CAIM: Cerca i Anàlisi d'Informació Massiva

FIB, Grau en Enginyeria Informàtica

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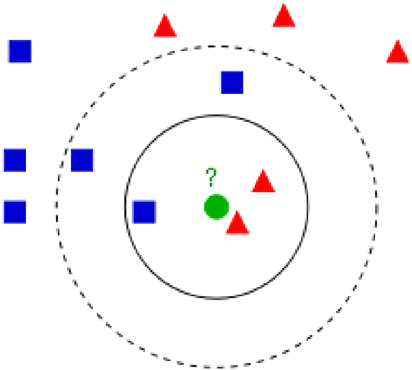
8. Locality Sensitive Hashing

Motivation, I

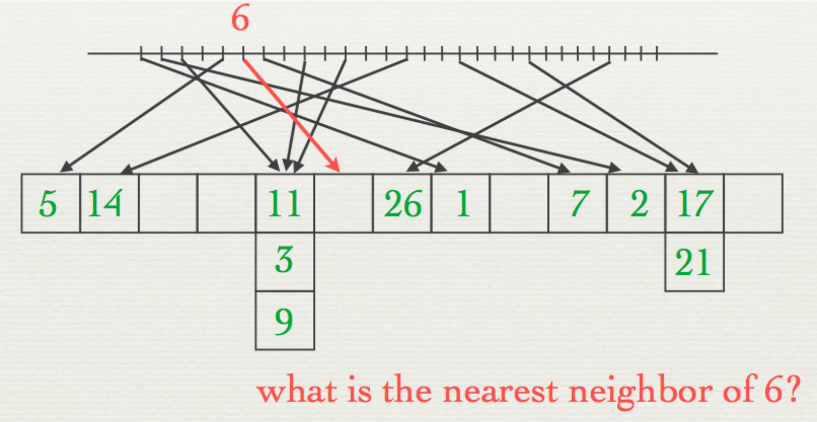
Find similar items in high dimensions, quickly

Could be useful, for example, in nearest neighbor algorithm..
but in a large, high dimensional dataset this may be difficult!



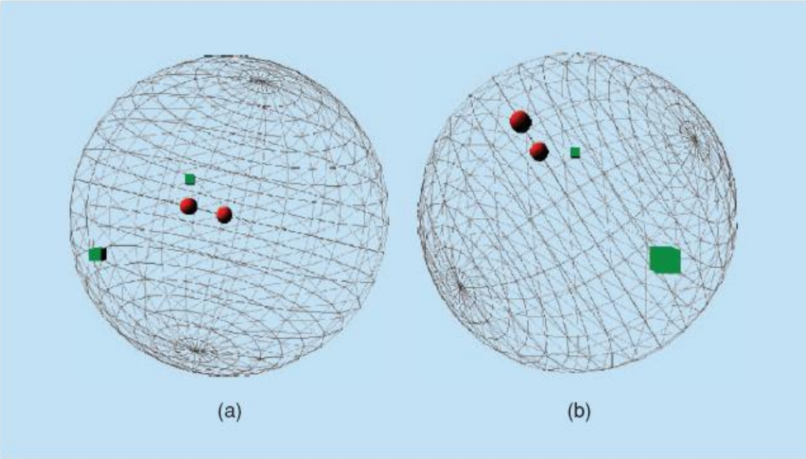
Motivation, II

Hashing is good for checking existence, not nearest neighbors



Motivation, III

Main idea: want hashing functions that map similar objects to nearby positions using *projections*



[FIG1] Two examples showing projections of two close (circles) and two distant (squares) points onto the printed page.

Different types of hashing functions

Perfect hashing

- Provide 1-1 mapping of objects to bucket ids
- Any two different objects mapped to different buckets (no collisions)

Universal hashing

- A family of functions $\mathcal{F} = \{h : U \to [n]\}$ is called *universal* if $P[h(x) = h(y)] \leq \frac{1}{n}$ for all $x \neq y$
- i.e. probability of collision for different objects is at most $1/n$

Locality sensitive hashing (lsh)

- Collision probability for *similar* objects is high enough
- Collision probability for *dissimilar* objects is low

Locality sensitive hashing functions

Definition

A family $\mathcal{F}$ is called $(s, c \cdot s, p_1, p_2)$-sensitive if for any two objects x and y we have:

- If $s(x, y) \geq s$, then $P[h(x) = h(y)] \geq p_1$
- If $s(x, y) \leq c \cdot s$, then $P[h(x) = h(y)] \leq p_2$

where the probability is taken over chosing h from $\mathcal{F}$, and $c < 1$, $p_1 > p_2$

How to use LSH to find nearest neighbor

The main idea

Pick a hashing function h from appropriate family $\mathcal{F}$

Preprocessing

- Compute $h(x)$ for all objects x in our available dataset

On arrival of query q

- Compute $h(q)$ for query object
- Sequentially check nearest neighbor in "bucket" $h(q)$

Locality sensitive hashing I

An example for bit vectors

- Objects are vectors in $\{0, 1\}^d$
- Distances are measured using Hamming distance

$$d(x, y) = \sum_{i=1}^{d} |x_i - y_i|$$

- Similarity is measured as nr. of common bits divided by length of vector

$$s(x, y) = 1 - \frac{d(x, y)}{d}$$

- For example, if $x = 10010$ and $y = 11011$, then $d(x, y) = 2$ and $s(x, y) = 1 - 2/5 = 0.6$

Locality sensitive hashing II

An example for bit vectors

- Consider the following "hashing family": sample the i-th bit of a vector, i.e. $\mathcal{F} = \{f_i | i \in [d]\}$ where $f_i(x) = x_i$
- Then, the probability of collision

$$P[h(x) = h(y)] = s(x, y)$$

  (the probability is taken over chosing a random $h \in \mathcal{F}$)
- Hence $\mathcal{F}$ is (s, cs, s, cs)-sensitive (with $c < 1$ so that $s > cs$ as required)

Locality sensitive hashing III

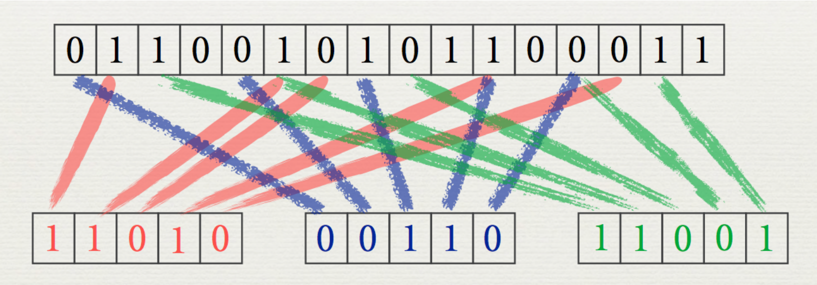
An example for bit vectors

- If gap between s and cs is too small (between p_1 and p_2), we can amplify it:
  - By stacking together k hash functions
    - $h(x) = (h_1(x), .., h_k(x))$ where $h_i \in \mathcal{F}$
    - Probability of collision of similar objects decreases to s^k
    - Probability of collision of dissimilar objects decreases even more to $(cs)^k$
  - By repeating the process m times
    - Probability of collision of similar objects increases to $1 - (1 - s)^m$
  - Choosing k and m appropriately, can achieve a family that is $(s, cs, 1 - (1 - s^k)^m, 1 - (1 - (cs)^k)^m)$-sensitive

Locality sensitive hashing IV

An example for bit vectors

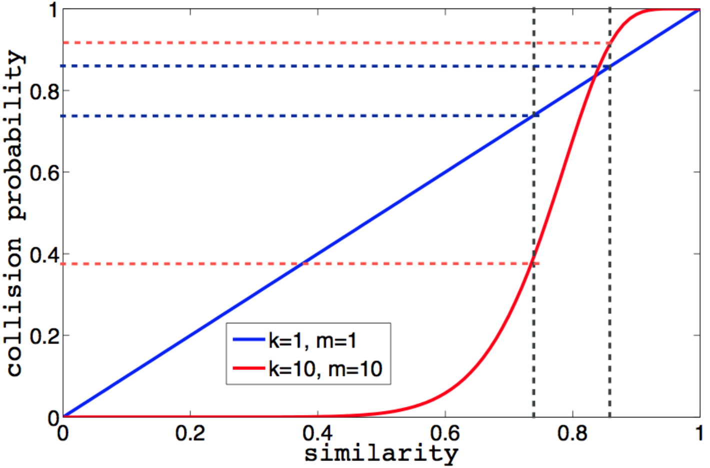
Here, $k = 5, m = 3$



Locality sensitive hashing V

An example for bit vectors

Collision probability is $1 - (1 - s^k)^m$



Similarity search becomes..

Pseudocode

Preprocessing

- Input: set of objects X
- for $i = 1..m$
  - for each $x \in X$
    - stack k hash functions and form $x_i = (h_1(x), .., h_k(x))$
    - store x in bucket given by $f(x_i)$

On query time

- Input: query object q
- $Z = \emptyset$
- for $i = 1..m$
  - stack k hash functions and form $q_i = (h_1(q), .., h_k(q))$
  - $Z_i = \{$ objects found in bucket $f(q_i)\}$
  - $Z = Z \cup Z_i$
- Output all $z \in Z$ such that $s(q, z) \geq s$

For objects in $[1..M]^d$

The idea is to represent each coordinate in unary form

- For example, if $M = 10$ and $d = 2$, then $(5, 2)$ becomes $(1111100000, 1100000000)$
- In this case, we have that the L_1 distance of two points in $[1..M]^d$ is

$$d(x, y) = \sum_{i=1}^{d} |x_i - y_i| = \sum_{i=1}^{d} d_{Hamming}(u(x), u(y))$$

  so we can concatenate vectors in each coordinate into one single dM bit-vector
- In fact, one does not need to *store* these vectors, they can be computed on-the-fly

Generalizing the idea..

- If we have a family of hash functions such that for all pairs of objects x, y

$$P[h(x) = h(y)] = s(x, y) \tag{1}$$

- We can then amplify the gap of probabilities by stacking k functions and repeating m times
- .. and so the core of the problem becomes to find a similarity function s and hash family satisfying (1)

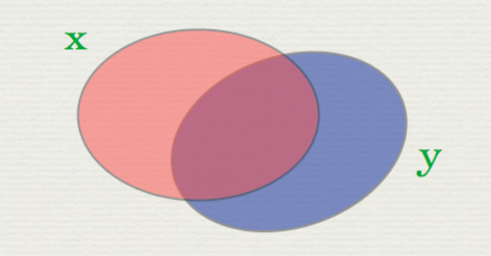
Another example: finding similar *sets* I

Using the Jaccard coefficient as similarity function

Jaccard coefficient

For pairs of sets x and y from a ground set U
(i.e. $x \subseteq U, y \subseteq U$) is

$$J(x, y) = \frac{|x \cap y|}{|x \cup y|}$$

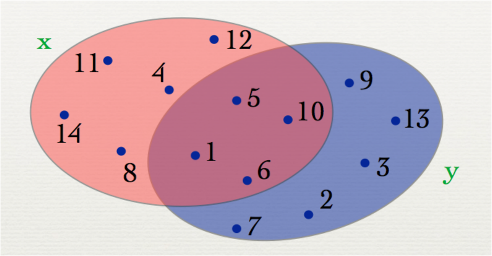


Another example: finding similar *sets* II

Using the Jaccard coefficient as similarity function

Main idea

- Suppose elements in U are ordered (randomly)
- Now, look at the smallest element in each of the sets
- The more similar x and y are, the more likely it is that their smallest element coincides



Another example: finding similar *sets* III

Using the Jaccard coefficient as similarity function

So, define family of hash functions for Jaccard coefficient:

- Consider a random permutation $r : U \rightarrow [1..|U|]$ of elements in U
- For a set $x = \{x_1, .., x_l\}$, define $h_r(x) = min_i\{r(x_i)\}$
- Let $\mathcal{F} = \{h_r | r$ is a permutation$\}$
- And so: $P[h(x) = h(y)] = J(x, y)$ as desired!

Scheme known as *min-wise independent permutation* hashing, in practice inefficient due to the cost of storing random permutations.