Algebraic Necessary Condition for Tractability of Valued CSP

joint work with Marcin Kozik

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MAX-CUT

\[ G = (V, E) \]

- a set of variables: \( V \)
- a set of their possible values: \( \{0, 1\} \)
- minimise: \( \sum_{(x,y) \in E} \varrho_{\text{XOR}}(x, y) \)

\[ \varrho_{\text{XOR}}(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \]
MAX-CUT

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*weighted relation*
MAX-CUT

\[ G = (V, E) \]

- a set of variables: \( V \)
- a set of their possible values: \( \{0, 1\} \)
- minimise: \( \sum_{(x,y) \in E} \varnothing_\text{XOR}(x, y) \)

\[ \varnothing_\text{XOR}(x, y) = \begin{cases} 
1 & \text{if } x = y \\
0 & \text{otherwise} 
\end{cases} \]
2-coloring

\[ G = (V, E) \]

- a set of variables: \( V \)
- a set of their possible values: \( \{0, 1\} \)
- minimise: \( \sum_{(x,y) \in E} \varrho \neq (x, y) \)

\[ \varrho \neq (x, y) = \begin{cases} 0 & \text{if } x \neq y \\ \infty & \text{otherwise} \end{cases} \]
VCSP Instance

An instance of the VCSP:

- a finite set of variables: $V = \{x_1, \ldots, x_n\}$
- a finite set of their possible values: $D$
- an objective function:

$$\varrho_1(x_{1,1}, \ldots, x_{1,m_1}) + \ldots + \varrho_k(x_{k,1}, \ldots, x_{k,m_k}).$$

**Goal:** find an assignment that minimises the function
Complexity

\( \Gamma \) - a fixed set of weighted relations called a *language*

\( \text{VCSP}(\Gamma) \) - the objective function is a sum of functions from \( \Gamma \)

- \( \text{VCSP}(\{ \rho_{\text{XOR}} \}) \)
- \( \text{VCSP}(\{ \rho_{\neq} \}) \)
Complexity

Γ - a fixed set of weighted relations called a *language*

VCSP(Γ) - the objective function is a sum of functions from Γ

- VCSP(\{\varrho_{XOR}\}) - NP-hard (MAX-CUT)
- VCSP(\{\varrho_{\neq}\}) - PTime (2-coloring)
\[ \Gamma \] - a fixed set of weighted relations called a *language*

\[ \text{VCSP}(\Gamma) \] - the objective function is a sum of functions from \( \Gamma \)

- \( \text{VCSP}(\{ \varrho_{\text{XOR}} \}) \) - NP-hard (MAX-CUT)
- \( \text{VCSP}(\{ \varrho_{\neq} \}) \) - PTime (2-coloring)

**Main goal:** classify the complexity of problems \( \text{VCSP}(\Gamma) \)
Valued Constraint Satisfaction Problems

- MAX-CUT
- Minimum Vertex Cover
- 3-coloring
- 2-SAT
Valued Constraint Satisfaction Problems

- MAX-CUT
- Minimum Vertex Cover
- 3-coloring
- 2-SAT
Constraint Satisfaction Problems

3-coloring

2-SAT
every weighted relation: \( \rho: D^n \rightarrow \{0, \infty\} \)
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objective function takes values 0 or $\infty$
every weighted relation: $\varrho : D^n \rightarrow \{0, \infty\}$

objective function takes values 0 or $\infty$

**Goal:** Is there an assignment with cost 0?
Constraint Satisfaction Problems

every weighted relation: \( \varrho : D^n \rightarrow \{0, \infty\} \)

objective function takes values 0 or \( \infty \)

goal: Is there an assignment with cost 0?

3-coloring

2-SAT

effectively a relation

a satisfying assignment

Joanna Ochremiak Algebraic Necessary Condition for Tractability of Valued CSP,
Constraint Satisfaction Problems

[Bulatov, Krokhin, Jeavons]

Algebraic necessary condition for tractability of valued CSP,
Constraint Satisfaction Problems

[Bulatov, Krokhin, Jeavons]

NP-complete

algebraic condition
Constraint Satisfaction Problems

[Bulatov, Krokhin, Jeavons]

algebraic condition

NP-complete

PTime?

Algebraic Dichotomy Conjecture
Valued Constraint Satisfaction Problems

- Maximum Cut (MAX-CUT)
- Minimum Vertex Cover
- 3-coloring
- 2-SAT

Algebraic condition
Valued Constraint Satisfaction Problems

NP-hard

algebraic condition
Valued Constraint Satisfaction Problems

NP-hard

PTime?

algebraic condition

Dichotomy Conjecture
Polymorphisms

\[ \varrho : D^n \rightarrow \mathbb{Q} \cup \{\infty\} \quad \text{such that} \quad \varrho(x_1) < \infty \]

\[ f : D^k \rightarrow D \quad \text{such that} \quad \varrho(f(x_1, \ldots, x_k), \ldots, f(x_1, \ldots, x_k)) < \infty \]
polymorphism of $\varrho$

$$\varrho : D^n \to \mathbb{Q} \cup \{\infty\}$$

$$f : D^k \to D$$

$x_1 = (x_1^1, \ldots, x_1^n)$ such that $\varrho(x_1) < \infty$

$\vdots$

$x_k = (x_k^1, \ldots, x_k^n)$ such that $\varrho(x_k) < \infty$

$$\varrho(f(x_1^1, \ldots, x_k^1), \ldots, f(x_1^n, \ldots, x_k^n)) < \infty$$
Weighted polymorphisms

Polymorphisms characterize the complexity of CSP.
Weighted polymorphisms

Polymorphisms characterize the complexity of CSP.

_weighted polymorphism_ - probability distribution on the set of polymorphisms

**Theorem** [Cohen, Cooper, Creed, Jeavons, Živný]

Weighted polymorphisms characterize the complexity of VCSP.
Cyclic polymorphism

cyclic polymorphism - for every \( x_1, \ldots, x_k \in D \)

\[
f(x_1, x_2, \ldots, x_k) = f(x_2, \ldots, x_k, x_1)
\]
Constraint Satisfaction Problems

3-coloring

2-SAT

algebraic condition
Constraint Satisfaction Problems

[Barto, Kozik]

3-coloring

2-SAT

no cyclic polymorphism
Valued Constraint Satisfaction Problems

- MINIMUM VERTEX COVER
- MAX-CUT
- 3-COLOURING
- 2-SAT

algebraic condition
Valued Constraint Satisfaction Problems

- Minimum Vertex Cover
- MAX-CUT
- 3-coloring
- 2-SAT

- no cyclic polymorphism
- has positive probability

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Algebraic Necessary Condition for Tractability of Valued CSP,
Valued Constraint Satisfaction Problems

- NP-hard
- PTime?
- No cyclic polymorphism
- Has positive probability

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Algebraic Necessary Condition for Tractability of Valued CSP,
Thank you