Splitting on Demand in SAT Modulo Theories

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Overview of the talk

- Introduction to SMT
  - Eager approach
  - Lazy approach: Boolean engine DPLL(X) + T-solver
- Inside the T-solver
  - What does DPLL(X) need from T-solver?
  - Splitting on Demand
- Use of Splitting on Demand for Nelson-Oppen
- Conclusions

Introduction to SMT

- Some problems are more naturally expressed in other logics than propositional logic, e.g:
  - Software verification needs reasoning about equality, arithmetic, data structures, ...
- SMT consists of deciding the satisfiability of a (ground) FO formula with respect to a background theory T
- Example (Equality with Uninterpreted Functions – EUF):

$$g(a)=c \quad \wedge \quad (\ f(g(a)) \neq f(c) \ \vee \ g(a)=d\ ) \quad \wedge \quad c \neq d$$

- Wide range of applications:
  - Predicate abstraction
  - Model checking
  - Equivalence checking
  - Static analysis
  - Scheduling
  - ...

SMT - Eager approach vs lazy approach

EAGER APPROACH:

- Methodology: translate problem into equisatisfiable propositional formula and use off-the-shelf SAT solver [Bryant, Velev, Pnueli, Lahiri, Seshia, Strichman, ...]
- Why "eager"? Search uses all theory information from the beginning
- Tools: UCLID [Lahiri, Seshia and Bryant]

LAZY APPROACH:

- Methodology: integration of a SAT-solver with a theory solver
- Why "lazy"? Theory information used lazily when checking T-consistency of propositional models
- Tools: CVC-Lite, Yices, MathSAT, TSAT+, Barcelogic ...

SMT - Lazy approach example

Consider EUF and

$$\underbrace{g(a)=c}_{1} \quad \wedge \quad (\underbrace{f(g(a))\neq f(c)}_{\overline{2}} \vee \underbrace{g(a)=d}_{3}) \quad \wedge \quad \underbrace{c\neq d}_{\overline{4}}$$

- Send $\{1, \ \overline{2}\vee 3, \ \overline{4}\}$ to SAT solver

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  - SAT solver returns model $[1,\ \ \overline{2},\ \ \overline{4}]$

SMT - Lazy approach example

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- Send $\{1,\ \overline{2} \lor 3,\ \overline{4}\}$ to SAT solver
  - SAT solver returns model $[1,\ \overline{2},\ \overline{4}]$
  - Theory solver says T-inconsistent

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- Send $\{1,\ \ \overline{2}\vee 3,\ \ \overline{4},\ \ \overline{1}\vee 2\vee 4,\ \ \overline{1}\vee\overline{2}\vee\overline{3}\vee 4\}$ to SAT solver
  - SAT solver detects it UNSATISFIABLE

SMT - Lazy approach optimizations

Several optimizations for enhancing efficiency:

- Check T-consistency only of full propositional models

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- Given a T-inconsistent assignment M, identify a T-inconsistent subset $M_0 \subseteq M$ and add $\neg M_0$ as a clause

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- Upon a T-inconsistency, add clause and restart

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- Upon a T-inconsistency, use the conflicting clause $\neg M_0$ to backjump to some point where the assignment was still T-consistent, as in SAT-solvers.

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What does DPLL(X) need from T-Solver?

- T-consistency check of a set of literals M, with:
  - Explain of T-inconsistency: find (small) T-inconsistent subset of M [minimal wrt. size?, wrt. $\subseteq$?]
  - Incrementality: if l is added to M, check for $M\,l$ faster than reprocessing $M\,l$ from scratch.
- Theory propagation: find input T-consequences of M, with:
  - Explain T-Propagate of l: find (small) subset of M that T-entails l (needed in conflict analysis).
- Backtrack n: undo last n literals added

> PAPER FOCUSES only on T-consistency checks

A T-Solver for EUF

- Theory solvers can usually be described using inference rules
- The input **conjunction** of literals $\mathcal{C}$ is saturated wrt the rules
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A congruence closure algorithm (a solver for EUF) can be described with the following rules:

Reflexitivy:

$$\frac{}{t = t}$$

Symmetry:

$$\frac{u = t}{t = u}$$

Transitivity:

$$\frac{t = u \quad u = v}{t = v}$$

Monotonicity:

$$\frac{t_1 = u_1 \quad \dots \quad t_n = u_n}{f(t_1, \dots, t_n) = f(u_1, \dots, u_n)}$$

Contradiction:

$$\frac{t = u \quad t \neq u}{\perp}$$

A T-solver for difference logic (in $\mathbb{R}$)

Atoms are of the form $x \bowtie y + d$, being x and y variables, d a real constant and $\bowtie \in \{<, \leq\}$, or of the form $x = y + d$.

Transitivity:

$$\frac{x \leq z + c \quad z \bowtie y + d}{x \bowtie y + (c + d)} \qquad \frac{x < z + c \quad z \bowtie y + d}{x < y + (c + d)}$$

Equality treatment:

$$\frac{x \leq y + c \quad y \leq x - c}{x = y + c} \qquad \frac{x = y + c}{x \leq y + c, y \leq x - c}$$

Contradiction:

$$\frac{x < x + c}{\perp} \;\; (\text{if } c \leq 0) \qquad \frac{x = y + c \qquad x \neq y + c}{\perp}$$

A T-solver for difference logic (in $\mathbb{Z}$)

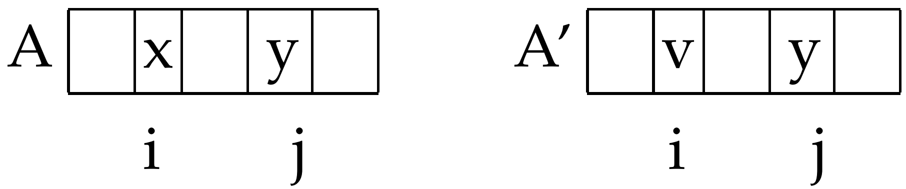
- Consider the unsatisfiable set of literals
$$\{1 \leq x - y,\ \ x - y \leq 2,\ \ x \neq y + 1,\ \ x \neq y + 2\}$$
- Saturation wrt the previous inference rules only adds
$$\{y \leq y + 1\}.$$
- To obtain a (refutationally) complete inference system:
  - Add splitting rule:
$$\frac{x \neq y + c}{x < y + c \qquad x > y + c}$$
  - Or add splitting rules of the form:
$$\frac{c \leq x - y \quad x - y \leq (c + k)}{x - y = c \qquad x - y = c + 1 \qquad \ldots \qquad x - y = c + k}$$
- This may give an exponential amount of work, but problem is NP-hard anyway.

Other theories requiring case-splitting

T-solvers requiring internal case-splitting are common:

- Theory of arrays:

$$\frac{read(write(A,i,v),j) = read(A,j)}{i \neq j \qquad\qquad i = j\,,\; read(A,j) = v}$$



- Fragments of set theory:

$$\frac{S_1 \neq S_2}{e \in S_1, e \notin S_2 \qquad e \notin S_1, e \in S_2}$$

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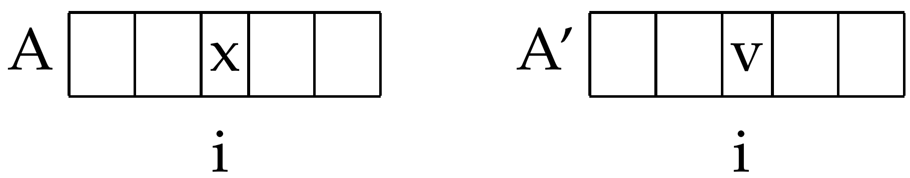

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Our proposal: splitting on demand

INFORMALLY:

- IDEA: pass theory case-splits to the DPLL engine as clauses
- BENEFITS:
  - Split-backtrack infrastructure is not duplicated
  - Allow flexibility in T-reasoning (cheap computations first)

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FORMALLY:

- Given initial state $\emptyset \parallel F$, consider $\mathcal{L}$ the **finite** set of all literals that might need case splitting.
- Modify T-Learn: also clauses with literals form $\mathcal{L}$ may be learned.
- $\mathcal{L}$ avoids termination problems (under certain conditions)
- T-solvers complete only when all atoms in $\mathcal{L}$ are decided

Example

Consider again Diff. Logic over $\mathbb{Z}$ and the formula:

$$\underbrace{x \leq y+1}_{1} \quad \wedge \quad (\underbrace{x < y}_{2} \vee \underbrace{x \neq y+1}_{\overline{3}}) \quad \wedge \quad \underbrace{x \neq y}_{\overline{4}}$$

$$\emptyset \parallel \quad 1, \;\; 2 \vee \overline{3}, \;\; \overline{4} \qquad\qquad \Rightarrow \qquad (\text{UnitPropagate x } 2)$$

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$$1\,\overline{4} \;\|\; 1,\; 2 \vee \overline{3},\; \overline{4} \quad \Rightarrow \quad (T\text{-Learn with } 5 \equiv x > y)$$

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\emptyset & \| & 1,\; 2 \vee \overline{3},\; \overline{4} & & \Rightarrow & (\text{UnitPropagate x } 2) \\
1\,\overline{4} & \| & 1,\; 2 \vee \overline{3},\; \overline{4} & & \Rightarrow & (T\text{-Learn with } 5 \equiv x > y) \\
1\,\overline{4} & \| & 1,\; 2 \vee \overline{3},\; \overline{4}, & 4 \vee 2 \vee 5 & \Rightarrow & (\text{Decide})
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1\ \overline{4}\ 5 & \| & 1,\ \ 2 \vee \overline{3},\ \ \overline{4},\ \ 4 \vee 2 \vee 5 & \Rightarrow & (\text{T-Propagate x 2}) \\
1\ \overline{4}\ 5\ \overline{2}\ 3 & \| & 1,\ \ 2 \vee \overline{3},\ \ \overline{4},\ \ 4 \vee 2 \vee 5 & \Rightarrow & (\text{Backjump})
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$1\ \overline{4}$	$\parallel$	$1,\ 2 \vee \overline{3},\ \overline{4}$	$\Rightarrow$	(T-Learn with $5 \equiv x > y$)
$1\ \overline{4}$	$\parallel$	$1,\ 2 \vee \overline{3},\ \overline{4},\ 4 \vee 2 \vee 5$	$\Rightarrow$	(Decide)
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Nelson-Oppen: combination of theories

- SMT problems usually involve more than one theory:

$$a = b+2 \;\wedge\; A = write(B, a+1, 4) \;\wedge\; (\, read(A, b+3) = 2 \vee f(a-1) \neq f(b+1)\,)$$

- Combination problem:

  INPUT:
  - Two theories T_1 and T_2.
  - A T_1-solver and a T_2-solver

  OUTPUT:
  - A $(T_1 \cup T_2)-$solver

- Nelson-Oppen provides a combination procedure if:
  - Theories are signature disjoint
  - Theories are stably-infinite

Nelson-Oppen: example

$$\Gamma = \{ f(f(x) - f(y)) \neq f(z), \quad x \leq y, \quad y + z \leq x, \quad z \geq 0 \}$$

1. Purify literals: introduce new variables
   $w_1 = f(x), w_2 = f(y), w_3 = w_1 - w_2$
2. Now we get
   $\Gamma_{\mathbb{R}} = \{x \leq y, y + z \leq x, z \geq 0, w_3 = w_1 - w_2\}$ and
   $\Gamma_{\mathbb{E}} = \{f(w_3) \neq f(z), w_1 = f(x), w_2 = f(y)\}$
   with shared variables $\{x, y, z, w_1, w_2, w_3\}$.
3. N-O: Γ SAT in the combined theory iff exists arrangement $\mathcal{A}$ (for each pair of shared variables, say whether they are equal or distinct) such that $\Gamma_{\mathbb{R}} \wedge \mathcal{A}$ is $T_{\mathbb{R}}$-SAT and $\Gamma_{\mathbb{E}} \wedge \mathcal{A}$ is $T_{\mathbb{E}}$-SAT.

Ideal implementation: T-solvers exchange entailed equations until fix point or unsatisfiability is detected by a single T-solver.

Nelson-Oppen: example(2)

$\Gamma_{\mathbb{R}} = \{x \leq y, \quad y + z \leq x, \quad z \geq 0, \quad w_3 = w_1 - w_2\}$

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with shared variables $\{x, y, z, w_1, w_2, w_3\}$.

Arrangement $\mathcal{A}$ (init. empty) is seen by both solvers:

- $T_{\mathbb{R}}$-solver detects $x = y$ is entailed (and added to $\mathcal{A}$)

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- $T_{\mathbb{E}}$-solver detects $\Gamma_{\mathbb{E}} \wedge \mathcal{A}$ is $T_{\mathbb{E}}$-unsat

But exchanging entailed equalities does not always suffice:

$$\Gamma_{\mathbb{Z}} = \{1 \leq x - y, \quad x - y \leq 2, \quad w_1 = y + 1, \quad w_2 = y + 2\}$$

$$\Gamma_{\mathbb{E}} = \{f(x) \neq f(w_1), \quad f(x) \neq f(w_2)\}$$

is UNSAT, but no equation is entailed.

Nelson-Oppen with non-convex theories

Why didn't it work with

$\Gamma_{\mathbb{Z}} = \{1 \leq x - y, \quad x - y \leq 2, \quad w_1 = y + 1, \quad w_2 = y + 2\}$

$\Gamma_{\mathbb{E}} = \{f(x) \neq f(w_1), \quad f(x) \neq f(w_2)\}$?

Nelson-Oppen with non-convex theories

Why didn't it work with

$\Gamma_{\mathbb{Z}} = \{1 \leq x - y, \quad x - y \leq 2, \quad w_1 = y + 1, \quad w_2 = y + 2\}$

$\Gamma_{\mathbb{E}} = \{f(x) \neq f(w_1), \quad f(x) \neq f(w_2)\}$?

- $\Gamma_{\mathbb{Z}}$ does not entail any equation between shared variables
- But $\Gamma_{\mathbb{Z}} \models_T \; x = w_1 \vee x = w_2$ (non-convex theory)
- For non-convex theories, DISJUNCTIONS of equalities should be communicated. Possibilities:
  - Send clauses from solver to solver
  - Force DPLL(X) to split on equalities between shared variables [DTC]
  - Send clauses from solvers to DPLL(X) only as necessary [DTC,Splitting on Demand]

Overview of the talk

- Introduction to SMT
  - Eager approach
  - Lazy approach: Boolean engine DPLL(X) + T-solver
- Inside the T-solver
  - What does DPLL(X) need from T-solver?
  - Splitting on Demand
- Use of Splitting on Demand for Nelson-Oppen
- Conclusions $\Leftarrow$

Conclusions

- Expensive theories easily dealt with the appropriate infrastructure
- This infrastructure allows greater flexibility
- Nelson-Oppen easily accommodated