CAIM: Cerca i Anàlisi d’Informació Massiva
FIB, Grau en Enginyeria Informàtica

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Fall 2016
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8. Locality Sensitive Hashing
Motivation, I

Find similar items in high dimensions, quickly

Could be useful, for example, in nearest neighbor algorithm. But in a large, high dimensional dataset this may be difficult!
Motivation, II
Hashing is good for checking existence, not nearest neighbors
Motivation, III
Main idea: want hashing functions that map similar objects to nearby positions using *projections*.
Different types of hashing functions

Perfect hashing

- Provide 1-1 mapping of objects to bucket ids
- Any two different objects mapped to different buckets (no collisions)

Universal hashing

- A family of functions $\mathcal{F} = \{ h : U \to [n] \}$ is called universal if $P[h(x) = h(y)] \leq \frac{1}{n}$ for all $x \neq y$
- i.e. probability of collision for different objects is at most $1/n$

Locality sensitive hashing (Lsh)

- Collision probability for similar objects is high enough
- Collision probability for dissimilar objects is low
A family $\mathcal{F}$ is called $(s, c \cdot s, p_1, p_2)$-sensitive if for any two objects $x$ and $y$ we have:

- If $s(x, y) \geq s$, then $P[h(x) = h(y)] \geq p_1$
- If $s(x, y) \leq c \cdot s$, then $P[h(x) = h(y)] \leq p_2$

where the probability is taken over choosing $h$ from $\mathcal{F}$, and $c < 1$, $p_1 > p_2$
How to use LSH to find nearest neighbor

The main idea

Pick a hashing function $h$ from appropriate family $\mathcal{F}$

Preprocessing

- Compute $h(x)$ for all objects $x$ in our available dataset

On arrival of query $q$

- Compute $h(q)$ for query object
- Sequentially check nearest neighbor in “bucket” $h(q)$
Locality sensitive hashing I
An example for bit vectors

- Objects are vectors in \( \{0, 1\}^d \)
- Distances are measured using Hamming distance

\[
d(x, y) = \sum_{i=1}^{d} |x_i - y_i|
\]

- Similarity is measured as nr. of common bits divided by length of vector

\[
s(x, y) = 1 - \frac{d(x, y)}{d}
\]

- For example, if \( x = 10010 \) and \( y = 11011 \), then \( d(x, y) = 2 \) and \( s(x, y) = 1 - 2/5 = 0.6 \)
Consider the following “hashing family”: sample the $i$-th bit of a vector, i.e. $\mathcal{F} = \{f_i | i \in [d]\}$ where $f_i(x) = x_i$

Then, the probability of collision

$$P[h(x) = h(y)] = s(x, y)$$

(the probability is taken over choosing a random $h \in \mathcal{F}$)

Hence $\mathcal{F}$ is $(s, cs, s, cs)$-sensitive (with $c < 1$ so that $s > cs$ as required)
Locality sensitive hashing III
An example for bit vectors

- If gap between $s$ and $cs$ is too small (between $p_1$ and $p_2$), we can amplify it:
  - By stacking together $k$ hash functions
    - $h(x) = (h_1(x), \ldots, h_k(x))$ where $h_i \in F$
    - Probability of collision of similar objects decreases to $s^k$
    - Probability of collision of dissimilar objects decreases even more to $(cs)^k$
  - By repeating the process $m$ times
    - Probability of collision of similar objects increases to $1 - (1 - s)^m$
  - Choosing $k$ and $m$ appropriately, can achieve a family that is $(s, cs, 1 - (1 - s^k)^m, 1 - (1 - (cs)^k)^m)$-sensitive
Here, $k = 5$, $m = 3$
Locality sensitive hashing V
An example for bit vectors

Collision probability is \(1 - (1 - s^k)^m\)
Similarity search becomes...

Pseudocode

Preprocessing

- Input: set of objects $X$
- for $i = 1..m$
  - for each $x \in X$
    - stack $k$ hash functions and form $x_i = (h_1(x), \ldots, h_k(x))$
    - store $x$ in bucket given by $f(x_i)$

On query time

- Input: query object $q$
- $Z = \emptyset$
- for $i = 1..m$
  - stack $k$ hash functions and form $q_i = (h_1(q), \ldots, h_k(q))$
  - $Z_i = \{\text{objects found in bucket } f(q_i)\}$
  - $Z = Z \cup Z_i$
- Output all $z \in Z$ such that $s(q, z) \geq s$
For objects in $[1..M]^d$

The idea is to represent each coordinate in unary form

- For example, if $M = 10$ and $d = 2$, then $(5, 2)$ becomes $(1111100000, 1100000000)$

- In this case, we have that the $L_1$ distance of two points in $[1..M]^d$ is

$$d(x, y) = \sum_{i=1}^{d} |x_i - y_i| = \sum_{i=1}^{d} d_{Hamming}(u(x), u(y))$$

so we can concatenate vectors in each coordinate into one single $dM$ bit-vector

- In fact, one does not need to store these vectors, they can be computed on-the-fly
Generalizing the idea..

- If we have a family of hash functions such that for all pairs of objects \( x, y \)
  \[
P[h(x) = h(y)] = s(x, y) \tag{1}
  \]

- We can then amplify the gap of probabilities by stacking \( k \) functions and repeating \( m \) times

- .. and so the core of the problem becomes to find a similarity function \( s \) and hash family satisfying (1)
Another example: finding similar sets

Using the Jaccard coefficient as similarity function

**Jaccard coefficient**

For pairs of sets \( x \) and \( y \) from a ground set \( U \) (i.e. \( x \subseteq U, y \subseteq U \)) is

\[
J(x, y) = \frac{|x \cap y|}{|x \cup y|}
\]
Another example: finding similar sets II
Using the Jaccard coefficient as similarity function

Main idea

- Suppose elements in $U$ are ordered (randomly)
- Now, look at the smallest element in each of the sets
- The more similar $x$ and $y$ are, the more likely it is that their smallest element coincides
Another example: finding similar sets III
Using the Jaccard coefficient as similarity function

So, define family of hash functions for Jaccard coefficient:

- Consider a random permutation $r : U \rightarrow [1..|U|]$ of elements in $U$
- For a set $x = \{x_1, .., x_l\}$, define $h_r(x) = \min_i\{r(x_i)\}$
- Let $\mathcal{F} = \{h_r | r \text{ is a permutation}\}$
- And so: $P[h(x) = h(y)] = J(x, y)$ as desired!

Scheme known as min-wise independent permutation hashing, in practice inefficient due to the cost of storing random permutations.