Mathematical Logic and Linguistics

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BGSMath Course
Class 3

Tree-based hypersequent calculus

Tree-based hypersequent calculus

We shall motivate, present, illustrate and analyse a conservative extension of the Lambek calculus called *displacement calculus* (Morrill & Valentín 2010; Morrill, Valentín & Fadda 2011).

Recall Lambek calculus

Recall Lambek calculus

Logic of strings

Recall Lambek calculus

Logic of strings

$$\boxed{\quad\alpha\quad} + \boxed{\quad\beta\quad}$$

Recall Lambek calculus

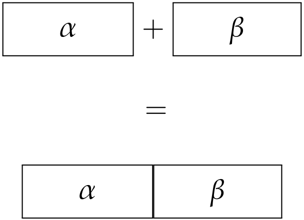
Logic of strings

$$\boxed{\alpha} + \boxed{\beta}$$

$$=$$

Recall Lambek calculus

Logic of strings



Syntactic types

The set $\mathcal{F}$ of types is defined in terms of a set $\mathcal{P}$ of primitive types by:

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$$\mathcal{F} \quad := \quad \mathcal{P}$$

$$\begin{array}{lllrcll}
\mathcal{F} & ::= & \mathcal{F}/\mathcal{F} & T(C/B) & = & T(B)\to T(C) & \text{over} \\
\mathcal{F} & ::= & \mathcal{F}\backslash\mathcal{F} & T(A\backslash C) & = & T(A)\to T(C) & \text{under} \\
\mathcal{F} & ::= & \mathcal{F}\bullet\mathcal{F} & T(A\bullet B) & = & T(A)\& T(B) & \text{continuous product} \\
\mathcal{F} & ::= & I & T(I) & = & \top & \text{continuous unit}
\end{array}$$

Syntactical interpretation

$$
\begin{aligned}
[[C/B]] &= \{s_1 |\ \forall s_2 \in [[B]], s_1{+}s_2 \in [[C]]\} \\
[[A\backslash C]] &= \{s_2 |\ \forall s_1 \in [[A]], s_1{+}s_2 \in [[C]]\} \\
[[A{\bullet}B]] &= \{s_1{+}s_2 |\ s_1 \in [[A]]\ \&\ s_2 \in [[B]]\} \\
[[I]] &= \{0\}
\end{aligned}
$$

Sequents

The set $\mathcal{O}$ of configurations is defined by the following, where Λ is the empty configuration:

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A sequent has the form:

$$\mathcal{O} \Rightarrow \mathcal{F}$$

Sequent calculus

The logical rules are as follows, where $\Delta(\Gamma)$ signifies context configuration Δ with a distinguished subconfiguration Γ.

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$$\frac{\Gamma \Rightarrow B \qquad \Delta(C) \Rightarrow D}{\Delta(C/B, \Gamma) \Rightarrow D}\,/L \qquad \frac{\Gamma, B \Rightarrow C}{\Gamma \Rightarrow C/B}\,/R$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta(C) \Rightarrow D}{\Delta(\Gamma, A\backslash C) \Rightarrow D}\,\backslash L \qquad \frac{A, \Gamma \Rightarrow C}{\Gamma \Rightarrow A\backslash C}\,\backslash R$$

$$\frac{\Delta(A, B) \Rightarrow D}{\Delta(A\bullet B) \Rightarrow D}\,\bullet L \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A\bullet B}\,\bullet R$$

$$\frac{\Delta(\Lambda) \Rightarrow A}{\Delta(I) \Rightarrow A}\,IL \qquad \frac{}{\Lambda \Rightarrow I}\,IR$$

Descriptive inadequacy of Lambek calculus

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Discontinuous idioms

Descriptive inadequacy of Lambek calculus

Discontinuous idioms

- Mary gave the man the cold shoulder

Descriptive inadequacy of Lambek calculus

Discontinuous idioms

- Mary gave the man the cold shoulder

Medial relativisation

Descriptive inadequacy of Lambek calculus

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- the man that Mary saw today

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Cross serial dependencies . . .

Displacement calculus

Displacement calculus

Logic of strings with holes

Displacement calculus

Logic of strings with holes — append and plug

Displacement calculus

Logic of strings with holes — append and plug

$$\boxed{\quad\alpha\quad} + \boxed{\quad\beta\quad}$$

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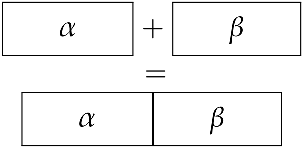

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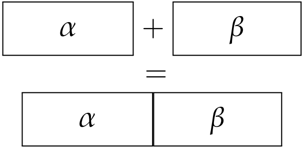

append $+ : L_i, L_j \to L_{i+j}$

Displacement calculus

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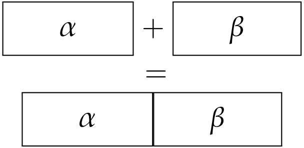

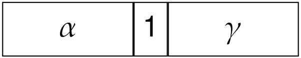

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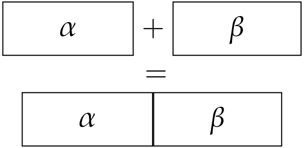

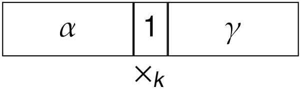

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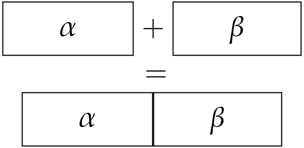

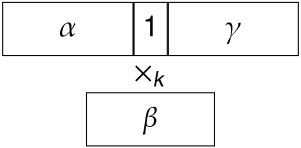

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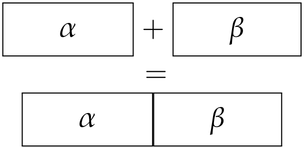
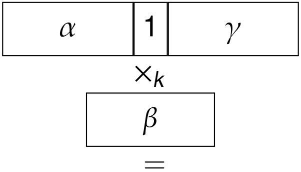
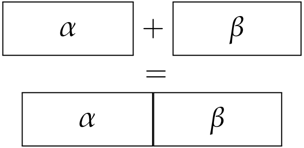


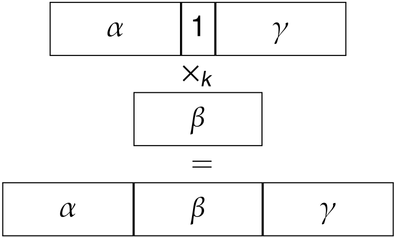


append $+ : L_i, L_j \rightarrow L_{i+j}$

Displacement calculus

Logic of strings with holes — append and plug





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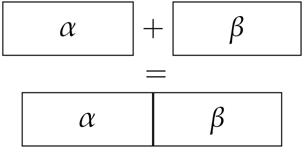

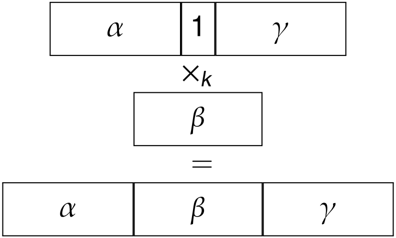

append $+ : L_i, L_j \to L_{i+j}$ plug $\times_k : L_{i+1}, L_j \to L_{i+j}$

Syntactic types

The syntactic types are sorted $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$ according to the number of holes $0, 1, 2, \ldots$ their expressions contain.

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The sets $\mathcal{F}_i$ of types of sort i are defined in terms of sets $\mathcal{P}_i$ of primitive types of sort i by:

$$
\begin{array}{rcl}
\mathcal{F}_i & ::= & \mathcal{P}_i
\end{array}
$$

$$
\begin{array}{rcllcll}
\mathcal{F}_i & ::= & \mathcal{F}_{i+j}/\mathcal{F}_j & T(C/B) & = & T(B) \to T(C) & \text{over} \\
\mathcal{F}_j & ::= & \mathcal{F}_i \backslash \mathcal{F}_{i+j} & T(A\backslash C) & = & T(A) \to T(C) & \text{under} \\
\mathcal{F}_{i+j} & ::= & \mathcal{F}_i \bullet \mathcal{F}_j & T(A \bullet B) & = & T(A) \& T(B) & \text{continuous product} \\
\mathcal{F}_0 & ::= & I & T(I) & = & \top & \text{continuous unit}
\end{array}
$$

$$
\begin{array}{rcllcll}
\mathcal{F}_{i+1} & ::= & \mathcal{F}_{i+j} \uparrow_k \mathcal{F}_j & T(C \uparrow_k B) & = & T(B) \to T(C) & \text{circumfix} \\
\mathcal{F}_j & ::= & \mathcal{F}_{i+1} \downarrow_k \mathcal{F}_{i+j} & T(A \downarrow_k C) & = & T(A) \to T(C) & \text{infix} \\
\mathcal{F}_{i+j} & ::= & \mathcal{F}_{i+1} \odot_k \mathcal{F}_j & T(A \odot_k B) & = & T(A) \& T(B) & \text{discontinuous product} \\
\mathcal{F}_1 & ::= & J & T(J) & = & \top & \text{discontinuous unit}
\end{array}
$$

Syntactical interpretation

$$
\begin{array}{rcl}
[[C/B]] & = & \{s_1 |\ \forall s_2 \in [[B]], s_1 + s_2 \in [[C]]\} \\
[[A\backslash C]] & = & \{s_2 |\ \forall s_1 \in [[A]], s_1 + s_2 \in [[C]]\} \\
[[A \bullet B]] & = & \{s_1 + s_2 |\ s_1 \in [[A]]\ \&\ s_2 \in [[B]]\} \\
[[I]] & = & \{0\} \\
 & & \\
[[C \uparrow_k B]] & = & \{s_1 |\ \forall s_2 \in [[B]], s_1 \times_k s_2 \in [[C]]\} \\
[[A \downarrow_k C]] & = & \{s_2 |\ \forall s_1 \in [[A]], s_1 \times_k s_2 \in [[C]]\} \\
[[A \odot_k B]] & = & \{s_1 \times_k s_2 |\ s_1 \in [[A]]\ \&\ s_2 \in [[B]]\} \\
[[J]] & = & \{1\}
\end{array}
$$

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$$\begin{array}{lcl} \mathcal{O} & ::= & \Lambda \mid \mathcal{T}, \mathcal{O} \\ \mathcal{T} & ::= & 1 \mid \mathcal{F}_0 \mid \mathcal{F}_{i>0}\{\underbrace{\mathcal{O} : \ldots : \mathcal{O}}_{i\,\mathcal{O}'\mathrm{s}}\} \end{array}$$

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Where A is a type, sA is its sort.

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Where Γ is a configuration, its sort $s\Gamma$ is the number of holes (1's) it contains.

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Sequents Σ are defined by:

$$\mathcal{O} \Rightarrow \mathcal{F} \text{ such that } s\mathcal{O} = s\mathcal{F}$$

Sequent calculus

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The figure $\overrightarrow{A}$ of a type A is defined by:

Sequent calculus

The figure $\overrightarrow{A}$ of a type A is defined by:

$$\overrightarrow{A} = \begin{cases} A & \text{if } sA = 0 \\ A\{\underbrace{1 : \ldots : 1}_{sA\ 1\text{'s}}\} & \text{if } sA > 0 \end{cases}$$

Where Γ is a configuration of sort i and $\Delta_1, \ldots, \Delta_i$ are configurations, the *fold* $\Gamma \otimes \langle \Delta_1, \ldots, \Delta_i \rangle$ is the result of replacing the successive holes in Γ by $\Delta_1, \ldots, \Delta_i$ respectively.

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Where Γ is of sort i, the notation $\Delta\langle\Gamma\rangle$ abbreviates $\Delta_0(\Gamma \otimes \langle \Delta_1, \ldots, \Delta_i \rangle)$, i.e. a context configuration Δ (which is externally Δ_0 and internally $\Delta_1, \ldots, \Delta_i$) with a potentially discontinuous distinguished subconfiguration Γ.

Continuous logical rules

$$\frac{\Gamma \Rightarrow B \qquad \Delta\langle \overrightarrow{C} \rangle \Rightarrow D}{\Delta\langle \overrightarrow{C/B}, \Gamma \rangle \Rightarrow D} /L \qquad \frac{\Gamma, \overrightarrow{B} \Rightarrow C}{\Gamma \Rightarrow C/B} /R$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta\langle \overrightarrow{C} \rangle \Rightarrow D}{\Delta\langle \Gamma, \overrightarrow{A\backslash C} \rangle \Rightarrow D} \backslash L \qquad \frac{\overrightarrow{A}, \Gamma \Rightarrow C}{\Gamma \Rightarrow A\backslash C} \backslash R$$

$$\frac{\Delta\langle \overrightarrow{A}, \overrightarrow{B} \rangle \Rightarrow D}{\Delta\langle \overrightarrow{A \bullet B} \rangle \Rightarrow D} \bullet L \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A \bullet B} \bullet R$$

$$\frac{\Delta\langle \Lambda \rangle \Rightarrow A}{\Delta\langle \overrightarrow{I} \rangle \Rightarrow A} IL \qquad \frac{}{\Lambda \Rightarrow I} IR$$

Where Δ is a configuration of sort $i > 0$ and Γ is a configuration, the kth metalinguistic wrap $\Delta\,|_k\,\Gamma$, $1 \leq k \leq i$, is given by:

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$$\Delta\,|_k\,\Gamma =_{df} \Delta \otimes \langle \underbrace{1, \ldots, 1}_{k-1\ 1\text{'s}}, \Gamma, \underbrace{1, \ldots, 1}_{i-k\ 1\text{'s}} \rangle$$

i.e. $\Delta\,|_k\,\Gamma$ is the configuration resulting from replacing by Γ the kth hole in Δ.

Discontinuous logical rules

Discontinuous logical rules

$$\frac{\Gamma \Rightarrow B \qquad \Delta\langle \overrightarrow{C} \rangle \Rightarrow D}{\Delta\langle \overrightarrow{C{\uparrow_k}B} \,|_k\, \Gamma \rangle \Rightarrow D} \uparrow_k L \qquad \frac{\Gamma \,|_k\, \overrightarrow{B} \Rightarrow C}{\Gamma \Rightarrow C{\uparrow_k}B} \uparrow_k R$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta\langle \overrightarrow{C} \rangle \Rightarrow D}{\Delta\langle \Gamma \,|_k\, \overrightarrow{A{\downarrow_k}C} \rangle \Rightarrow D} \downarrow_k L \qquad \frac{\overrightarrow{A} \,|_k\, \Gamma \Rightarrow C}{\Gamma \Rightarrow A{\downarrow_k}C} \downarrow_k R$$

$$\frac{\Delta\langle \overrightarrow{A} \,|_k\, \overrightarrow{B} \rangle \Rightarrow D}{\Delta\langle \overrightarrow{A{\odot_k}B} \rangle \Rightarrow D} \odot_k L \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1 \,|_k\, \Gamma_2 \Rightarrow A{\odot_k}B} \odot_k R$$

$$\frac{\Delta\langle 1 \rangle \Rightarrow A}{\Delta\langle \overrightarrow{J} \rangle \Rightarrow A} JL \qquad \frac{}{1 \Rightarrow J} JR$$

Examples

Mary gave the man the cold shoulder

- *gave*+1+*the*+*cold*+*shoulder*: $(N\backslash S)\uparrow N$

the man Mary saw today

- *that*: $(CN\backslash CN)/((S\uparrow N)\odot I)$