# IRRS: Information Retrieval and Recommender Systems

FIB, Master in Data Science

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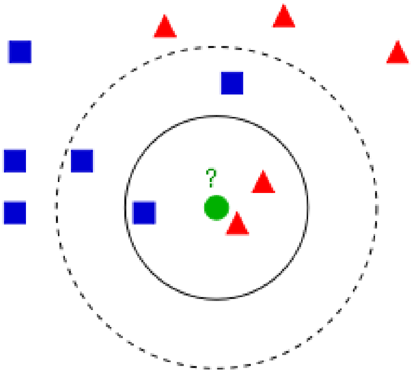
# 8. Locality Sensitive Hashing

# Motivation, I

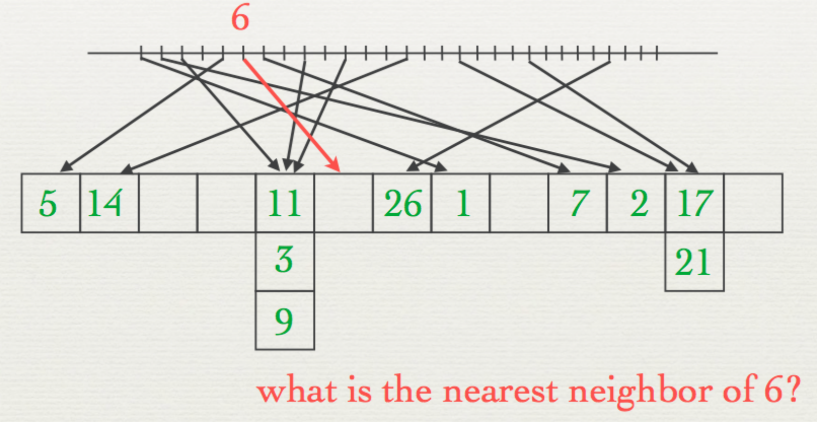
## Find similar items in high dimensions, quickly

Could be useful, for example, in nearest neighbor algorithm..
but in a large, high dimensional dataset this may be difficult!



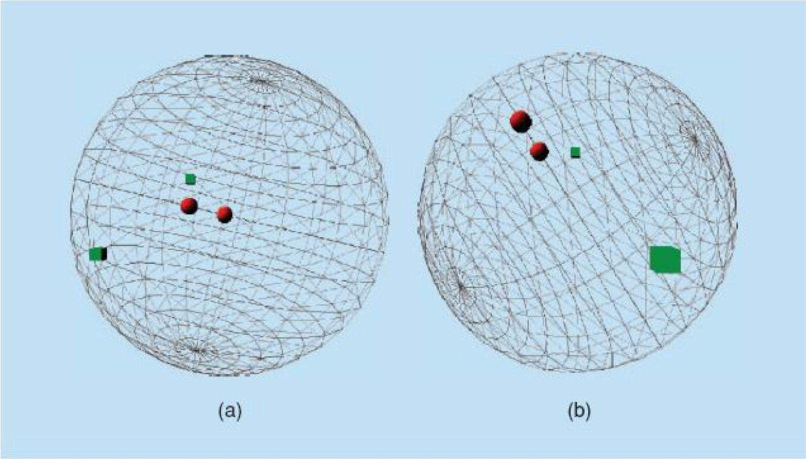
# Motivation, II

## Hashing is good for checking existence, not nearest neighbors



# Motivation, III

Main idea: want hashing functions that map similar objects to nearby positions using *projections*




[FIG1] Two examples showing projections of two close (circles) and two distant (squares) points onto the printed page.

# Different types of hashing functions

## Perfect hashing

- Provide 1-1 mapping of objects to bucket ids
- Any two different objects mapped to different buckets (no collisions)

## Universal hashing

- A family of functions $\mathcal{F} = \{h : U \to [n]\}$ is called *universal* if $P[h(x) = h(y)] \leq \frac{1}{n}$ for all $x \neq y$
- i.e. probability of collision for different objects is at most $1/n$

## Locality sensitive hashing (lsh)

- Collision probability for *similar* objects is high enough
- Collision probability for *dissimilar* objects is low

# Locality sensitive hashing functions

## Definition

A family $\mathcal{F}$ is called $(s, c \cdot s, p_1, p_2)$-sensitive if for any two objects $x$ and $y$ we have:

- If $s(x, y) \geq s$, then $P[h(x) = h(y)] \geq p_1$
- If $s(x, y) \leq c \cdot s$, then $P[h(x) = h(y)] \leq p_2$

where the probability is taken over chosing $h$ from $\mathcal{F}$, and $c < 1$, $p_1 > p_2$

# How to use LSH to find nearest neighbor

## The main idea

Pick a hashing function $h$ from appropriate family $\mathcal{F}$

### Preprocessing

- Compute $h(x)$ for all objects $x$ in our available dataset

### On arrival of query $q$

- Compute $h(q)$ for query object
- Sequentially check nearest neighbor in "bucket" $h(q)$

# Locality sensitive hashing I

## An example for bit vectors

- Objects are vectors in $\{0,1\}^d$
- Distances are measured using Hamming distance

$$d(x,y) = \sum_{i=1}^{d} |x_i - y_i|$$

- Similarity is measured as nr. of common bits divided by length of vector

$$s(x,y) = 1 - \frac{d(x,y)}{d}$$

- For example, if $x = 10010$ and $y = 11011$, then $d(x,y) = 2$ and $s(x,y) = 1 - 2/5 = 0.6$

# Locality sensitive hashing II

## An example for bit vectors

- Consider the following "hashing family": sample the $i$-th bit of a vector, i.e. $\mathcal{F} = \{f_i | i \in [d]\}$ where $f_i(x) = x_i$
- Then, the probability of collision

$$P[h(x) = h(y)] = s(x, y)$$

  (the probability is taken over chosing a random $h \in \mathcal{F}$)
- Hence $\mathcal{F}$ is $(s, cs, s, cs)$-sensitive (with $c < 1$ so that $s > cs$ as required)

# Locality sensitive hashing III

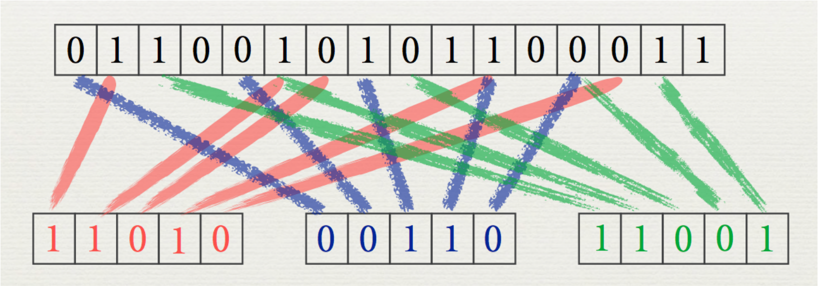
## An example for bit vectors

- If gap between $s$ and $cs$ is too small (between $p_1$ and $p_2$), we can amplify it:
  - By stacking together $k$ hash functions
    - $h(x) = (h_1(x), .., h_k(x))$ where $h_i \in \mathcal{F}$
    - Probability of collision of similar objects decreases to $s^k$
    - Probability of collision of dissimilar objects decreases even more to $(cs)^k$
  - By repeating the process $m$ times
    - Probability of collision of similar objects increases to $1-(1-s)^m$
  - Choosing $k$ and $m$ appropriately, can achieve a family that is $(s, cs, 1-(1-s^k)^m, 1-(1-(cs)^k)^m)$-sensitive

# Locality sensitive hashing IV

## An example for bit vectors

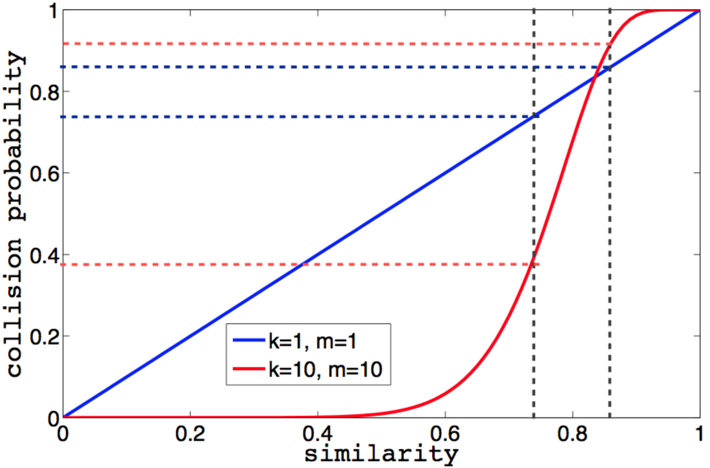
Here, $k = 5, m = 3$



# Locality sensitive hashing V

## An example for bit vectors

Collision probability is $1 - (1 - s^k)^m$



# Similarity search becomes..

## Pseudocode

### Preprocessing

- Input: set of objects $X$
- for $i = 1..m$
  - for each $x \in X$
    - stack $k$ hash functions and form $x_i = (h_1(x), .., h_k(x))$
    - store $x$ in bucket given by $f(x_i)$

### On query time

- Input: query object $q$
- $Z = \emptyset$
- for $i = 1..m$
  - stack $k$ hash functions and form $q_i = (h_1(q), .., h_k(q))$
  - $Z_i = \{$ objects found in bucket $f(q_i)\}$
  - $Z = Z \cup Z_i$
- Output all $z \in Z$ such that $s(q, z) \geq s$

# For objects in $[1..M]^d$

The idea is to represent each coordinate in unary form

- For example, if $M = 10$ and $d = 2$, then $(5, 2)$ becomes $(1111100000, 1100000000)$
- In this case, we have that the $L_1$ distance of two points in $[1..M]^d$ is

$$d(x,y) = \sum_{i=1}^{d} |x_i - y_i| = \sum_{i=1}^{d} d_{Hamming}(u(x), u(y))$$

  so we can concatenate vectors in each coordinate into one single $dM$ bit-vector
- In fact, one does not need to *store* these vectors, they can be computed on-the-fly

# Generalizing the idea..

- If we have a family of hash functions such that for all pairs of objects $x, y$

$$P[h(x) = h(y)] = s(x, y) \tag{1}$$

- We can then amplify the gap of probabilities by stacking $k$ functions and repeating $m$ times
- .. and so the core of the problem becomes to find a similarity function $s$ and hash family satisfying (1)

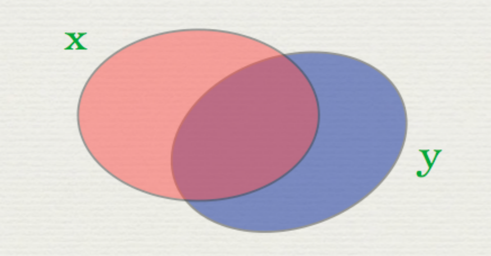
# Another example: finding similar *sets* I

Using the Jaccard coefficient as similarity function

## Jaccard coefficient

For pairs of sets $x$ and $y$ from a ground set $U$ (i.e. $x \subseteq U, y \subseteq U$) is

$$J(x, y) = \frac{|x \cap y|}{|x \cup y|}$$

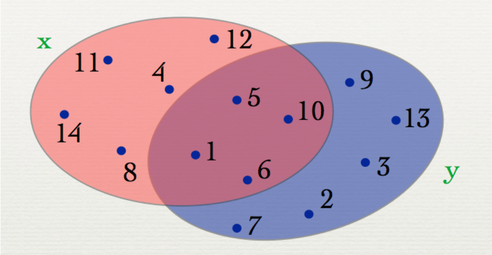


# Another example: finding similar *sets* II

Using the Jaccard coefficient as similarity function

## Main idea

- Suppose elements in $U$ are ordered (randomly)
- Now, look at the smallest element in each of the sets
- The more similar $x$ and $y$ are, the more likely it is that their smallest element coincides



# Another example: finding similar *sets* III

## Using the Jaccard coefficient as similarity function

So, define family of hash functions for Jaccard coefficient:

- Consider a random permutation $r : U \rightarrow [1..|U|]$ of elements in $U$
- For a set $x = \{x_1, .., x_l\}$, define $h_r(x) = min_i\{r(x_i)\}$
- Let $\mathcal{F} = \{h_r | r$ is a permutation$\}$
- And so: $P[h(x) = h(y)] = J(x, y)$ as desired!

Scheme known as *min-wise independent permutation* hashing, in practice inefficient due to the cost of storing random permutations.