Deciding FO-definable CSP instances

joint work with Bartek Klin, Eryk Kopczyński
and Szymon Toruńczyk

Joanna Ochremiak

University of Warsaw (moving to UPC Barcelona)

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Atoms

\[ \mathcal{A} = \{a, b, c, \ldots\} \text{ - countably infinite set of } \textit{atoms} \]
**Graph colorability**

\( G \) - an **infinite**, undirected graph:

- vertices indexed by ordered pairs of distinct atoms: \( x_{ab}, x_{ad}, \ldots \)
- edges: \( x_{ab} - x_{bc}, \) where \( a \) and \( c \) are distinct

Subgraph of \( G \):

![Subgraph diagram]

**Question:** Is the infinite graph \( G \) three-colorable?
Systems of linear equations over $\mathbb{Z}_2$

$E$ - an **infinite** system of linear equations over $\mathbb{Z}_2$

- variables indexed by ordered pairs of distinct atoms: $x_{ab}, x_{ad}, ...$
- equations:

$$x_{ab} + x_{ba} = 1, \text{ where } a \text{ and } b \text{ are distinct}$$
$$x_{ab} + x_{bc} + x_{ca} = 0, \text{ where } a, b \text{ and } c \text{ are distinct}$$

**Question:** Does the system $E$ have a solution?
Systems of linear equations over $\mathbb{Z}_2$

\[
\begin{align*}
ab + ba & = 1 \\
ab + bc + ca & = 0 \\
ba + ac + cb & = 0 \\
bc + cd + db & = 0 \\
ca + ae + ec & = 0 \\
ac + cd + da & = 0 \\
bc + be + ec & = 0 \\
db + be + ed & = 0 \\
ae + ed + da & = 0 \\
0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 1
\end{align*}
\]
A CSP instance $\mathcal{I} = (V, T, C)$:

- a set of variables: $V = \{x, y, \ldots\}$
- a set of their possible values: $T$
- a set of constraints: $C$
Constraint Satisfaction Problem

$G$ - an infinite, undirected graph:
- vertices indexed by ordered pairs of distinct atoms: $x_{ab}, x_{ad}, \ldots$
- edges: $x_{ab} — x_{bc}$, where $a$ and $c$ are distinct

**Question:** Is this graph three-colorable?

$\Pi_G$ - a CSP instance:
- variables: vertices $V = \{x_{ab} \mid a, b \in A \text{ distinct}\}$
- values: possible colors $T = \{1, 2, 3\}$
- constraints: $C = \{((x_{ab}, x_{bc}), R) \mid a, b, c \in A \text{ distinct}\}$

For each edge $x_{ab} — x_{bc}$ there is a constraint: $((x_{ab}, x_{bc}), R)$

$R = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$.

**Question:** Is there a solution?
Classical Constraint Satisfaction Problem

\[ \mathbb{T} = (T, R_1, R_2, \ldots, R_n) - \text{a fixed finite template} \]

**Problem:** CSP\(_{\text{fin}}(\mathbb{T})

**Input:** a finite CSP instance \( \mathbb{I} \) over \( \mathbb{T} \)

**Decide:** Does \( \mathbb{I} \) have a solution?

What kind of instances do we consider?
Definable instances

- variables indexed by tuples of atoms
- constraints defined by a first-order formula over $(\mathbb{A}, =)$

Set of variables in $\mathbb{I}_G$:
$\{x_{ab} \mid a, b \in \mathbb{A}, a \neq b\}$. 
Definable instances

- variables indexed by tuples of atoms
- constraints defined by a first-order formula over $(\mathbb{A}, =)$

Set of variables in $\mathbb{I}_G$:
\[ \{x_{ab} \mid a, b \in \mathbb{A}, a \neq b\}. \]

Set of constraints in $\mathbb{I}_G$:
\[ \{((x_{ab}, x_{bc}), R) \mid a, b, c \in \mathbb{A}, a \neq b \land a \neq c \land b \neq c\}. \]
\[ \mathbb{T} = (T, R_1, R_2, \ldots, R_n) - \text{a fixed finite template} \]

**Problem:** \( \text{CSP}_{inf}(\mathbb{T}) \)

**Input:** a **definable** CSP instance \( \mathbb{I} \) over \( \mathbb{T} \)

**Decide:** Does \( \mathbb{I} \) have a solution?
Theorem. If $\text{CSP}_{\text{fin}}(\mathbb{T})$ is $C$-complete then $\text{CSP}_{\text{inf}}(\mathbb{T})$ is $\text{Exp}(C)$-complete.
Complexity

3-colorability of finite graphs – NP-complete

\[\downarrow\]

3-colorability of definable graphs – \(\text{NExp}\text{-complete}\)
CSP_{inf}(\mathbb{T}) is decidable

**Theorem.** It is decidable whether a definable instance \( \mathcal{I} \) over a finite template \( \mathbb{T} \) has a solution.

Uses Ramsey theorem and topological dynamics.

**Proof idea:** Look for regular solutions.
Atom permutations

$\text{Aut}(\mathbb{A}, =)$ acts on set of variables in $\mathbb{I}_G$: 
\[ \{x_{ab} \mid a, b \in \mathbb{A}, a \neq b\}. \]

$\pi$ - a permutation of atoms 
$\pi(x_{ab}) = x_{\pi(a)\pi(b)}$

\[
\begin{align*}
  a & \mapsto b \\
  b & \mapsto c \\
  c & \mapsto a
\end{align*}
\]
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\[\begin{align*}
a & \mapsto b \\
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\end{align*}\]
Invariant assignments

\[ \text{Aut}(\mathbb{A}, =) \text{ acts on the set of assignments } f : V \to T \]

\[ f \quad x \mapsto t \]

\[ \pi \cdot f \quad \pi(x) \mapsto t \]

fixpoint \leftrightarrow \textit{invariant} assignment
In invariant assignments

\[ x_{ab} + x_{ba} = 1, \text{ where } a \text{ and } b \text{ are distinct} \]
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\[ x_{ab} + x_{ba} = 1, \text{ where } a \text{ and } b \text{ are distinct} \]

There is no invariant solution.
Monotone-invariant assignments

Fix a linear order on atoms $(\mathbb{A}, \leq)$ isomorphic to $(\mathbb{Q}, \leq)$.

$\text{Aut}(\mathbb{A}, \leq)$ acts on the set of assignments $f : V \rightarrow T$

fixpoint $\leftrightarrow \text{monotone-invariant} \text{ assignment}$
Monotone-invariant assignments

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Monotone-invariant assignments

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\[ \text{Aut}(\mathbb{A}, \leq) \]

\[ e < b < a < c < d \]
Monotone-invariant assignments

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\[ \text{Aut(A, } \leq \text{)} \]

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Monotone-invariant assignments

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\[ \text{Aut}(\mathbb{A}, \leq) \]

\[ e < b < a < c < d \]
Monotone-invariant assignments

- There are finitely many monotone-invariant assignments $f : V \to T$.
- Monotone-invariant assignments $f : V \to T$ can be represented in a finite way (by first order formulas using $\leq$).

**Fact.** It is decidable whether a definable instance $\mathcal{I}$ over a finite template $\mathcal{T}$ has a monotone-invariant solution.
CSP\(^{\inf}\)(\(\mathbb{T}\)) is decidable

**Theorem.** A definable instance \(\mathbb{I}\) has a solution if and only if it has a monotone-invariant solution.

**Theorem [Pestov].** Every continuous action of the topological group Aut(\(\mathbb{Q}, \leq\)) on a nonempty compact space has a fixpoint.
CSP_{inf}(\mathbb{T}) is decidable

**Theorem.** A definable instance $I$ has a solution if and only if it has a monotone-invariant solution.

**Proof.**
Sol($I$, $\mathbb{T}$) – the set of solutions (possibly empty)

**Theorem [Pestov].** Every continuous action of the topological group Aut($\mathbb{Q}$, $\leq$) on a nonempty compact space has a fixpoint.
CSP\textsubscript{inf}(\mathbb{T}) \text{ is decidable}

**Theorem.** A definable instance \( \mathbb{I} \) has a solution if and only if it has a monotone-invariant solution.

**Proof.**

Sol(\( \mathbb{I}, \mathbb{T} \)) – the set of solutions (possibly empty)

\text{Aut}(\mathbb{A}, \leq) \text{ acts on Sol}(\mathbb{I}, \mathbb{T}) \text{ (solutions are mapped to solutions)}

**Theorem [Pestov].** Every continuous action of the topological group Aut(\( \mathbb{Q}, \leq \)) on a nonempty compact space has a fixpoint.
CSP_{inf}(\mathbb{T}) is decidable

**Theorem.** A definable instance $\mathbb{I}$ has a solution if and only if it has a monotone-invariant solution.

**Proof.**
Sol($\mathbb{I}, \mathbb{T}$) – the set of solutions (possibly empty)
Aut($\mathbb{A}, \leq$) acts on Sol($\mathbb{I}, \mathbb{T}$) (solutions are mapped to solutions)
Sol($\mathbb{I}, \mathbb{T}$) $\subseteq \mathbb{T}^\mathbb{I}$ – a compact space

**Theorem [Pestov].** Every continuous action of the topological group Aut($\mathbb{Q}, \leq$) on a nonempty compact space has a fixpoint.
Corollary. It is decidable whether a definable instance $I$ over a finite template $T$ has a solution.
<table>
<thead>
<tr>
<th>Complexity</th>
<th>Complexity</th>
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<tbody>
<tr>
<td>C</td>
<td>Exp(C)</td>
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<tr>
<td>P</td>
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<td>NP</td>
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<td>L</td>
<td>PSpace</td>
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**Theorem.** If CSP\(_{\text{fin}}(\mathbb{T})\) is C-complete then CSP\(_{\text{inf}}(\mathbb{T})\) is Exp(C)-complete.
Locally Finite Constraint Satisfaction Problem

A template $\mathcal{T} = \{T, R_1, R_2, \ldots\}$ is \textit{locally finite} is every relation of $\mathcal{T}$ is finite.
Locally Finite Constraint Satisfaction Problem

\[ T = \{ T, R_1, R_2, \ldots \} \] - locally finite, definable template

**Problem:** \( \text{CSP}_{inf}(T) \)

**Input:** a definable CSP instance \( \Pi \) over \( T \)

**Decide:** Does \( \Pi \) have a solution?

**Theorem.** For any definable, locally finite template \( T \), it is decidable whether a given definable instance \( \Pi \) over \( T \) has a solution.

**Open:** What about definable instances over arbitrary definable templates?
Locally Finite Constraint Satisfaction Problem

\[ \mathbb{T} = \{ T, R_1, R_2, \ldots \} - \text{locally finite, definable template} \]

**Problem:** \( \text{CSP}_{\text{fin}}(\mathbb{T}) \)

**Input:** a finite CSP instance \( \mathcal{I} \) over \( \mathbb{T} \)

**Decide:** Does \( \mathcal{I} \) have a solution?
Generalized graph colorability

$G$ - a finite, undirected graph

We treat atoms as colors.

To each vertex we assign a set of $n$ possible colors.

\[
\{a \ b\} \quad \{b \ c\} \\
\{a \ c\} \quad \{d \ e\} \\
\{b \ c\} \quad \{d \ e\}
\]

**Question**: Can this graph be colored with atoms such that no two adjacent vertices share the same color?
Generalized graph colorability

$G$ - a finite, undirected graph

We treat atoms as colors.

To each vertex we assign a set of $n$ possible colors.

Question: Can this graph be colored with atoms such that no two adjacent vertices share the same color?
Locally Finite Constraint Satisfaction Problem

\[ T = \{ T, R_1, R_2, \ldots \} - \text{locally finite, definable template} \]

**Problem:** CSP\(_{\text{fin}}(T)\)

**Input:** a finite CSP instance \( \mathcal{I} \) over \( T \)

**Decide:** Does \( \mathcal{I} \) have a solution?

Obviously decidable.

What about the complexity?
Bounded width

Theorem [Larose, Zádori; Barto, Kozik] A finite template $\mathbb{T}$ has bounded width (solvable in Datalog) if and only if an instance $I_{bw}^\mathbb{T}$ over $\mathbb{T}$ has a solution.

$I_{bw}^\mathbb{T}$ has a solution iff $\mathbb{T}$ has certain polymorphisms.
**Corollary.** A locally finite template $\mathbb{T}$ has bounded width (so-solvable in Datalog) if and only if an instance $\Pi_{\mathbb{T}}^{bw}$ over $\mathbb{T}$ has a solution.

\[ \Pi_{\mathbb{T}}^{bw} \text{ is a definable instance computable from } \mathbb{T} \]
\[ \Downarrow \]
Effective characterization of locally finite templates of bounded width.
Thank you