3. Recursion, part 1

Programming and Algorithms II

Degree in Bioinformatics

Fall 2018

```
>>> f(1)
#
>>>
```

```
>>> f(2)
#
##
#
>>>
```

```
>>> f(3)
#
##
#
###
#
##
#
>>>
```

```
>>> f(4)          (TRY GUESSING!)
#
##
#
###
#
##
#
####
#
##
#
###
#
##
#
>>>
```

A recursive function

```
def f(n):
    if n == 1:
        print("#")
    else:
        f(n-1)
        print("#"*n)
        f(n-1)
```

A recursive function

```
def f(n):
    if n > 0:
        f(n-1)
        print("#"*n)
        f(n-1)
```

Recursive factorial

n! = n * (n-1) * (n-2) * … * 3 * 2 *1

But also:

n! = n * (n-1)!

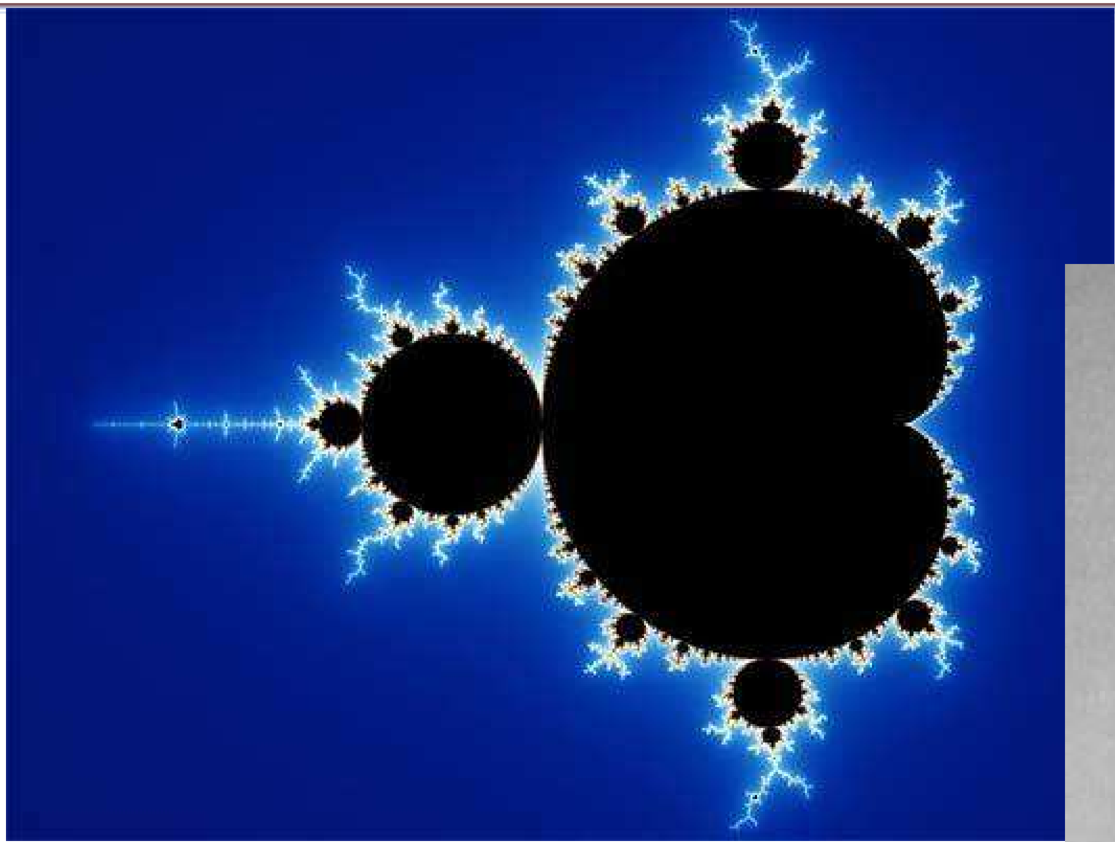
0! = 1

- "…" leads to a loop
- defining a function in terms of itself leads to recursion

Recursive factorial

```
def factorial(n):
    "returns n!, for any natural n>=0"
    if n == 0:
        return 1
    else:
        return n * factorial(n-1)
```

Defining in terms of itself?



Mandelbrot's set

Emmy Noether (1892-1935)

Base and recursive cases

Every recursive function must have:

- One or more base cases (no more calls)
- One or more recursive cases (more calls)

Conditions:

- If “size” is <=0 we must be in a base case
- A recursive call must decrease “size” by at least 1

Base and recursive cases

Size decreases by 1 or more at each call

When size is <= 0, we are in base case

This guarantees termination by the following property of natural numbers:

Every strictly decreasing sequence of natural numbers is finite

Not true for integers and real numbers

Base and recursive cases

n! = (n+1)! / (n+1)   True but not good

n! = n * (n-1) * (n-2)!   Correct but watch out:

```
def fact(n):
    if (n == 0): return 1
    else: return n*(n-1)*fact(n-2)
```

Computing integer powers

pow(x,y) = x * x * x * … * x (y times)

= x * pow(x,y-1) (assume x nonzero, y>0; 0^0 undef.)

Base case?

Running time O(y)

Observation:

$$x^{(2y)} = (x^2)^y$$

pow(x,y) = pow(x*x,y//2) if y is even

Slow powering

```
def pow(x,y):
    if y == 0:
        return 1
    else:
        return x * pow(x,y-1)
```

y decreases by 1 at each call

running time O(y)

Fast powering

```
def pow(x,y):
    if y == 0:
        return 1
    elif y % 2 == 1:
        return x * pow(x,y-1)
    else:
        return pow(x*x,y//2)
```

y is divided by 2 every 2 calls

2 log y calls maximum, O(1) ops per call

running time O(log y)

(alternatively, y in binary loses 1 bit at every /2)

Finding a zero of a continuous function

Given:

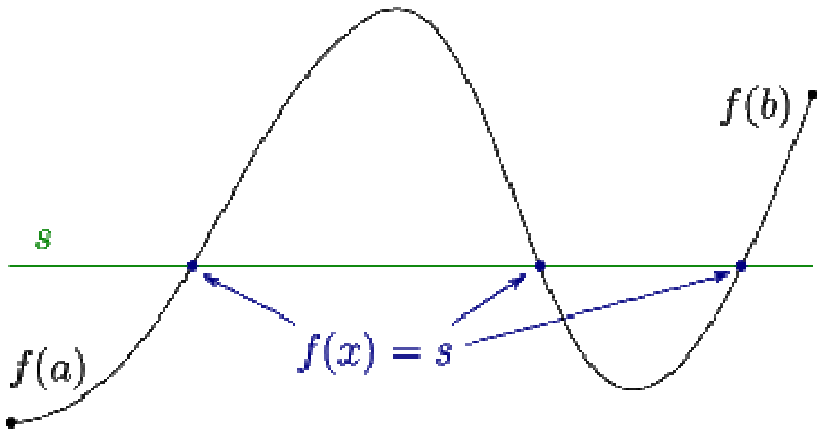
- a function f promised to be continuous
- a range [a,b] such that f(a) < 0 < f(b)
- a margin epsilon

Compute:

- a value x in [a,b] such that f has a zero in [x-epsilon,x+epsilon]

Such x exists

by Bolzano's theorem



Source: https://en.wikipedia.org/wiki/Intermediate_value_theorem

Finding zeros

```
def f(x):   return x*x – 2
>>> print(solver(f,0,4,0.000001))
1.4142136573791504

import math
>>>print(solver(math.sin,1,4,0.000001))
3.1415926218032837
>>>print(solver(math.cos,0,4,0.000001))
1.5707964897155762
```

Finding a zero of a continuous function

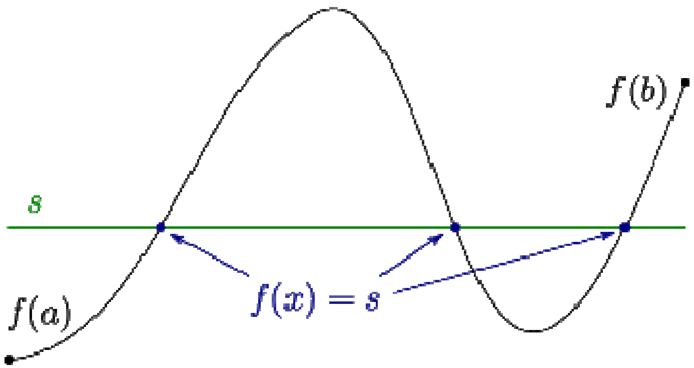
def solver(f, a, b, epsilon)

We keep the promise that f(a) and f(b) have different signs. So there must be a point in [a,b] where f(a) = 0

Similar to binary search:

Check the sign of f((a+b)/2)

Discard half the interval



Termination:

what decreases by at least 1 at each call?

Finding a zero of a continuous function

```
def solver(f, a, b, epsilon):
    c = (a+b)/2
    if abs(b-a) <= epsilon:
        return c
    else:
        if f(c) * f(a) < 0:
           return solver(f,a,c,epsilon)
        else:
           return solver(f,c,b,epsilon)
```

If we hit f(c) == 0, is there a problem?

Termination

Suppose that |b-a| is approx epsilon * 2^k

Then |b-a| at the next call is

epsilon * 2^k / 2 = epsilon * $2^{(k-1)}$

Number of calls = number of times we can divide (epsilon * 2^k) by 2 before we get to epsilon

… which is k

Isolating, we have |b-a| = epsilon * 2^k iff

k = log_2 (|b-a| / epsilon)

Exercise: Binary search

Give a recursive function of binary search

search(lst,x)

You need to generalize to

search(lst,i,j)

1. What is/are the base case/s?
2. How do you make recursive calls with “smaller” inputs?

Product of a list

$\text{prod}([v_1,v_2,v_3,\ldots,v_n]) = v_1 * v_2 * v_3 * \ldots * v_n$

$= v_1 * \text{prod}([v2,\ldots,v_n])$

$= \text{prod}([v_1,\ldots,v_{n-1}]) * v_n$

Product of a list

```
def prod(lst):
    “returns the product of lst”
    if len(lst) == 0:
        return 1
    else:
        return lst[0] * prod(lst[1:])
```

Inefficiency: extracting & copying lst[1:]

Product of a list

```
def prod(lst):
    return rprod(lst,0)

def rprod(lst,i):
    "returns the product of lst[i..len(lst)-1]"
    if (i == len(lst)):
        return 1
    else:
        return lst[i] * rprod(lst,i+1)
```

Product of a list

```
def prod(lst):
    return rprod(lst,len(lst)-1)

def rprod(lst,i):
    "returns the product of lst[0..i]"
    if (i == -1):
        return 1
    else:
        return rprod(lst,i-1) * lst[i]
```

How is recursion executed?

Stack of calls made, with parameters

(example in blackboard)

Local variables local to the call

A new copy is created at each call

To note: recursive factorial uses memory O(n) while iterative factorial uses memory O(1)

What is better, loops or recursion?

Every loop can be simulated with recursion

(next slides)

Recursion can be simulated with a loop and a stack

(not trivial with multiple recursion)

Tail recursion can be replaced with a loop, with no stack

(next slides)

Conclusions

- Loops and recursion have the same power in theory (if you can add a stack to loops)
- Often choice depends on elegance / naturality
- But some problems have natural multiple recursion solutions, complex iterative solutions
- Some languages automatically turn tail recursion to a loop
- Python DOES NOT optimize for tail recursion. You have to do the transformation by hand, if you want
- Remember that recursion may have “hidden memory usage”: stack of calls O(1) in loop may turn to O(n) in recursion
- So if tail recursive, in Python probably prefer loops
- Python has other ways (continuations, iterators, generators…)

Exercises

- Given a list of digits, compute the integer it represents [3,7,2,9] -> the int 3279
- Write a recursive version of binary search
- Write a recursive version of selection sort (auxiliary function to find min also recursive)
- Given a list of integers, say if there is a number that equals the sum of all numbers before it in the list
  - First version is probably $O(len(lst)^2)$. Why?
  - Plus: Think of a second version O(len(lst)). Add new parameter or result

Optional for theoretically-minded people

Loop to recursion

```
def f(x):
    y = a(x)
    while cond(x,y):
        (x,y) = b(x,y)
    return c(x,y)
```

```
def f(x):
    y = a(x)
    return rec_f(x,y)

def rec_f(x,y):
    if cond(x,y):
        return c(x,y)
    else:
        (x,y) = b(x,y)
        return rec_f(x,y)
```

Loop to recursion

```
def fact(n):
    f = 1
    while n > 0:
        f = f * n
        n = n - 1
    return f
```

```
def fact(n):
    f = 1
    return rec_fact(n,f)

def rec_fact(n,f):
    if n == 0:
        return f
    else:
        (f,n) = (f*n,n-1)
        return
            rec_fact(n,f)
```

Tail recursion to loop

```
def f(x):
    if cond(x):
        return c(x)
    else:
        z = b(x)
        return f(z)
```

```
def f(x):
    while cond(x):
        x = b(x)
    return c(x)
```

Tail recursion:

Nothing done after recursive call

Some languages (& modern compilers) do this transformation for you when they see tail recurssion.

Not Python. It complicates recursión

Exercise: do the transformation “by hand” for binary search and selection sort