Resolution in Propositional Logic

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Overview of the session

- Inference rules
  - Resolution
  - Ordered resolution
  - Practical remarks
Inference rules

- They allow one to deduce new formulas from given ones.

- Given an inference rule $R$ and a set of formulas $S$, we define:

  - The closure of $S$ under $R$, denoted $R(S)$, is the set of all formulas that can be obtained in zero or more deduction steps from $S$ using $R$.

  - More formally, for $i \geq 0$

    \[
    S_0 = S \\
    S_{i+1} = S_i \cup R_1(S_i)
    \]

    and $R(S) = \bigcup_{i=0}^{\infty} S_i$

    where $R_1(S_i)$ is the set of all formula obtained from $S_i$ in exactly one application of $R$. 

Resolution in Propositional Logic – p. 3
Inference rules - Closure

Resolution in Propositional Logic – p. 4
Inference rules - properties

- \( R \) is correct iff \( F \in R(S) \) implies \( S \models F \)
  - That is, the closure only contains logical consequences (but maybe not all of them)

- \( R \) is complete iff \( S \models F \) implies \( F \in R(S) \)
  - That is, the closure contains all logical consequences (but maybe something more)

- Ideally, we want correct and complete inference rules

- A weaker notion of completeness is refutational completeness:
  \[ S \text{ unsatisfiable} \implies \square \in R(S) \]

- If \( R \) is correct and refutationally completely, then
  \[ S \text{ unsatisfiable} \iff \square \in R(S) \]

**EXERCISE:** prove the last property
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The resolution inference rule is the following:

\[
p \lor C \quad \neg p \lor D \\
\hline
C \lor D
\]

We will see that:

- Resolution is **Correct**
- Not complete
- Refutationally complete

If \( S \) is a finite set of clauses, then \( Res(S) \) is also finite

Hence, given a set of clauses \( S \), its satisfiability is checked by:

1. Computing \( Res(S) \)
2. If \( \Box \in Res(S) \) Then UNSAT ; Else SAT
EXERCISE: prove that

- If $S$ is finite, $Res(S)$ is also finite
- Resolution is not complete
- Resolution is correct
- Resolution is refutationally complete
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Ordered resolution

- The proof of refutational completeness introduces ordered resolution
- Given clauses $S$ and a total ordering on the variables in $S$:
  \[ p_1 < p_2 < p_3 < \ldots \]

  we can define ordered resolution:
  \[
  \frac{p \lor C}{C \lor D} \quad \frac{\neg p \lor D}{C \lor D}
  \]

  if $p > q$ for all var. $q \in C \lor D$

- It is easy to see that:
  - If $S$ is finite, $ResOrd(S)$ is also finite
  - It is correct (because resolution is)
  - It is refutationally complete (same proof suffices)

- Hence, it is better from the practical point of view
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Practical Remarks

- In practice, even ordered resolution is not efficient enough
- SAT engines based on resolution not used in practice
- However, resolution plays a crucial role in DPLL