R can be integrated with the Extended DPLL Modulo Theories framework.

Proof. We first show that \mathcal{L} satisfies Definition 2. It is easy to see that Properties 1 and 2 in the definition are satisfied. Because $\mathcal{D}(\Phi)$ considers derivations

starting with any subset of Φ , Property 3 must also be satisfied. Finally, because \mathcal{L} is a fixed point of \mathcal{L}_i , it must be idempotent.

Now, we must show that whenever a state $M \parallel F$ is reached that is final with respect to Basic DPLL Modulo Theories, the theory solver can do one of the following: determine that M is T -consistent; determine that M is T -inconsistent; or introduce a new clause via Extended T -Learn that enables some Basic DPLL Modulo Theories step.

Given a state $M \parallel F$, we simply apply rules from R to M according to strategy S . If \perp is derived, then by soundness, M is T -inconsistent. If an irreducible set of literals is derived, then by completeness, M must be T -consistent. If a splitting rule is reached, and $\Gamma, \Delta, \Delta_1, \dots, \Delta_n$ are instantiated with $\phi, \delta, \delta_1, \dots, \delta_n$ respectively, there are three possibilities:

1. For all i , $M \models \neg\delta_i$. In this case, we apply Extended T -Learn to learn $\neg(\delta) \vee \bigvee_{i=1}^n \delta_i$, which will result in one or more clauses that are conflicting in M , thus enabling either Fail or T -Backjump by Lemma 2.
2. For some i , $M \models \delta_i$, or else δ_i is undefined in M and $M \models \neg\delta_j$ for every $j \neq i$. In either case, no split is necessary and we simply proceed by applying rules of R to ϕ, δ_i .
3. The final case is when at least two of the δ_i are undefined in M . Then we apply Extended T -Learn to learn $\neg(\delta) \vee \bigvee_{i=1}^n \delta_i$ which is guaranteed to contain at least one clause that is not satisfied by M , thus enabling Decide. \square

Example 3. The theory of arrays is a well-known theory used in Satisfiability Modulo Theories that requires case splitting. One (sound and refining) rule-based decision procedure for this theory is given in [12]. A careful examination of the decision procedure reveals the following: (i) each term can be categorized as an *array* term, an *index* term, a *value* term, or a *set* term; (ii) no new array terms are ever introduced by the inference rules; (iii) at most one new index term for every pair of array terms is introduced; (iv) set terms are made up of some finite number of index terms; (v) the only new value terms introduced are of the form $read(a, i)$ where a is an array term and i is an index term. It follows that the total number of possible terms that can be generated starting with any finite set of literals is finite. Because there are only a finite number of predicates, it then follows that this set of rules is literal-bounded. \square

4 Application to Satisfiability Modulo Multiple Theories

In this section we focus on background theories T that are actually the union of two or more component theories T_1, \dots, T_n , each equipped with its own solver.

We first show how to obtain an Abstract DPLL Modulo Theories transition system for the combined theory T as a *refinement* of the system XT described in Section 2 using only the solvers of the theories T_i . Then we show how to refine the new DPLL(T) architecture into a DPLL(T_1, \dots, T_n) architecture in which each T_i -solver is directly integrated into the DPLL(X_1, \dots, X_n) engine.

We will work here in the context of first-order logic with equality. For the rest of the section we fix $n > 1$ *stably infinite* theories² T_1, \dots, T_n with respective, mutually disjoint signatures $\Sigma_1, \dots, \Sigma_n$. We will consider the theory $T = T_1 \cup \dots \cup T_n$ with signature $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_n$. We are interested in the T -satisfiability of ground formulas over the signature Σ extended with an infinite set K of *free constants*. For any signature Ω we will denote by $\Omega(K)$ the signature $\Omega \cup K$. We say that a ground clause or literal is *(i-)pure* if it has signature $\Sigma_i(K)$ where $i \in \{1, \dots, n\}$. Given a CNF formula F of signature $\Sigma(K)$, by abstracting subterms with fresh constants from K , it is possible to convert F in linear time into an equisatisfiable CNF formula, all of whose atoms are pure. See [13], for instance, for details on this purification procedure. From now on then, we will limit ourselves with no loss of generality to pure formulas.

Following the Nelson-Oppen combination method, the various solvers will cooperate by exchanging entailed equalities over *shared* constants. Let L be a set of pure literals over the signature $\Sigma(K)$. We say that a constant $k \in K$ is an *(ij)-shared constant* of L if it occurs in an i -pure and a j -pure literal of L for some distinct i and j . For $i = 1, \dots, n$, we denote by L^i the set of all the $\Sigma_i(K)$ -literals of L and by $\mathcal{S}_i(L)$ the set of all equalities between distinct ij -shared constants of L for every $j \neq i$. Note that for every $j \neq i$, $L^j \cap L^i$ contains at most equalities or the negation of equalities from $\mathcal{S}_i(L)$. An *arrangement for* L is a set containing for each equality $e \in \bigcup_i \mathcal{S}_i(L)$ either e or $\neg e$ (but not both), and nothing else.

The extended Abstract DPLL Modulo theories framework can be refined to take into account that T is a combined theory by imposing the following additional requirements on the XT system.

Refinement 1. We consider only derivations starting with states of the form $\emptyset \parallel F$, where each atom of F is a pure $\Sigma(K)$ -atom.

Refinement 2. We consider only applications $M \parallel F \implies Ml \parallel F$ of T -Propagate and applications $M \parallel F \implies M \parallel F, C$ of Extended T -Learn where l and each literal of C are pure.

Refinement 1 and 2 maintain the invariant that all the literals occurring in a state are pure, and so can be fed to the corresponding local solvers. Given these minimal requirements, it will be sufficient for T -Propagate to propagate only literals l that are i -pure for some $i = 1, \dots, n$ and such that $M^i \models_{T_i} l$, where the entailment $M^i \models_{T_i} l$ is determined by the T_i -solver. Similarly, Extended T -Learn will rely on the local solvers only to learn T_i -lemmas, i.e., i -pure clauses C such that $\emptyset \models_{T_i} \gamma_F(C)$. Note that we do allow lemmas C consisting of pure literals from different theories and such that $F \models_T \gamma_F(C)$, as lemmas of this sort can be computed even if one only has local solvers (consider for example the backjump clauses generated by standard conflict analysis mechanisms).

Refinement 3. The suitable literal-generating function \mathcal{L} maps every finite set L of pure $\Sigma(K)$ -literals to a finite set of pure $\Sigma(K)$ -literals including $\bigcup_i \mathcal{S}_i(L)$.

² A theory T is stably infinite if every T -consistent quantifier-free formula F over T 's signature is satisfiable in an infinite model of T .

To use the various solvers together in a refutationally complete way for T -consistency, it is necessary to make them agree on an arrangement. To do this efficiently, they should be able to share the entailed (disjunctions) of equalities of shared constants. Refinement 3 then essentially states that theory lemmas can include shared equalities.

4.1 From DPLL(T) to DPLL(T_1, \dots, T_n)

Assuming the previous refinements at the abstract level, we now show in concrete how the DPLL(T) architecture can be specialized to use the various local solvers directly and to facilitate cooperation among them. Here we define a local requirement on each T_i -solver that does not even need refutational completeness for T_i -consistency. If M is an assignment consisting of pure literals and $i \in \{1, \dots, n\}$, we call (*default*) *completion* of M^i and denote by \widehat{M}^i the smallest extension of M^i falsifying every shared equation for M^i that is undefined in M , that is, $\widehat{M}^i = M^i \cup \{\neg e \mid e \in \mathcal{S}_i(M), e \text{ undefined in } M\}$.

Requirement 1. For each $i = 1, \dots, n$, if the solver for T_i , given a state $M \parallel F$, cannot determine that \widehat{M}^i is T_i -consistent, then it must either:

1. identify a T_i -inconsistent subset of M^i or
2. produce an i -pure clause C such that $\emptyset \models_{T_i} \gamma_F(C)$ and C contains at least one literal of $\mathcal{L}(M)$ undefined in M .

Note that, according to this last requirement, if a solver cannot determine the T_i -consistency of the completion \widehat{M}^i , it must be either because it has determined that a subset of \widehat{M}^i (possibly of M^i alone) is in fact inconsistent, or that it needs more information about some of the undefined literals of $\mathcal{L}(M)$ first. However, once every literal of $\mathcal{L}(M)$ is defined in M , including the equalities in $\mathcal{S}_i(M)$, the solver *must* be able to tell whether M^i is T_i -consistent or not. The latter is a minimal requirement for any solver to be used in a Nelson-Oppen style combination procedure.

Usually, however, it is desirable for Nelson-Oppen solvers to also be able to compute (disjunctions of) shared equalities entailed by a given set of literals, so that only these equalities can be propagated to the other solvers, and guessing is minimized (for instance, if one solver communicates that a is equal to either b or c , then the other solvers do not have to consider cases where a is equal to some fourth constant). Here we include that possibility, as illustrated by this example.

Assume for simplicity that for every M , $\mathcal{L}(M)$ is no more than $\bigcup_i \mathcal{S}_i(M)$, which entails that each T_i -solver is refutationally complete for T_i -consistency. Then consider an assignment M where M_i is T_i -consistent, for some i , and let e_1, \dots, e_n be equalities in $\mathcal{S}_i(M)$ undefined in M such that $l_1, \dots, l_m \models \bigvee_k e_k$ for some $\{l_1, \dots, l_m\} \subseteq M^i$. In this case \widehat{M}^i is clearly T_i -inconsistent. However, since M^i is by itself consistent, by Requirement 1 the T_i -solver must return a lemma containing one undefined literal of $\mathcal{L}(M)$. Now, if the solver can in fact compute the clause $l_1, \dots, l_m \models \bigvee_k e_k$, that clause will be the ideal lemma to

return. Otherwise, it is enough for the solver to return *any* lemma that contains at least one shared equality. Intuitively, this marks a progress in the computation because eventually that shared equality will become defined (for instance, by an application of `Decide`), reducing the number of undefined literals in $\mathcal{L}(M)$.

Requirement 1 and the earlier refinements are enough to guarantee that we can use the local T_i -solvers directly—as opposed to building a solver for the combined theory T —to generate derivations satisfying Theorem 2. The first thing we need for Theorem 2 is easy to see : it is always possible to derive from a state $\emptyset \parallel F$ a final state S with respect to Basic DPLL Modulo Theories (\implies_{B}). The second thing we need is that whenever the final state S has the form $M \parallel F'$, the assignment M is T -consistent. Although none of the local solvers is able to determine by itself whether M is T -consistent, it can do that in cooperation with the other solvers, thanks to Requirement 1 and the Nelson-Oppen combination result in [13]. It is then not difficult to show that, under the assumptions in this section, the following property holds (see [2] for a detailed proof).

Property 3. Every derivation of the form $\emptyset \parallel F \implies_{\text{XT}}^* M \parallel G$ where $M \parallel G$ is final wrt. Basic DPLL Modulo Theories can be extended in finitely many steps using the solvers for T_1, \dots, T_n to a derivation of the form $\emptyset \parallel F \implies_{\text{XT}}^* M \parallel G \implies_{\text{XT}}^* S$ where S is either *FailState* or a state $M' \parallel G'$ with a T -consistent M' .

4.2 Convex theories

We mentioned in Section 1 that the Nelson-Oppen method is particularly efficient if (i) the component theories T_i are all *convex*, i.e., such that for all sets L of ground $\Sigma_i(K)$ -literals and all sets E of ground K -equalities, $L \models_{T_i} \bigvee_{e \in E} e$ iff $L \models_{T_i} e$ for some $e \in E$ and (ii) the T_i solvers are able to compute T_i -entailed shared equalities. In that case, no expensive case splits over shared equalities are needed, and the equality propagation process can be stopped as soon as there are no more entailed shared equalities. This situation can be neatly modeled in our framework if we slightly revise the precondition of the T -Propagate rule on the state $M \parallel F$ to allow the propagation of entailed equalities occurring in $\bigcup_i \mathcal{S}(M)$ in addition to the literals occurring (possibly negated) in F .³ It is not difficult to see that this extension preserves all the properties of the framework.

When only some of the component theories are convex, an even better extension is to allow the propagation of entailed shared *disequalities* as well, because this too, even if not strictly needed, may ease the lemma generation burden of the solvers for the non-convex theories. With these extensions then, we can have solvers for convex theories use only T -Propagate to propagate entailed shared (dis)equalities to the other solvers, and solvers for non-convex theories use a flexible combination of T -Propagate, to propagate single shared (dis)equalities and Extended T -Learn, to propagate proper disjunctions of shared (dis)equalities.

³ Note that an equality between shared constants need not occur in F .

5 Conclusions and further work

We have proposed a new version of $DPLL(T)$ in which theory solvers can delegate all case splits to the DPLL engine. This can be done *on demand* for solvers that can encode their internal case splits into one or more clauses, possibly including new constants and literals. It results in drastically simpler theory solvers, and can lead, we believe, to more efficient overall systems. We have formalized this in an extension of Abstract DPLL and proved it correct. As one anonymous referee once put it: “this is worth emphasizing in an area notorious for gaps between published algorithms and their actual implementations.”

We have also introduced an efficient, we believe, $DPLL(T_1, \dots, T_n)$ architecture for combined theories, which also fits naturally into the extended Abstract DPLL framework. We think that the new insights gained by this formalization will help us and others when incorporating these ideas into our respective SMT solvers.

Future work involves the definition of theory-dependent splitting heuristics, and the ways in which the T_i -solvers should communicate with the engine about these heuristics.

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