An Alternative to CARMA Models via Iterations of Ornstein–Uhlenbeck Processes

Argimiro Arratia, Alejandra Cabaña, and Enrique M. Cabaña

Abstract We present a new construction of continuous ARMA processes based on iterating an Ornstein–Uhlenbeck operator $\mathcal{OU}_\kappa$ that maps a random variable $y(t)$ onto $\mathcal{OU}_\kappa y(t) = \int_{-\infty}^t \mathrm{e}^{-\kappa(t-s)} dy(s)$. This construction resembles the procedure to build an AR(p) from an AR(1) and derives in a parsimonious model for continuous autoregression, with fewer parameters to compute than the known CARMA obtained as a solution of a system of stochastic differential equations. We show properties of this operator, give state space representation of the iterated Ornstein–Uhlenbeck process and show how to estimate the parameters of the model.

1 Introduction

The link between discrete ARMA processes and stationary processes with continuous time has been of interest for many years; see, e.g., [3, 6, 7]. Also, there is a recent upsurge of interest in continuous time models, because they can be used in presence of irregularly spaced data, and in non Gaussian processes mainly due to the fact that jumps play an important role in realistic modeling in finance and other fields of applications. One approach is via the stochastic volatility model from [2], in which the volatility process V and the log asset price G satisfy:

$$dV(t) = -\lambda V(t) + d\Lambda(t) \quad \text{and} \quad dG(t) = (\gamma + \beta V(t))dt + \sqrt{V(t)}dW(t) + \rho d\Lambda(t),$$

A. Arratia (✉)
Universitat Politècnica de Catalunya, Barcelona, Spain
BGSMath, Barcelona, Spain
e-mail: argimiro@cs.upc.edu

A. Cabaña
Universitat Autònoma de Barcelona, Bellaterra, Spain
BGSMath, Barcelona, Spain
e-mail: acabana@mat.uab.cat

E.M. Cabaña
Universidad de la República, Montevideo, Uruguay
e-mail: ecabana@ccee.edu.uy

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where $\lambda > 0$, Λ is a non-decreasing Lévy process, and W is a standard Brownian motion independent of Λ. The volatility V is a Lévy driven Ornstein–Uhlenbeck (or OU) process, or a continuous time autoregression of order 1. The autocorrelations of V decay exponentially, hence they constitute a very restrictive family.

In order to include a wider family of covariances, econometric or physical models apply frequently linear combinations (superpositions) of OU processes driven by either uncorrelated or correlated noise $\sum_{j=1}^{p} a_j \int_{-\infty}^{t} \mathrm{e}^{-\kappa_j(t-s)} d\Lambda_j(s)$ (see, e.g., [8]); or models that replace the finite linear combination by a continuous version

$$\int_{s=-\infty}^{t} \int_{\Re(\kappa)>0} \mathrm{e}^{-\kappa(t-s)} d\Lambda(s,\kappa);$$

see [3–5]. In particular, Brockwell [4, 5] proposes to define CARMA processes via a state space representation of the formal equation $a(D)Y(t) = \sigma b(D)D\Lambda(t)$, where $\sigma > 0$ is a scale parameter, D denotes differentiation with respect to t, Λ is a second-order Lévy process, a is a polynomial of order p, and b is a polynomial of order $q \leq p-1$ with coefficient $b_q = 1$. When the zeroes of the AR polynomial are all different, he obtains a representation of the CARMA as a sum of Lévy driven Ornstein–Uhlenbeck processes. Brockwell estimates the CARMA parameters by adjusting an ARMA(p,q), $q < p$, to regularly spaced data and then obtain the parameters of the CARMA whose values at the observation times have the same distribution of the fitted ARMA.

We propose in [1] a parsimonious model for continuous autoregression, with fewer parameters, able to adjust slowly decaying covariances. The model is obtained by a procedure that resembles the one that allows to build an AR(p) from an AR(1), that we summarize in the sequel.

2 Iterated Lévy Driven Ornstein–Uhlenbeck Processes

The AR(p) process $X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \sigma\epsilon_t$ or $\phi(B)X_t = \sigma\epsilon_t$, where $\phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j = \prod_{j=1}^{p}(1 - z/\rho_j)$ has roots $\rho_j = \mathrm{e}^{\kappa_j}$, is obtained by applying the composition of the moving averages $\mathcal{MA}(1/\rho_j)$ to the noise, that is, $X_t = \sigma \prod_{j=1}^{p} \mathcal{MA}(1/\rho_j)\epsilon_t$, where $\mathcal{MA}(1/\rho)$ is the moving average that maps ϵ_t onto $\mathcal{MA}(1/\rho)\epsilon_t = \sum_{j=0}^{\infty} \frac{1}{\rho^j}\epsilon_{t-j}$.

Let us denote $\mathcal{MA}_\kappa = \mathcal{MA}(\mathrm{e}^{-\kappa})$. A continuous version of the operator $\mathcal{MA}_\kappa$ mapping ϵ_t onto $\mathcal{MA}_\kappa\epsilon_t = \sum_{l\leq t,\text{integer}} \mathrm{e}^{-\kappa(t-l)}\epsilon_l$ is $\mathcal{OU}_\kappa$ that maps $y(t)$ onto $\mathcal{OU}_\kappa y(t) = \int_{-\infty}^{t} \mathrm{e}^{-\kappa(t-s)} dy(s)$ and this suggests the use of the model OU(p) with a second order Lévy process Λ,

$$x_{\boldsymbol{\kappa},\sigma}(t) = \sigma \prod_{j=1}^{p} \mathcal{OU}_{\kappa_j}\Lambda(t) \quad \text{with parameters} \quad \boldsymbol{\kappa} = (\kappa_1,\ldots,\kappa_p), \sigma. \tag{1}$$

Theorem 1 (OU(p) as a superposition of OU(1)) *The* Ornstein–Uhlenbeck process (1) *can be written as a linear combination of p processes of order 1:*

(i) *when the components of* $\boldsymbol{\kappa}$ *are pairwise different, the linear combination is* $x_{\boldsymbol{\kappa},\boldsymbol{\sigma}} = \sum_{j=1}^{p} K_j(\boldsymbol{\kappa})\xi_{\kappa_j}$, *where* $\xi_{\kappa_j}(t) = \int_{-\infty}^{t} \mathrm{e}^{-\kappa_j(t-s)} d(\sigma\Lambda(s))$, *and coefficients are* $K_j(\boldsymbol{\kappa}) = 1/\prod_{\kappa_l \neq \kappa_j}(1-\kappa_l/\kappa_j)$*;*

(ii) *when* $\boldsymbol{\kappa}$ *has components* κ_h *repeated* p_h *times* ($h = 1, 2, \ldots, q$, $\sum_{h=1}^{q} p_h = p$) *the linear combination is* $x_{\boldsymbol{\kappa},\boldsymbol{\sigma}} = \sum_{h=1}^{q} K_h(\boldsymbol{\kappa}) \sum_{j=0}^{p_h-1} \binom{p_h-1}{j} \xi_{\kappa_h}^{(j)}$, *where* $\xi_{\kappa_h}^{(j)}(t) = \int_{-\infty}^{t} \mathrm{e}^{-\kappa_h(t-s)} \frac{(-\kappa_h(t-s))^j}{j!} d(\sigma\Lambda(s))$.

The autocovariances of $x_{\boldsymbol{\kappa},\sigma}$ *are*

$$\gamma_{\kappa,\sigma}(t)=\sum_{h'=1}^{q}\sum_{i'=0}^{p_{h'}-1}\sum_{h''=1}^{q}\sum_{i''=0}^{p_{h''}-1} K_{h'}(\boldsymbol{\kappa})\bar{K}_{h''}(\boldsymbol{\kappa})\binom{p_{h'}-1}{i'}\binom{p_{h''}-1}{i''}\gamma_{\kappa_{h'},\kappa_{h''},\sigma}^{(i',i'')}(t).$$

3 A State Space Representation of the OU(p) Process

The decomposition of the OU(p) process $x_{\boldsymbol{\kappa},\sigma}(t)$ as a linear combination of simpler processes of order 1, leads to an expression of the process by means of a state space model. State space modelling provides us with a unified approach for computing the likelihood of $x_{\boldsymbol{\kappa},\sigma}(t)$ through a Kalman filter, and with a tool to show that the covariances of $x_{\boldsymbol{\kappa},\sigma}(t)$ coincide with those of an ARMA($p, p-1$) whose coefficients can be computed from $\boldsymbol{\kappa}$.

In the sequel, in order to ease notation, we consider that the components of $\boldsymbol{\kappa}$ are all different. The decomposition of $x_{\boldsymbol{\kappa},\sigma}(t) = \sum_{j=1}^{p} K_j \xi_{\kappa_j}(t)$ as a linear combination of the OU(1) processes, given by Theorem 1, with innovations $\boldsymbol{\eta}_{\kappa}$ with components $\eta_{\kappa_j}(t) = \int_{t-1}^{t} \mathrm{e}^{-\kappa_j(t-s)} d\Lambda(s)$, provides a representation of the OU(p) process in the space of states $\xi_{\boldsymbol{\kappa}} = (\xi_{\kappa_1}, \ldots, \xi_{\kappa_p})^{\mathrm{tr}}$. The transitions in the state space are

$$\boldsymbol{\xi}_{\kappa}(t) = \mathrm{diag}(\mathrm{e}^{-\kappa_1}, \ldots, \mathrm{e}^{-\kappa_p})\boldsymbol{\xi}_{\kappa}(t-1) + \boldsymbol{\eta}_{\kappa}(t)$$

and

$$\boldsymbol{x}(t) = \boldsymbol{K}^{tr}(\boldsymbol{\kappa})\boldsymbol{\xi}(t).$$

The assumption $\mathbf{E}\Lambda(1)^2 = 1$ implies that the innovations have variance $\mathbf{Var}(\boldsymbol{\eta}_{\kappa,\tau}(t)) = ((v_{j,l}))$, where $v_{j,l} = \mathbf{E}\int_{t-1}^{t} \mathrm{e}^{-(\kappa_j+\bar{\kappa}_l)(t-s)} ds = (1-\mathrm{e}^{-(\kappa_j+\bar{\kappa}_l)})/(\kappa_j+\bar{\kappa}_l)$.

Now apply the AR operator $\prod_{j=1}^{p}(1-\mathrm{e}^{-\kappa_j}B)$ to x_{κ} and obtain

$$\prod_{j=1}^{p}(1-\mathrm{e}^{-\kappa_j}B)x_{\boldsymbol{\kappa}}(t) = \sum_{j=1}^{p} K_j G_j(B)\eta_{\kappa_j}(t) =: \zeta(t),$$

with $G_j(z) = \prod_{l\neq j}(1-\mathrm{e}^{-\kappa_l}z) := 1 - \sum_{l=1}^{p-1} g_{j,l} z^l$.

This process has the same second-order moments as the ARMA$(p, p-1)$. When Λ is a Wiener process, it is in fact an ARMA$(p, p-1)$,

$$\prod_{j=1}^{p}(1-\mathrm{e}^{-\kappa_j}B)x_{\kappa}(t)=\sum_{j=0}^{p-1}\theta_j\epsilon(t-j)=:\zeta'(t),$$

(ϵ is a white noise) when the covariances $c_j=\mathbf{E}\zeta(t)\bar{\zeta}(t-j)$ and $c'_j=\mathbf{E}\zeta'(t)\bar{\zeta}'(t-j)$ coincide. The covariances c'_j and c_j are given respectively by the generating functions

$$\left(\sum_{h=0}^{p-1}\theta_h z^h\right)\left(\sum_{k=0}^{p-1}\bar{\theta}_k z^{-h}\right)=\sum_{l=-p+1}^{p-1}c'_l z^l$$

and

$$J(z):=\sum_{j=1}^{p}\sum_{l=1}^{p}K_j\bar{K}_l G_j(z)\bar{G}_l(1/z)v_{j,l}=\sum_{l=-p+1}^{p-1}c_l z^l.$$

Since $J(z)$ can be computed once $\boldsymbol{\kappa}$ is known, the coefficients $\boldsymbol{\theta}=(\theta_0,\theta_1,\ldots,\theta_{p-1})$ are obtained by identifying the coefficients of the polynomials $z^{p-1}(\sum_{h=0}^{p-1}\theta_h z^h)(\sum_{k=0}^{p-1}\bar{\theta}_k z^{-h})$ and $z^{p-1}J(z)$.

A state space representation and its implications on the covariances of the OU process in the general case are slightly more complicated.

4 Estimation

Though $\gamma(t)$ depends continuously on $\boldsymbol{\kappa}$, the same does not happen with each term in the expression for the covariance, because of the lack of boundedness of the coefficients of the linear combination when two different values of the components of $\boldsymbol{\kappa}$ approach each other. Since we wish to consider real processes x and the process itself and its covariance $\gamma(t)$ depend only on the unordered set of the components of $\boldsymbol{\kappa}$, we shall reparameterize the process. With the notation

$$K_{j,i}=\frac{1}{(-\kappa_j)^i\prod_{l\neq j}(1-\kappa_l/\kappa_j)},$$

(in particular, $K_{j,0}$ is the same as K_j), the processes $x_i(t)=\sum_{j=1}^{p}K_{j,i}\xi_j(t)$ and the coefficients $\boldsymbol{\phi}=(\phi_1,\ldots,\phi_p)$ of the polynomial $g(z)=\prod_{j=1}^{p}(1+\kappa_j z)=1-\sum_{j=1}^{p}\phi_j z^j$ satisfy $\sum_{i=1}^{p}\phi_i x_i(t)=x(t)$. Therefore, the new parameter $\boldsymbol{\phi}=(\phi_1,\ldots,\phi_p)\in R^p$ is adopted.

4.1 The Gaussian Case

From the observations $\{\mu + x(i) \ : \ i = 0, 1, \ldots, n\}$, we obtain the likelihood L of the vector $x = (x(1)), \ldots, x(n))$,

$$\log L(\boldsymbol{x}; \boldsymbol{\phi}, \sigma) = -\tfrac{n}{2}\log(2\pi) - \tfrac{1}{2}\log(\det(V(\boldsymbol{\phi}, \sigma)) - \tfrac{1}{2}\boldsymbol{x}^{\mathrm{tr}}(V(\boldsymbol{\phi}, \sigma))^{-1}\boldsymbol{x},$$

with $V(\boldsymbol{\phi}, \sigma)$ equal to the $n \times n$ matrix with components $V_{h,i} = \gamma(|h-i|)$, that reduce to $\gamma(0)$ at the diagonal, $\gamma(1)$ at the first sub- and super-diagonals, and so on. Obtain via numerical optimisation the MLE $\hat{\boldsymbol{\phi}}$ of $\boldsymbol{\phi}$ and $\hat{\sigma}^2$ of σ^2. The estimations $\hat{\boldsymbol{\kappa}}$ follow by solving $\prod_{j=1}^{p}(1 + \hat{\kappa}_j z) = 1 - \sum_{j=1}^{p} \hat{\phi}_j z^j$.

The parameters $\boldsymbol{\kappa}, \sigma$ determine the Gaussian likelihood of $\mathcal{OU}_{\kappa,\sigma w}$, and are estimated by the values $\hat{\boldsymbol{\kappa}}$ and $\hat{\sigma}$ that maximize that likelihood.

4.2 A Non Gaussian Example

Let us assume that the noise is given by $\Lambda(t) = \sigma w(t) + c(N(t) - \lambda t)$, where w is a standard Wiener process and N is a Poisson process with intensity λ. The family of possible noises depends on the three parameters (σ, λ, c). In this case, the characteristic exponent has a simple form,

$$\psi_{\Lambda(1)}(iu) = -\frac{\sigma^2 u^2}{2} + \lambda(\mathrm{e}^{iuc} - iuc - 1)$$

hence,

$$\psi_{\eta}(iu) = \int_0^1 \left(-\frac{\sigma^2 u^2 g^2(s)}{2} + \lambda(\mathrm{e}^{iug(s)c} - iug(s)c - 1)\right) ds.$$

With $g_h = \int_0^1 g^h(s) ds$,

$$\psi_{\eta}(iu) = -\frac{\sigma^2 u^2 g_2}{2} + \lambda\left(-\frac{u^2 g_2 c^2}{2} - i\frac{u^3 g_3 c^3}{6} + \frac{u^4 g_4 c^4}{24} + \cdots\right).$$

Then we propose to estimate the parameters by equating the coefficients of u^2, u^3, u^4 in $\psi_{\eta}(iu)$ with the corresponding ones in the logarithm of the empirical characteristic function of the residuals. Assuming that the mean of the residuals $r_1, r_2, \ldots, r_n$ is zero, their empirical characteristic function is

$$\frac{1}{n}\sum_{h=1}^{n} \mathrm{e}^{iur_h} = 1 - \frac{1}{2}u^2 R_2 - \frac{1}{6}iu^3 R_3 + \frac{1}{24}u^4 R_4 + \cdots,$$

where $R_m = (\sum_{h=1}^{n} r_h^m)/n$. Then, the logarithm has the expansion

$$\log \frac{1}{n}\sum_{h=1}^{n} \mathrm{e}^{iur_h} = -\frac{1}{2}u^2R_2 - \frac{1}{6}iu^3R_3 + \frac{1}{24}u^4R_4 - \frac{1}{8}u^4R_2^2 + \cdots .$$

Consequently, the estimation equations are $(\sigma^2 + \lambda c^2)g_2 = R_2$, $\lambda c^3 g_3 = R_3$, and $\lambda c^4 g_4 = R_4 - 3R_2^2$, from which the estimators follow:

$$\tilde{c} = \frac{R_4 - 3R_2^2}{R_3}\frac{g_3}{g_4}, \qquad \tilde{\lambda} = \frac{R_3^4}{(R_4 - 3R_2^2)^3}\frac{g_4^3}{g_3^4}, \qquad \tilde{\sigma}^2 = \frac{R^2}{g_2} - \frac{R_3^2}{(R_4 - 3R_2^2)}\frac{g_4}{g_3^2}.$$

5 Conclusions

We have proposed a family of continuous time stationary processes, based on p iterations of the linear operator that maps a Lévy process onto an Ornstein–Uhlenbeck process. These operators have some nice properties, such as being commutative, and their p-compositions decompose as a linear combination of simple operators of the same kind. An OU(p) process depends on $p + 1$ parameters that can be easily estimated by either maximum likelihood (ML) or by matching correlations procedures. Matching correlation estimators provide a fair estimation of the covariances of the data, even if the model is not well specified. When sampled on equally spaced instants, the OU(p) family can be written as a discrete time state space model; i.e., a VARMA model in a space of dimension p. As a consequence, the families of OU(p) models are a parsimonious subfamily of the ARMA($p, p-1$) processes in the Gaussian case. Furthermore, the coefficients of the ARMA can be deduced from those of the corresponding OU(p). We have found time series data for which the ML-estimated OU model is able to capture a long term dependence that the ML-estimated ARMA model does not show; see [1]. This leads to recommend the inclusion of OU models as candidates to represent stationary series to the users interested in such kind of dependence.

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