IR: Information Retrieval

FIB, Master in Innovation and Research in Informatics

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7. Introduction to Network Analysis

Network Analysis, Part I

Today's contents

1. Examples of real networks
2. What do real networks look like?
    - real networks exhibit small diameter
        - .. and so does the Erdös-Rényi or random model
    - real networks have high clustering coefficient
        - .. and so does the Watts-Strogatz model
    - real networks' degree distribution follows a power-law
        - .. and so does the Barabasi-Albert or preferential attachment model

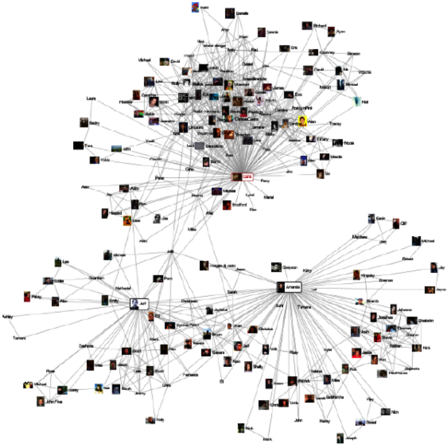
Examples of real networks

- Social networks
- Information networks
- Technological networks
- Biological networks

Social networks

Links denote social “interactions”

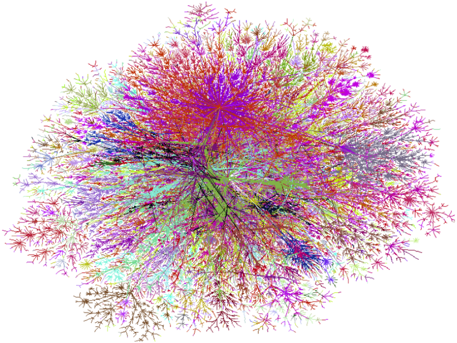
- friendship, collaborations, e-mail, etc.



Information networks

Nodes store information, links associate information

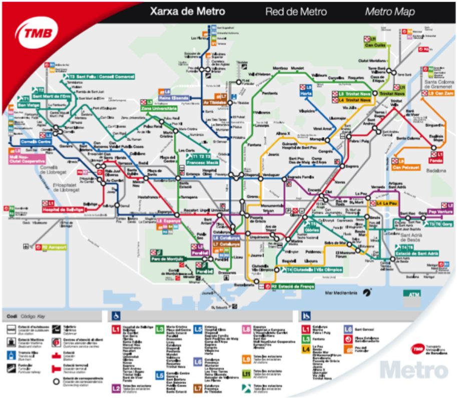
- citation networks, the web, p2p networks, etc.



Technological networks

Man-built for the distribution of a commodity

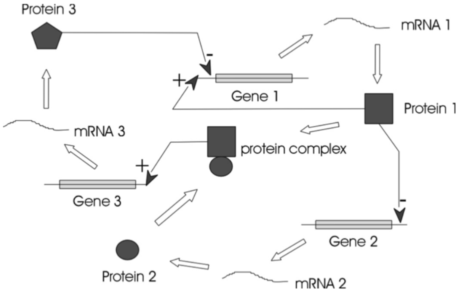
- telephone networks, power grids, transportation networks, etc.



Biological networks

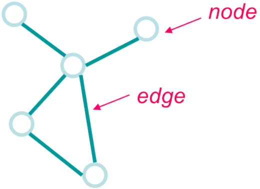
Represent biological systems

- protein-protein interaction networks, gene regulation networks, metabolic pathways, etc.



Representing networks

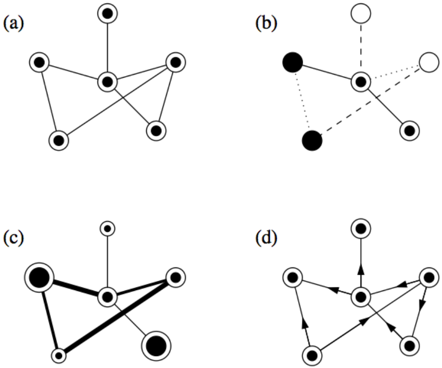
- Network $\equiv$ Graph
- Networks are just collections of "points" joined by "lines"



points	lines	
vertices	edges, arcs	math
nodes	links	computer science
sites	bonds	physics
actors	ties, relations	sociology

Types of networks

From [Newman, 2003]



(a) unweighted, undirected

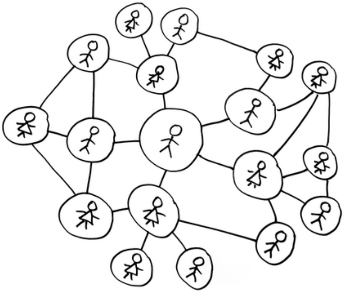
(b) discrete vertex and edge types, undirected

(c) varying vertex and edge weights, undirected

(d) directed

Small-world phenomenon

- A friend of a friend is also frequently a friend
- Only 6 hops separate any two people in the world



Measuring the small-world phenomenon, I

- Let d_{ij} be the shortest-path distance between nodes i and j
- To check whether “any two nodes are within 6 hops”, we use:
  - The diameter (longest shortest-path distance) as

$$d = \max_{i,j} d_{ij}$$

  - The average shortest-path length as

$$l = \frac{2}{n\,(n+1)} \sum_{i>j} d_{ij}$$

  - The harmonic mean shortest-path length as

$$l^{-1} = \frac{2}{n\,(n+1)} \sum_{i>j} d_{ij}^{-1}$$

From [Newman, 2003]

	network	type	n	m	z	ℓ	α	$C^{(1)}$	$C^{(2)}$	r	Ref(s).
social	film actors	undirected	449 913	25 516 482	113.43	3.48	2.3	0.20	0.78	0.208	20, 416
	company directors	undirected	7 673	55 392	14.44	4.60	–	0.59	0.88	0.276	105, 323
	math coauthorship	undirected	253 339	496 489	3.92	7.57	–	0.15	0.34	0.120	107, 182
	physics coauthorship	undirected	52 909	245 300	9.27	6.19	–	0.45	0.56	0.363	311, 313
	biology coauthorship	undirected	1 520 251	11 803 064	15.53	4.92	–	0.088	0.60	0.127	311, 313
	telephone call graph	undirected	47 000 000	80 000 000	3.16		2.1				8, 9
	email messages	directed	59 912	86 300	1.44	4.95	1.5/2.0		0.16		136
	email address books	directed	16 881	57 029	3.38	5.22	–	0.17	0.13	0.092	321
	student relationships	undirected	573	477	1.66	16.01	–	0.005	0.001	−0.029	45
	sexual contacts	undirected	2 810				3.2				265, 266
information	WWW `nd.edu`	directed	269 504	1 497 135	5.55	11.27	2.1/2.4	0.11	0.29	−0.067	14, 34
	WWW Altavista	directed	203 549 046	2 130 000 000	10.46	16.18	2.1/2.7				74
	citation network	directed	783 339	6 716 198	8.57		3.0/–				351
	Roget's Thesaurus	directed	1 022	5 103	4.99	4.87	–	0.13	0.15	0.157	244
	word co-occurrence	undirected	460 902	17 000 000	70.13		2.7		0.44		119, 157
technological	Internet	undirected	10 697	31 992	5.98	3.31	2.5	0.035	0.39	−0.189	86, 148
	power grid	undirected	4 941	6 594	2.67	18.99	–	0.10	0.080	−0.003	416
	train routes	undirected	587	19 603	66.79	2.16	–		0.69	−0.033	366
	software packages	directed	1 439	1 723	1.20	2.42	1.6/1.4	0.070	0.082	−0.016	318
	software classes	directed	1 377	2 213	1.61	1.51	–	0.033	0.012	−0.119	395
	electronic circuits	undirected	24 097	53 248	4.34	11.05	3.0	0.010	0.030	−0.154	155
	peer-to-peer network	undirected	880	1 296	1.47	4.28	2.1	0.012	0.011	−0.366	6, 354
biological	metabolic network	undirected	765	3 686	9.64	2.56	2.2	0.090	0.67	−0.240	214
	protein interactions	undirected	2 115	2 240	2.12	6.80	2.4	0.072	0.071	−0.156	212
	marine food web	directed	135	598	4.43	2.05	–	0.16	0.23	−0.263	204
	freshwater food web	directed	92	997	10.84	1.90	–	0.20	0.087	−0.326	272
	neural network	directed	307	2 359	7.68	3.97	–	0.18	0.28	−0.226	416, 421

But..

- Can we mimic this phenomenon in simulated networks ("models")?
- The answer is YES!

The (basic) random graph model

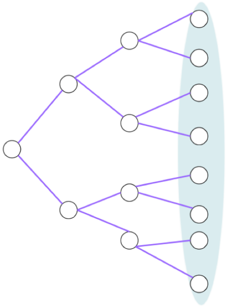
a.k.a. ER model

Basic $G_{n,p}$ Erdös-Rényi random graph model:

- parameter n is the number of vertices
- parameter p is s.t. $0 \leq p \leq 1$
- Generate and edge (i, j) independently at random with probability p

Measuring the diameter in ER networks

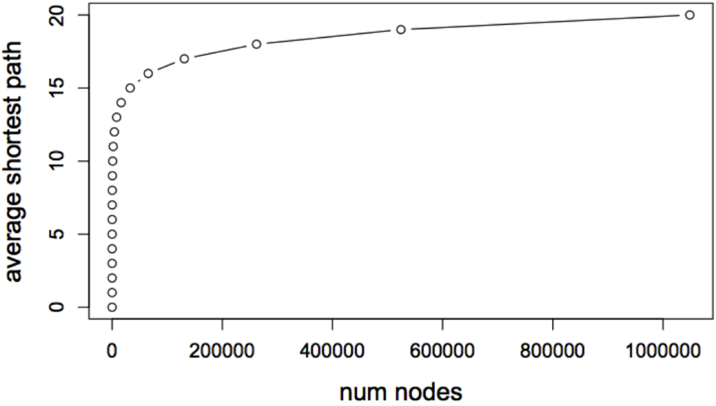
Want to show that the diameter in ER networks is small



- Let the average degree be z
- At distance l, can reach z^l nodes
- At distance $\frac{\log n}{\log z}$, reach all n nodes
- So, diameter is (roughly) $O(\log n)$

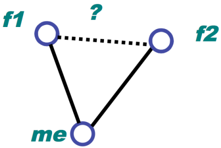
ER networks have small diameter

As shown by the following simulation



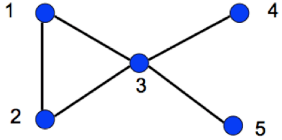
Measuring the small-world phenomenon, II

- To check whether “the friend of a friend is also frequently a friend”, we use:
  - The transitivity or clustering coefficient, which basically measures the probability that two of my friends are also friends



Global clustering coefficient

$$C = \frac{3 \times \text{number of triangles}}{\text{number of connected triples}}$$



$$C = \frac{3 \times 1}{8} = 0.375$$

Local clustering coefficient

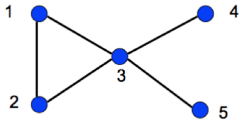
- For each vertex i, let n_i be the number of neighbors of i
- Let C_i be the fraction of pairs of neighbors that are connected within each other

$$C_i = \frac{\text{nr. of connections between } i\text{'s neighbors}}{\frac{1}{2} n_i \left(n_i - 1\right)}$$

- Finally, average C_i over all nodes i in the network

$$C = \frac{1}{n} \sum_i C_i$$

Local clustering coefficient example



- $C_1 = C_2 = 1/1$
- $C_3 = 1/6$
- $C_4 = C_5 = 0$
- $C = \frac{1}{5}(1 + 1 + 1/6) = 13/30 = 0.433$

From [Newman, 2003]

	network	type	n	m	z	ℓ	α	$C^{(1)}$	$C^{(2)}$	r	Ref(s).
social	film actors	undirected	449 913	25 516 482	113.43	3.48	2.3	0.20	0.78	0.208	20, 416
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ER networks do not show transitivity

- $C = p$, since edges are added independently
- Given a graph with n nodes and e edges, we can "estimate" p as

$$\hat{p} = \frac{e}{1/2\; n\; (n-1)}$$

- We say that clustering is high if $C \gg \hat{p}$
  - Hence, ER networks do not have high clustering coefficient since for them $C \approx \hat{p}$

ER networks do not show transitivity

Table 1: Clustering coefficients, C, for a number of different networks; n is the number of node, z is the mean degree. Taken from [146].

Network	n	z	C measured	C for random graph
Internet [153]	6,374	3.8	0.24	0.00060
World Wide Web (sites) [2]	153,127	35.2	0.11	0.00023
power grid [192]	4,941	2.7	0.080	0.00054
biology collaborations [140]	1,520,251	15.5	0.081	0.000010
mathematics collaborations [141]	253,339	3.9	0.15	0.000015
film actor collaborations [149]	449,913	113.4	0.20	0.00025
company directors [149]	7,673	14.4	0.59	0.0019
word co-occurrence [90]	460,902	70.1	0.44	0.00015
neural network [192]	282	14.0	0.28	0.049
metabolic network [69]	315	28.3	0.59	0.090
food web [138]	134	8.7	0.22	0.065

So ER networks do not have high clustering, but..

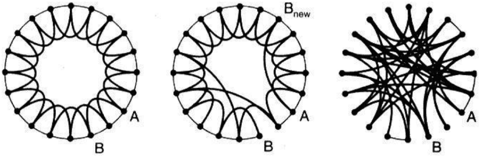
- Can we mimic this phenomenon in simulated networks ("models"), while keeping the diameter small?
- The answer is YES!

The Watts-Strogatz model, I

From [Watts and Strogatz, 1998]

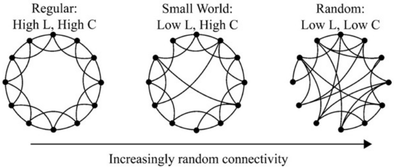
Reconciling two observations from real networks:

- High clustering: my friend's friends are also my friends
- small diameter



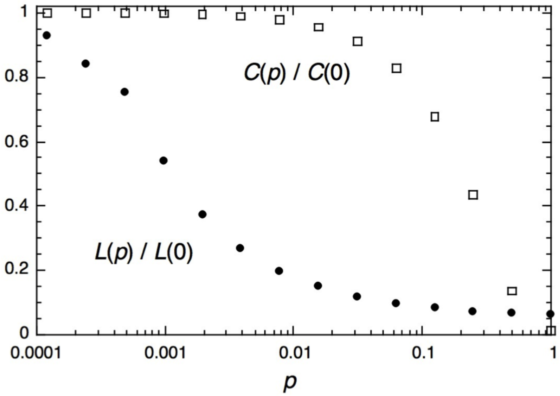
The Watts-Strogatz model, II

- Start with all n vertices arranged on a ring
- Each vertex has intially 4 connections to their closest nodes
  - mimics local or geographical connectivity
- With probability p, rewire each local connection to a random vertex
  - $p = 0$ high clustering, high diameter
  - $p = 1$ low clustering, low diameter (ER model)
- What happens in between?
  - As we increase p from 0 to 1
    - Fast decrease of mean distance
    - Slow decrease in clustering



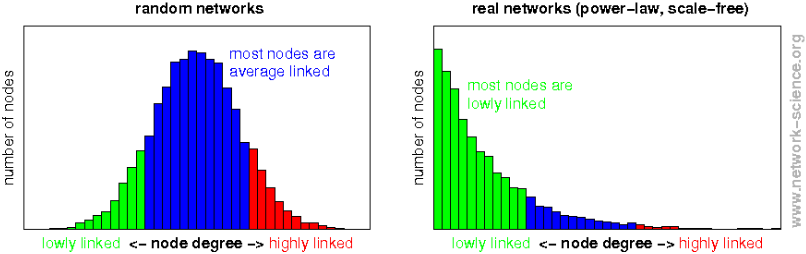
The Watts-Strogatz model, III

For an appropriate value of $p \approx 0.01$ (1 %), we observe that the model achieves high clustering and small diameter



Degree distribution

Histogram of nr of nodes having a particular degree

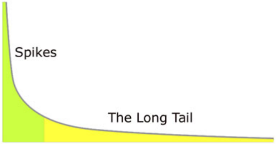


$$f_k = \text{fraction of nodes of degree } k$$

Scale-free networks

The degree distribution of most real-world networks follows a power-law distribution

$$f_k = ck^{-\alpha}$$



- "heavy-tail" distribution, implies existence of hubs
- hubs are nodes with very high degree

Random networks are not scale-free!

For random networks, the degree distribution follows the binomial distribution (or Poisson if n is large)

$$f_k = \binom{n}{k} p^k (1-p)^{(n-k)} \approx \frac{z^k e^{-z}}{k!}$$

- Where $z = p(n-1)$ is the mean degree
- Probability of nodes with very large degree becomes exponentially small
  - so no hubs

So ER networks are not scale-free, but..

- Can we obtained scale-free simulated networks?
- The answer is YES!

Preferential attachment

- "Rich get richer" dynamics
  - The more someone has, the more she is likely to have
- Examples
  - the more friends you have, the easier it is to make new ones
  - the more business a firm has, the easier it is to win more
  - the more people there are at a restaurant, the more who want to go

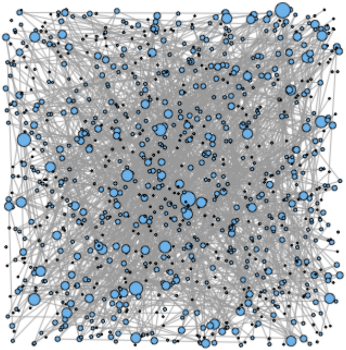
Barabási-Albert model

From [Barabási and Albert, 1999]

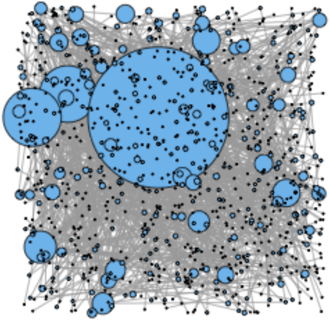
- "Growth" model
  - The model controls how a network grows over time
- Uses preferential attachment as a guide to grow the network
  - new nodes prefer to attach to well-connected nodes
- (Simplified) process:
  - the process starts with some initial subgraph
  - each new node comes in with m edges
  - probability of connecting to existing node i is proportional to i's degree
  - results in a power-law degree distribution with exponent $\alpha = 3$

ER vs. BA

Experiment with 1000 nodes, 999 edges ($m_0 = 1$ in BA model).



random



preferential attachment

In summary..

phenomenon	real networks	ER	WS	BA
small diameter	yes	yes	yes	yes
high clustering	yes	no	yes	yes[1]
scale-free	yes	no	no	yes

[1] clustering coefficient is higher than in random networks, but not as high as for example in WS networks

Network Analysis, Part II

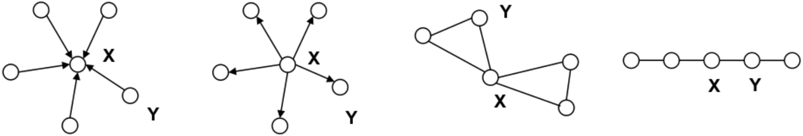
Today's contents

1. Centrality
    - Degree centrality
    - Closeness centrality
    - Betweenness centrality
2. Community finding algorithms
    - Hierarchical clustering
        - Agglomerative
        - Girvan-Newman
    - Modularity maximization: Louvain method

Centrality in Networks

Centrality is a node's measure w.r.t. others

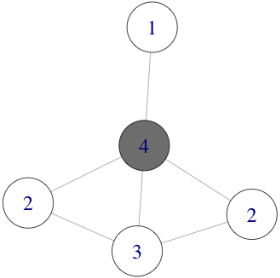
- A central node is *important* and/or *powerful*
- A central node has an *influential position in the network*
- A central node has an *advantageous position in the network*



Degree centrality

Power through connections

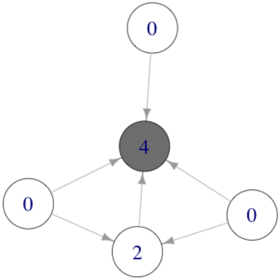
$$degree\_centrality(i) \stackrel{def}{=} k(i)$$



Degree centrality

Power through connections

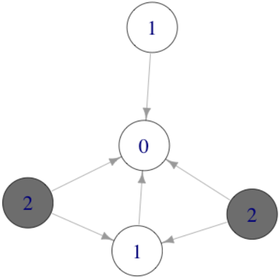
$$in\_degree\_centrality(i) \overset{def}{=} k_{in}(i)$$



Degree centrality

Power through connections

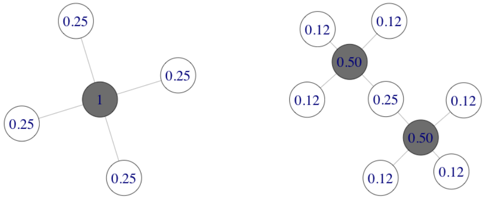
$$out\_degree\_centrality(i) \stackrel{def}{=} k_{out}(i)$$



Degree centrality

Power through connections

By the way, there is a *normalized* version which divides the centrality of each degree by the maximum centrality value possible, i.e. $n-1$ (so values are all between 0 and 1).

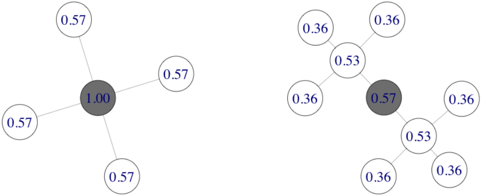


But look at these examples, does degree centrality look OK to you?

Closeness centrality

Power through proximity to others

$$closeness\_centrality(i) \stackrel{def}{=} \left(\frac{\sum_{j \neq i} d(i,j)}{n-1} \right)^{-1} = \frac{n-1}{\sum_{j \neq i} d(i,j)}$$



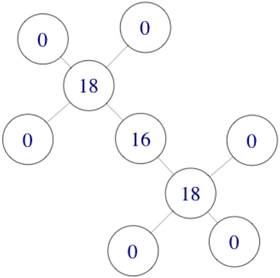
Here, what matters is to be close to everybody else, i.e., to be easily reachable or have the power to quickly reach others.

Betweenness centrality

Power through brokerage

A node is important if it lies in many shortest-paths

- so it is essential in passing information through the network



Betweenness centrality

Power through brokerage

$$betweenness\_centrality(i) \overset{def}{=} \sum_{j<k} \frac{g_{jk}(i)}{g_{jk}}$$

Where

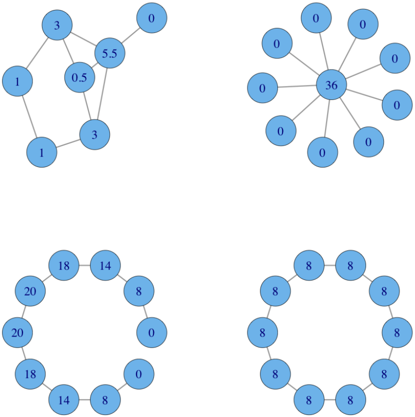
- g_{jk} is the number of shortest-paths between j and k, and
- $g_{jk}(i)$ is the number of shortest-paths through i

Oftentimes it is normalized:

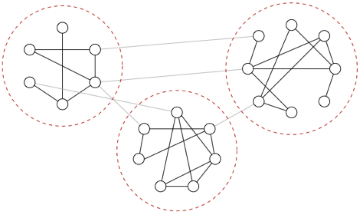
$$norm\_betweenness\_centrality(i) \overset{def}{=} \frac{betweenness\_centrality(i)}{\binom{n-1}{2}}$$

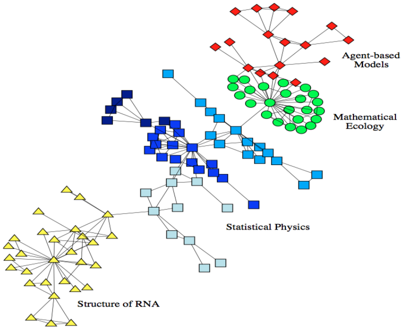
Betweenness centrality

Examples (non-normalized)

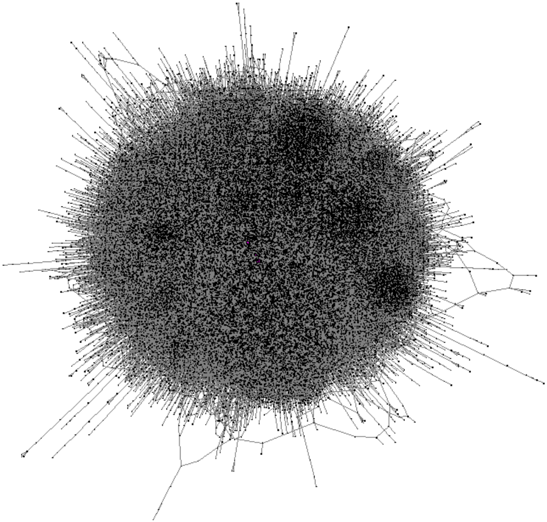


What is community structure?



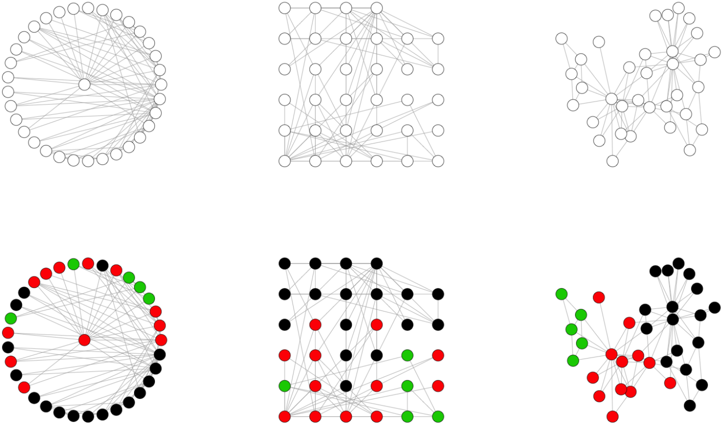


Why is community structure important?



.. but don't trust visual perception

it is best to use objective algorithms



Main idea

A community is *dense* in the inside but *sparse* w.r.t. the outside

No universal definition! But some ideas are:

- A community should be *densely connected*
- A community should be *well-separated* from the rest of the network
- Members of a community should be *more similar* among themselves than with the rest

Most common..

nr. of intra-cluster edges $>$ nr. of inter-cluster edges

Some definitions

Let $G = (V, E)$ be a network with $|V| = n$ nodes and $|E| = m$ edges. Let C be a subset of nodes in the network (a "cluster" or "community") of size $|C| = n_c$. Then

- *intra-cluster density*:

$$\delta_{int}(C) = \frac{\text{nr. internal edges of } C}{n_c(n_c - 1)/2}$$

- *inter-cluster density*:

$$\delta_{ext}(C) = \frac{\text{nr. inter-cluster edges of } C}{n_c(n - n_c)}$$

A community should have $\delta_{int}(C) > \delta(G)$, where $\delta(G)$ is the average edge density of the whole graph G, i.e.

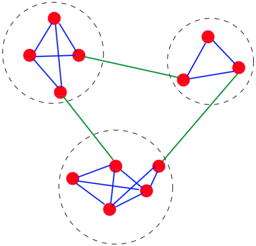
$$\delta(G) = \frac{\text{nr. edges in } G}{n(n-1)/2}$$

Most algorithms search for tradeoffs between *large* $\delta_{int}(C)$ and *small* $\delta_{ext}(C)$

- e.g. optimizing $\sum_C \delta_{int}(C) - \delta_{ext}(C)$ over all communities C

Define further:

- $m_c =$ nr. edges within cluster $C = |\{(u,v)|u,v \in C\}|$
- $f_c =$ nr. edges in the frontier of $C = |\{(u,v)|u \in C, v \notin C\}|$



- $n_{c_1} = 4, m_{c_1} = 5, f_{c_1} = 2$
- $n_{c_2} = 3, m_{c_2} = 3, f_{c_2} = 2$
- $n_{c_3} = 5, m_{c_3} = 8, f_{c_3} = 2$

Community quality criteria

- **conductance**: fraction of edges leaving the cluster $\frac{f_c}{2m_c+f_c}$
- **expansion**: nr of edges per node leaving the cluster $\frac{f_c}{n_c}$
- **internal density**: a.k.a. "intra-cluster density" $\frac{m_c}{n_c(n_c-1)/2}$
- **cut ratio**: a.k.a. "inter-cluster density" $\frac{f_c}{n_c(n-n_c)}$
- **modularity**: difference between nr. of edges in C and the expected nr. of edges $E[m_c]$ of a random graph with the same degree distribution

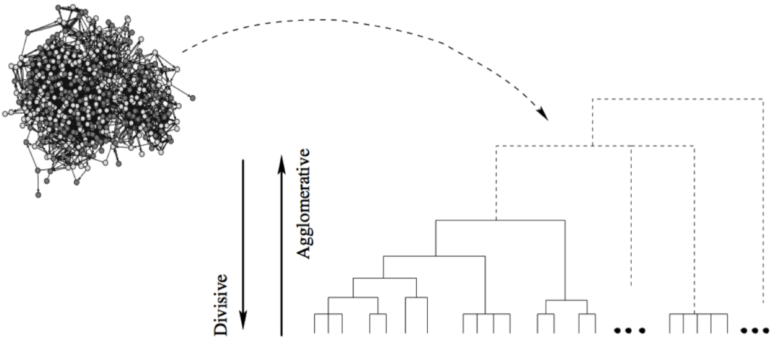
$$\frac{1}{4m}(m_c - E[m_c])$$

Methods we will cover

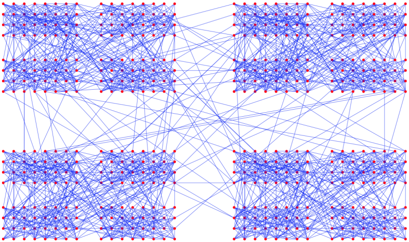
- Hierarchical clustering
  - Agglomerative
  - Divisive (Girvan-Newman algorithm)
- Modularity maximization algorithms
  - Louvain method

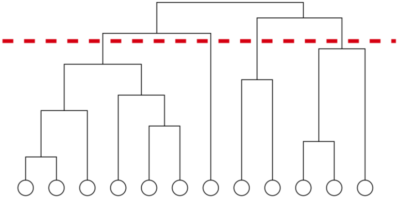
Hierarchical clustering

From hairball to *dendogram*



Suitable if input network has hierarchical structure





Agglomerative hierarchical clustering [Newman, 2010]

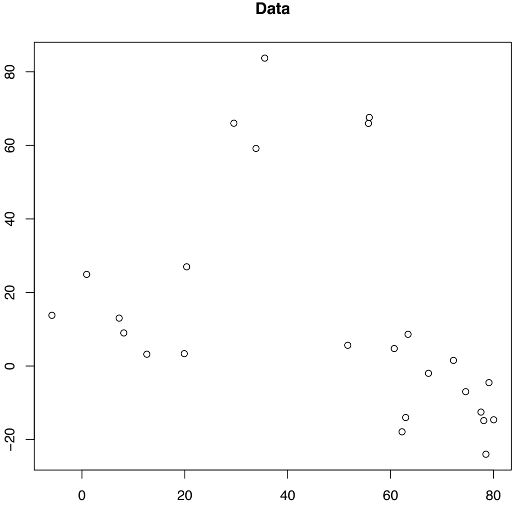
Ingredients

- Similarity measure between nodes
- Similarity measure between *sets of nodes*

Pseudocode

1. Assign each node to its own cluster
2. Find the cluster pair with highest similarity and join them together into a cluster
3. Compute new similarities between new joined cluster and others
4. Go to step 2 until all nodes form a single cluster

Example



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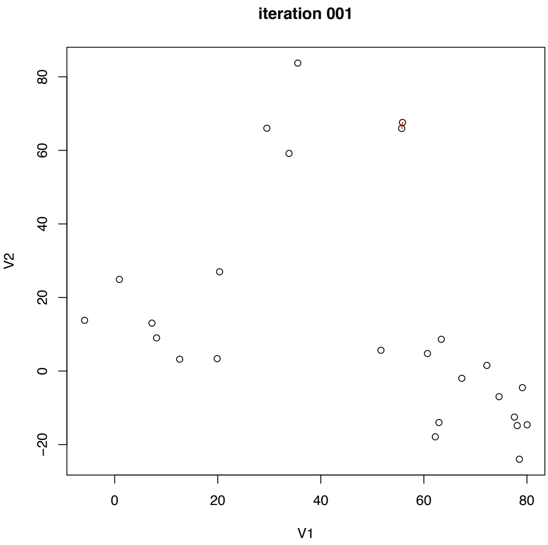

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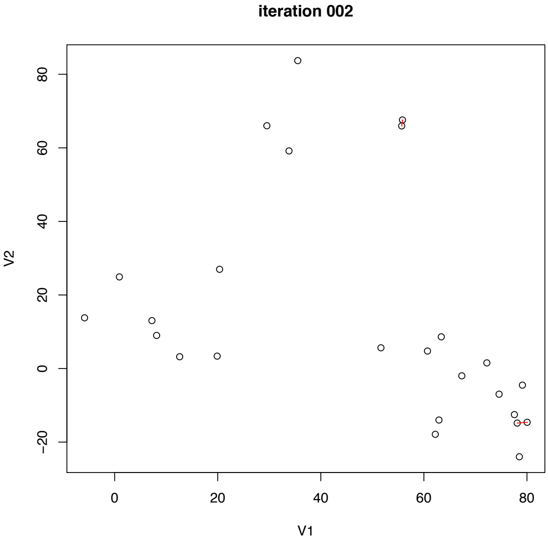

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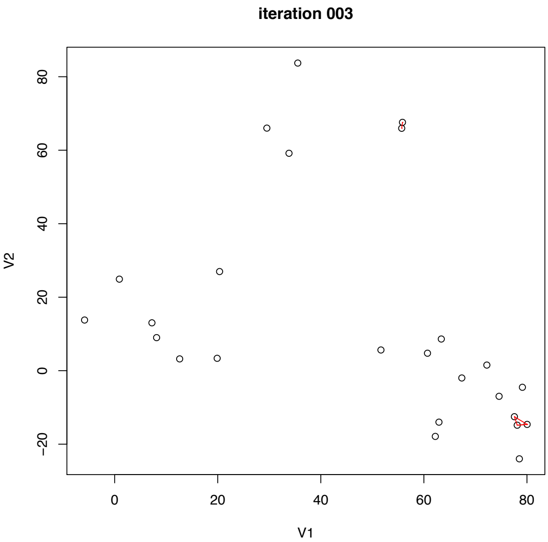

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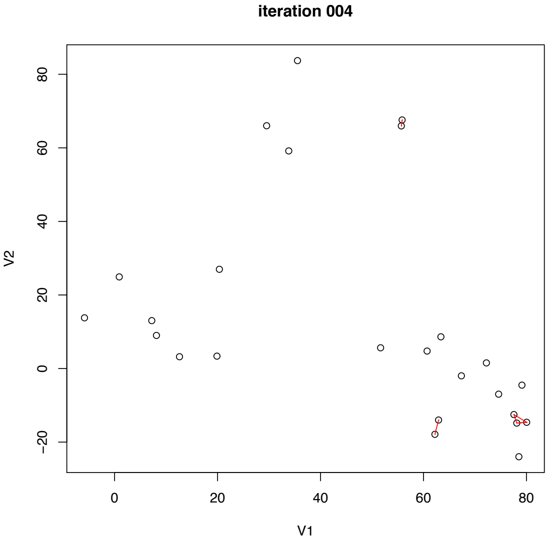

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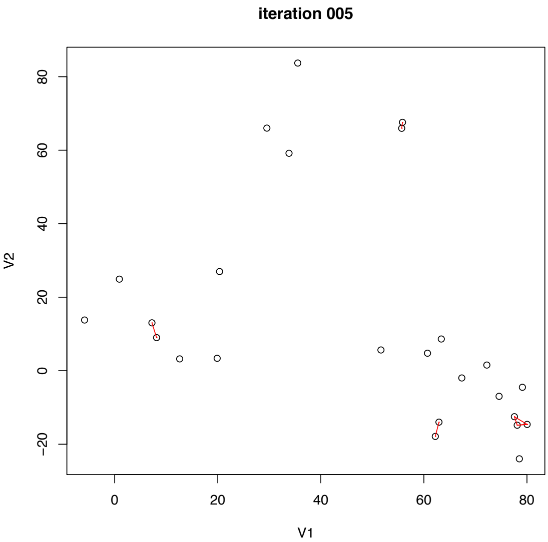

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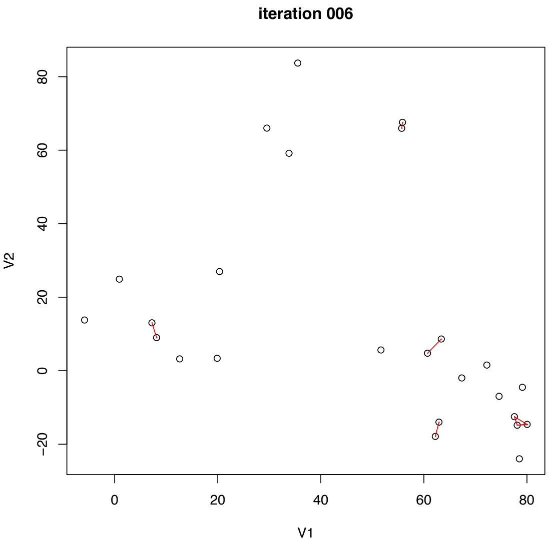

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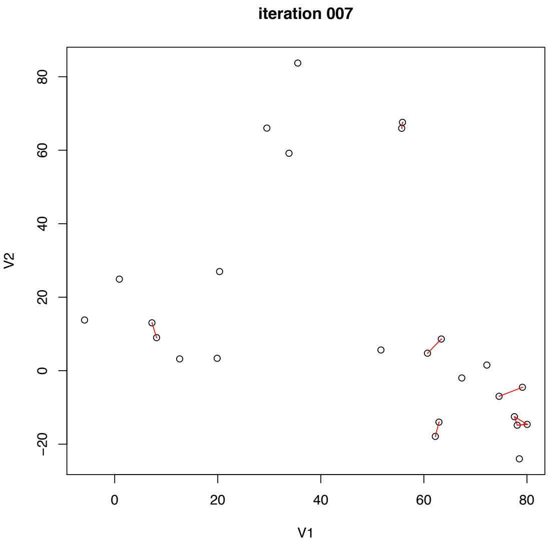

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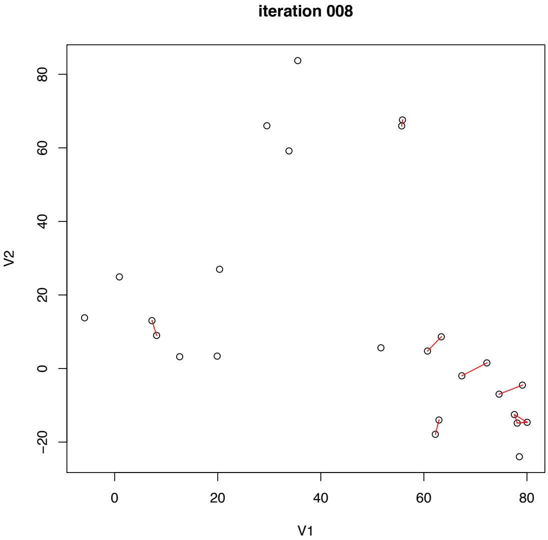

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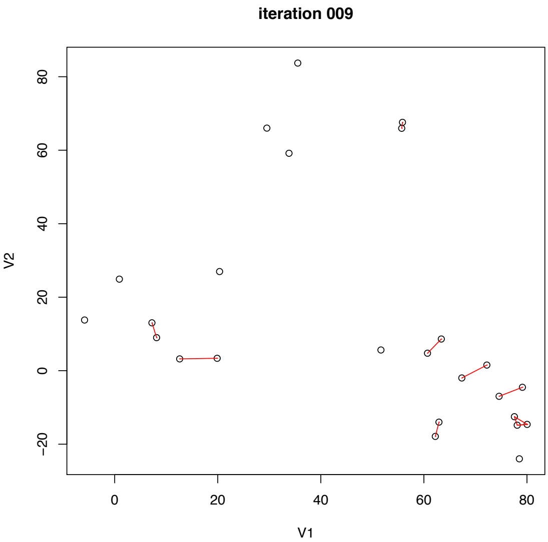

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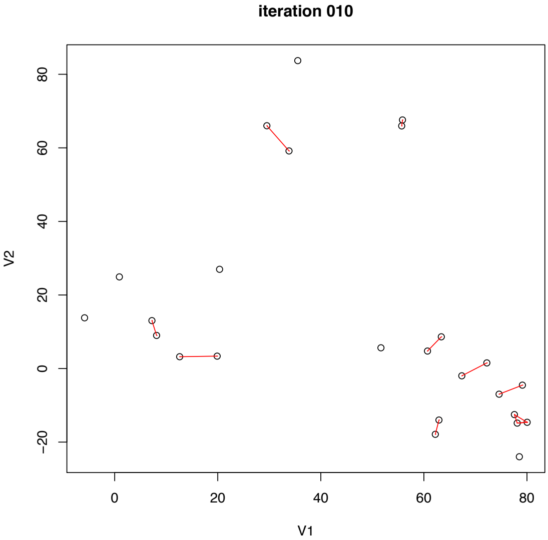

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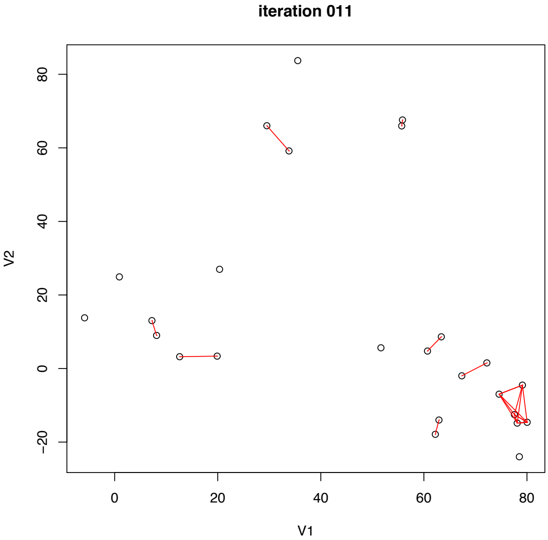

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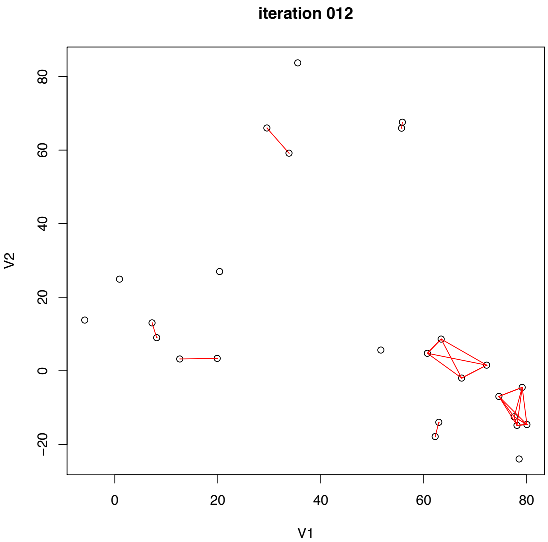

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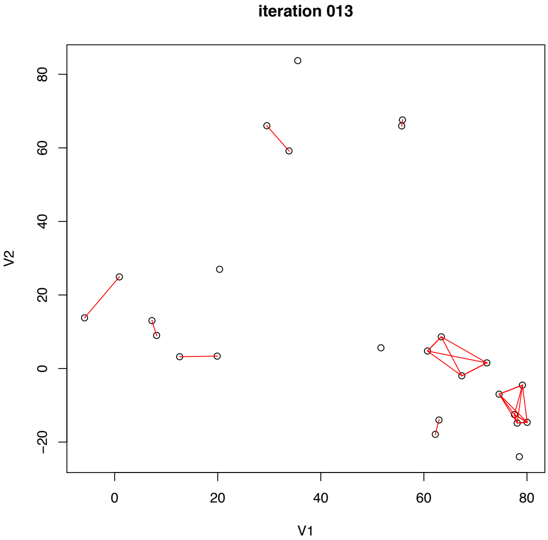

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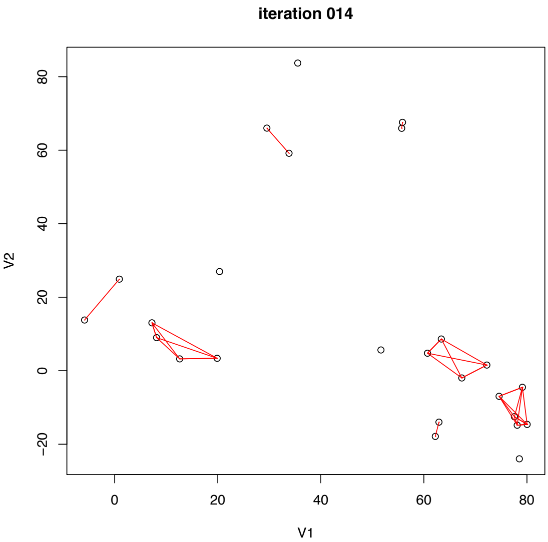

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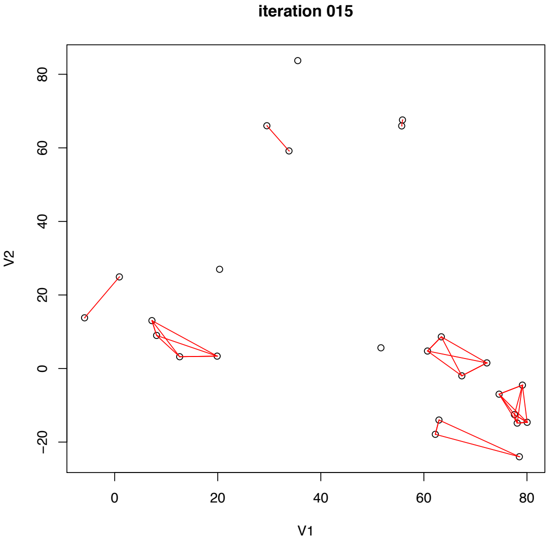

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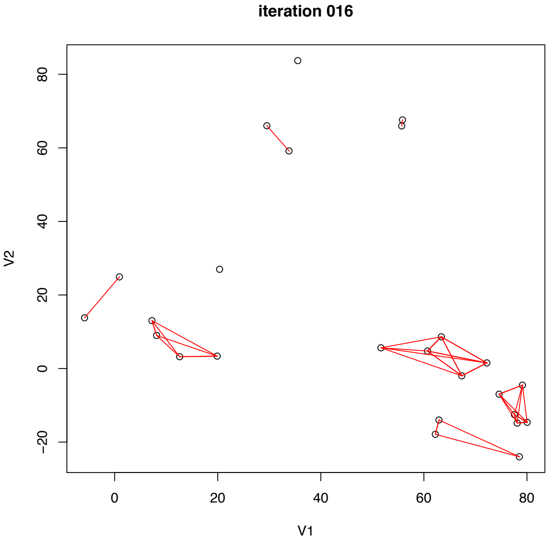

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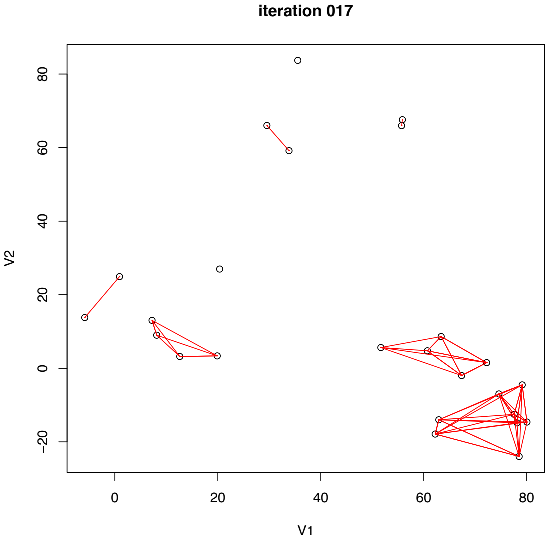

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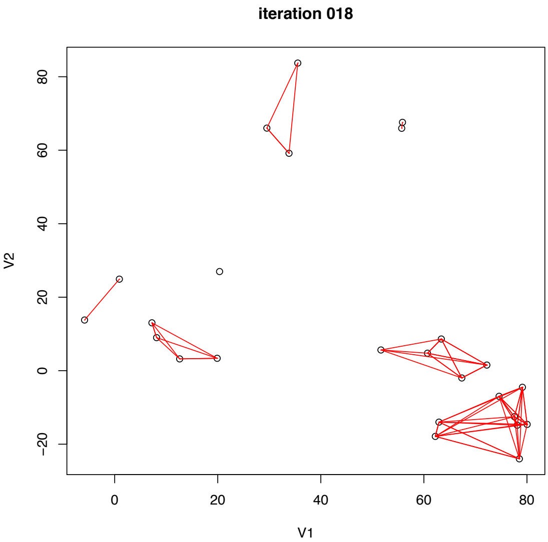

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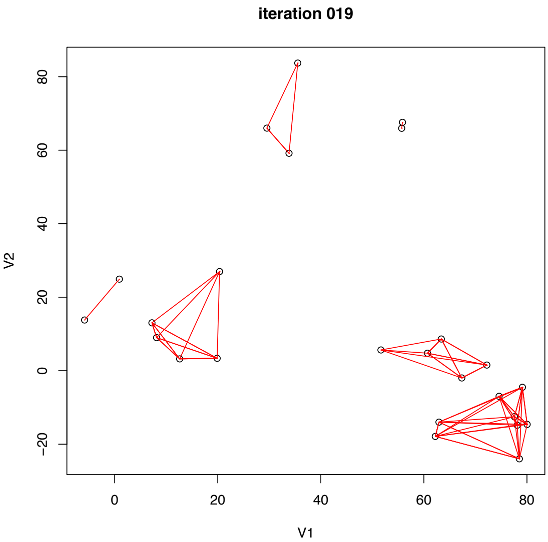

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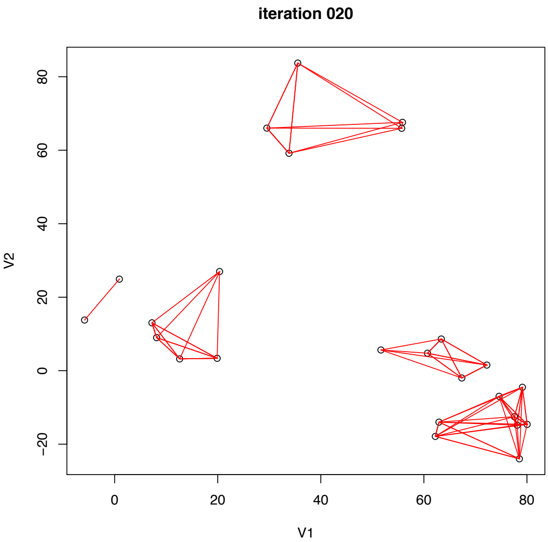

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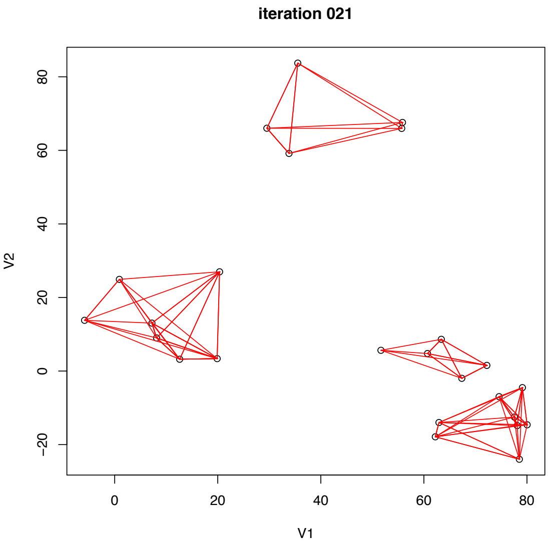

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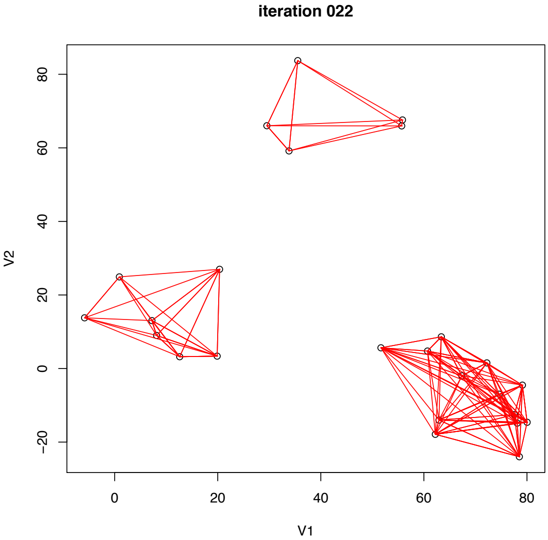

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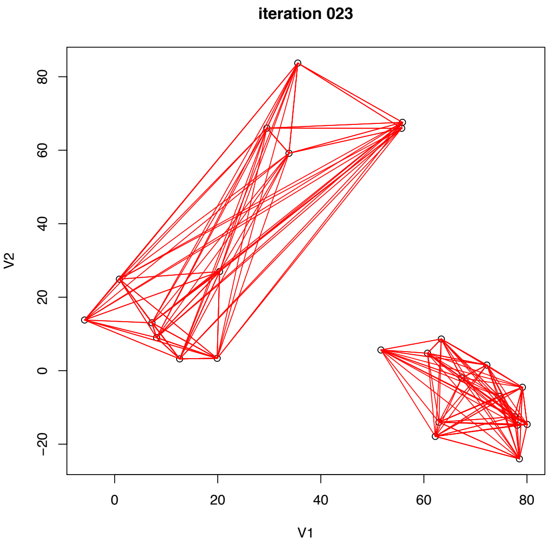

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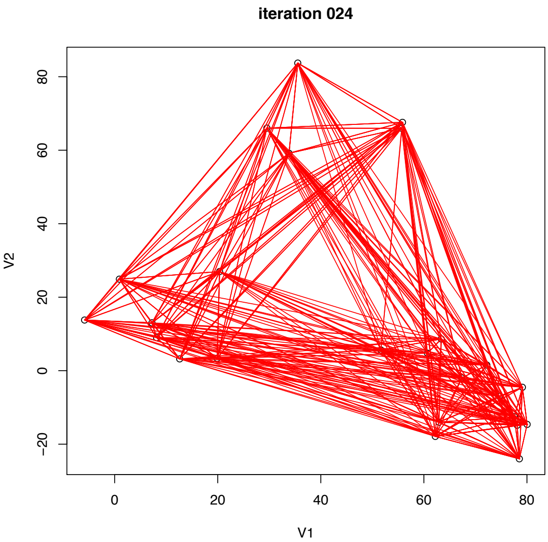


Similarity measures w_{ij} for nodes I

Let $\mathbf{A}$ be the adjacency matrix of the network, i.e. $A_{ij} = 1$ if $(i, j) \in E$ and 0 otherwise.

- **Jaccard index:**

$$w_{ij} = \frac{|\Gamma(i) \cap \Gamma(j)|}{|\Gamma(i) \cup \Gamma(j)|}$$

  where $\Gamma(i)$ is the set of neighbors of node i

- **Cosine similarity:**[2]

$$w_{ij} = \frac{\sum_k A_{ik} A_{kj}}{\sqrt{\sum_k A_{ik}^2}\sqrt{\sum_k A_{jk}^2}} = \frac{n_{ij}}{\sqrt{k_i k_j}}$$

  where:

  - $n_{ij} = |\Gamma(i) \cap \Gamma(j)| = \sum_k A_{ik} A_{kj}$, and
  - $k_i = \sum_k A_{ik}$ is the degree of node i

Similarity measures w_{ij} for nodes II

- **Euclidean distance:** (or rather Hamming distance since A is binary)

$$d_{ij} = \sum_k (A_{ik} - A_{jk})^2$$

- **Normalized Euclidean distance:**[3]

$$d_{ij} = \frac{\sum_k (A_{ik} - A_{jk})^2}{k_i + k_j} = 1 - 2\frac{n_{ij}}{k_i + k_j}$$

- **Pearson correlation coefficient**

$$r_{ij} = \frac{cov(A_i, A_j)}{\sigma_i \sigma_j} = \frac{\sum_k (A_{ik} - \mu_i)(A_{jk} - \mu_j)}{n \sigma_i \sigma_j}$$

where $\mu_i = \frac{1}{n}\sum_k A_{ik}$ and $\sigma_i = \sqrt{\frac{1}{n}\sum_k (A_{ik} - \mu_i)^2}$

---

[2] From the equation $\mathbf{xy} = |\mathbf{x}||\mathbf{y}|\cos\theta$

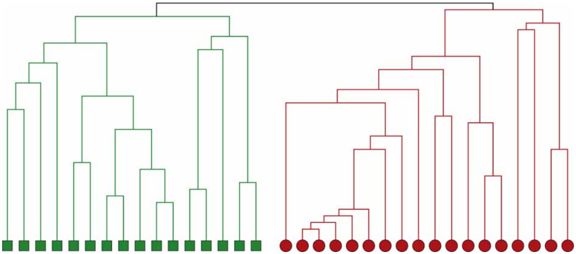
[3] Uses the idea that the maximum value of d_{ij} is when there are no common neighbors and then $d_{ij} = k_i + k_j$

Similarity measures for sets of nodes

- Single linkage: $s_{XY} = \max_{x \in X, y \in Y} s_{xy}$
- Complete linkage: $s_{XY} = \min_{x \in X, y \in Y} s_{xy}$
- Average linkage: $s_{XY} = \dfrac{\sum_{x \in X, y \in Y} s_{xy}}{|X| \times |Y|}$

Agglomerative hierarchical clustering on Zachary's network

Using average linkage



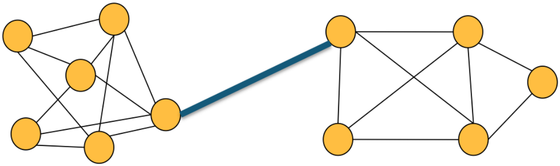
The Girvan-Newman algorithm

A *divisive* hierarchical algorithm [Girvan and Newman, 2002]

Edge betweenness

The betweenness of an edge is the nr. of shortest-paths in the network that pass through that edge

It uses the idea that “bridges” between communities must have high edge betweenness

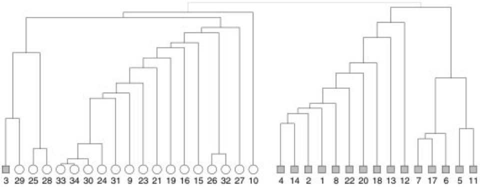


The Girvan-Newman algorithm

Pseudocode

1. Compute betweenness for all edges in the network
2. Remove the edge with highest betweenness
3. Go to step 1 until no edges left

Result is a dendogram



Definition of modularity [Newman, 2010]

Using a *null* model

Random graphs are not expected to have community structure, so we will use them as null models.

$Q =$ (nr. of intra-cluster communities) – (expected nr of edges)

In particular:

$$Q = \frac{1}{2m} \sum_{ij} (A_{ij} - P_{ij})\, \delta(C_i, C_j)$$

where P_{ij} is the expected number of edges between nodes i and j under the null model, C_i is the community of vertex i, and $\delta(C_i, C_j) = 1$ if $C_i = C_j$ and 0 otherwise.

How do we compute P_{ij}?

Using the "configuration" null model

The "configuration" random graph model choses a graph with the same degree distribution as the original graph uniformly at random.

- Let us compute P_{ij}
- There are $2m$ *stubs* or half-edges available in the configuration model
- Let p_i be the probability of picking at random a stub incident with i
$$p_i = \frac{k_i}{2m}$$
- The probability of connecting i to j is then $p_i p_j = \frac{k_i k_j}{4m^2}$
- And so $P_{ij} = 2m p_i p_j = \frac{k_i k_j}{2m}$

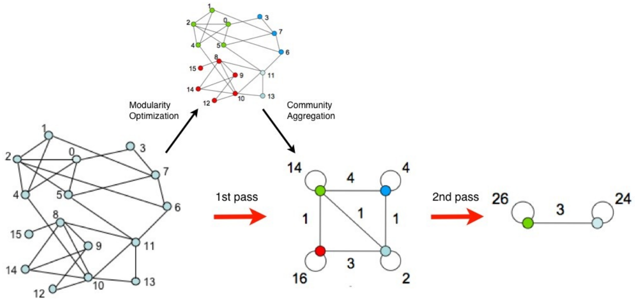
Properties of modularity

$$Q = \frac{1}{2m} \sum_{ij} \left(A_{ij} - \frac{k_i k_j}{2m} \right) \delta(C_i, C_j)$$

- Q depends on nodes in the same clusters only
- Larger modularity means better communities (better than random intra-cluster density)
- $Q \leq \frac{1}{2m} \sum_{ij} A_{ij} \, \delta(C_i, C_j) \leq \frac{1}{2m} \sum_{ij} A_{ij} \leq 1$
- Q may take negative values
  - partitions with large negative Q implies existence of cluster with small internal edge density and large inter-community edges

The Louvain method [Blondel et al., 2008]

Considered state-of-the-art



Pseudocode

1. Repeat until local optimum reached
   1.1 Phase 1: partition network greedily using modularity
   1.2 Phase 2: agglomerate found clusters into new nodes

The Louvain method

Phase 1: optimizing modularity

Pseudocode for phase 1

1. Assign a different community to each node
2. For each node i
   - For each neighbor j of i, consider removing i from its community and placing it to j's community
   - Greedily chose to place i into community of neighbor that leads to highest modularity gain
3. Repeat until no improvement can be done

The Louvain method

Phase 2: agglomerating clusters to form new network

Pseudocode for phase 2

1. Let each community C_i form a new node i
2. Let the edges between new nodes i and j be the sum of edges between nodes in C_i and C_j in the previous graph (notice there are self-loops)

The Louvain method

Observations

- The output is also a hierarchy
- Works for weighted graphs, and so modularity has to be generalized to

$$Q^w = \frac{1}{2W} \sum_{ij} \left(W_{ij} - \frac{s_i s_j}{2W} \right) \delta(C_i, C_j)$$

where W_{ij} is the weight of undirected edge (i, j), $W = \sum_{ij} W_{ij}$ and $s_i = \sum_k W_{ik}$.

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