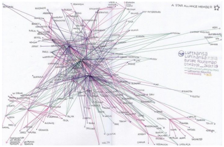
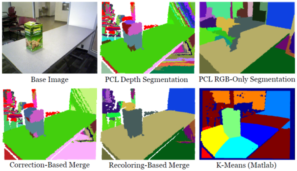
Max-flow and min-cut

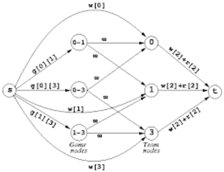
Max-Flow and Min-Cut

Two important algorithmic problems, which yield a beautiful duality

Myriad of non-trivial applications, it plays an important role in the optimization of many problems:
Network connectivity, airline schedule (extended to all means of transportation), image segmentation, bipartite matching, distributed computing, data mining, $\cdots\cdots$



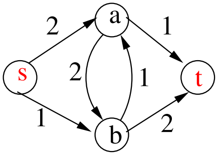




Flow Networks

Network diagraph $G = (V, E)$ s.t. it has

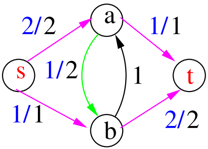
- source vertex $s \in V$
- sink vertex $t \in V$
- edge capacities $c : E \to \mathbb{R}^+ \cup \{0\}$



Flow $f : V \times V \to \mathbb{R}^+ \cup \{0\}$ s.t.
Kirchoff's laws:

- $\forall (u, v) \in E$, $0 \le f(u, v) \le c(u, v)$,
- (Flow conservation) $\forall v \in V - \{s, t\}$, $\sum_{u \in V} f(u, v) = \sum_{z \in V} f(v, z)$
- The value of a flow

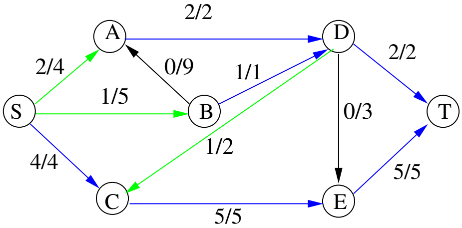
$$|f| = \sum_{v \in V} f(s, v) = f(s, V) = f(V, t).$$



Value |f|=3

The Maximum flow problem

INPUT: Given a flow network $(G = (V, E), s, t, c)$
QUESTION: Find a flow of maximum value on G.



The value of the max-flow is $7 = 4 + 1 + 2 = 5 + 2$.

Notice: Although the flow exiting s is not maximum, the flow going into t is maximum (= max. capacity).
Therefore the total flow is maximum.

The $s - t$ cut

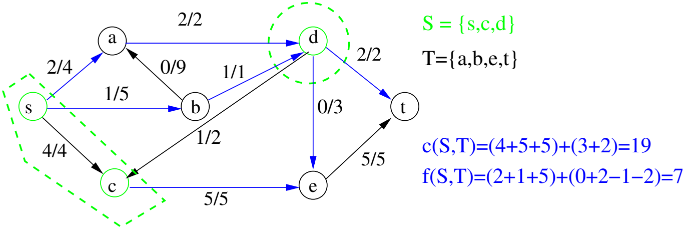
Given $(G = (V, E), s, t, c)$ a $s - t$ cut is a partition of $V = S \cup T$ $(S \cap T = \emptyset)$, with $s \in S$ and $t \in T$.

The flow across the cut:
$f(S) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{v \in S} \sum_{u \in T} f(v, u)$.

The capacity of the cut: $c(S) = \sum_{u \in S} \sum_{v \in T} c(u, v)$

capacity of cut (S, T) = sum of weights leaving S.

Notice because of the capacity constrain: $f(S) \leq c(S)$



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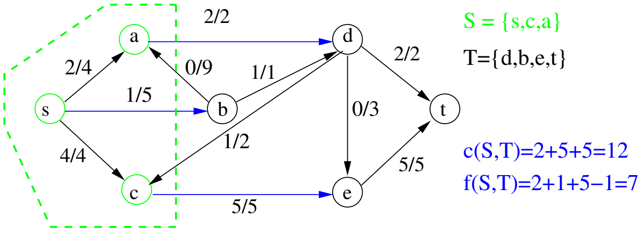
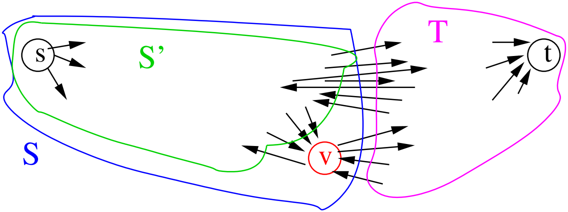


Notation

Given $v \in G$ and cut (S, T) and a $v \in S$, let $S' = S - \{v\}$. Then

- Denote $f(S', T)$ flow between S' and T (without going by v).
  i.e. $f(S', T) = \sum_{u \in S'} \sum_{w \in T} f(u, w) - \sum_{w \in T} \sum_{u \in S'} f(w, u)$ with $(u, w) \in E$ and $(u, w) \in E$,
- denote $f(v, T)$ flow $v \to T$ i.e. $f(v, T) = \sum_{u \in T} f(v, u)$,
- denote $f(T, v)$ flow $T \to v$ i.e. $f(T, v) = \sum_{u \in T} f(u, v)$,
- denote $f(S', v)$ flow $S' \to v$ i.e. $f(S', v) = \sum_{u \in S'} f(u, v)$,
- denote $f(v, S')$ flow $v \to S'$ i.e. $f(v, S') = \sum_{u \in S'} f(v, u)$,



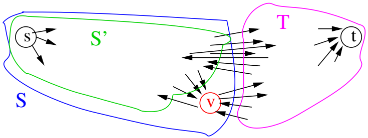
Any $s - t$ cut has the same flow

Theorem

Given (G, s, t, c) the flow through any $s - t$ cut (S, T) is $f(S) = |f|$.

Proof (Induction on $|S|$)

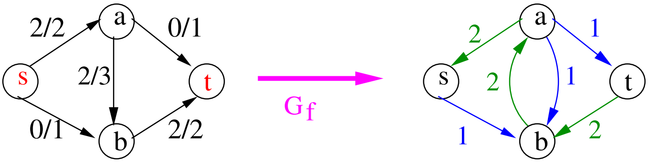
- If $S = \{s\}$ then $f(S) = |f|$.
- Assume it is true for $S' = S - \{v\}$, i.e. $f(S') = |f|$.
  Notice $f(S') = f(S', T) + f(S', v) - f(v, S')$. Moreover from the flow conservation, $f(S', v) + f(T, v) = f(v, S') + f(v, T)$
  $\Rightarrow \underbrace{f(v, T) - f(T, v) = f(S', v) - f(v, S')}_{*}$
- Then $f(S) = f(S', T) + f(v, T) - f(T, v)$, using $(*)$
  $f(S) = f(S') = |f|$ $\square$



Residual network

Given a network $(G = (V, E), s, t, c)$ together with a flow f on it, the residual network, $(G_f = (V, E_f), c_f)$ is the network with the same vertex set and edge set:

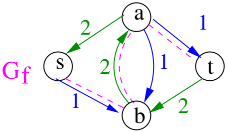
- if $c(u, v) - f(u, v) > 0$ then $(u, v) \in E_f$ and $c_f(u, v) = c(u, v) - f(u, v) > 0$ (forward edges), and
- if $f(u, v) > 0$ then $(v, u) \in E_f$ and $c_f(v, u) = f(u, v)$ (backward edges). i.e. there are $f(u, v)$ units of flow we can undo, by pushing flow backward. Notice, if $c(u, v) = f(u, v)$ then there is only a backward edge.
- the c_f are denoted residual capacity.



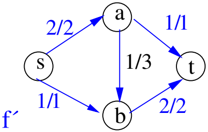
Residual network: Augmenting paths

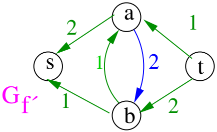
Given $G = (V, E)$ and a flow f on G, an augmenting path P is any simple path in G_f (using forward and backward edges, but $P : s \rightsquigarrow t$).

Given $f : s \rightsquigarrow t$ in G and P in G_f define the bottleneck (P, f) to be the minimum residual capacity of any edge in P, with respect to f.



P: dotted line





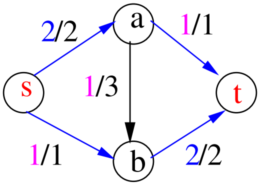
Residual network: Augmenting paths

Given $G = (V, E)$ and a flow f on G, an augmenting path P is any simple path in G_f.

Given f $s \to t$ in G and P in G_f define the bottleneck (P, f) to be the minimum residual capacity of any edge in P.

Augment(P, f)
b=bottleneck (P, f)
for each $(u, v) \in P$ **do**
  if (u, v) is forward edge in G **then**
    Increase $f(u, v)$ in G by b
  else
    Decrease $f(u, v)$ in G by b
  end if
end for
return f

f'



Residual network: Augmenting paths

Lemma
Consider $f' =$Augment(P, f), then f' is a flow in G.

Proof: We have to prove that (1) $\forall e \in E$, $0 \leq f(e) \leq c(e)$ and that $\forall v$ flow to $v =$ flow out of v.

- Capacity law Forward edges $(u, v) \in P$ we increase $f(u, v)$ by b, as $b \leq c(u, v) - f(u, v)$ then $f'(u, v) = f(u, v) + b \leq c(u, v)$.
  Backward edges $(u, v) \in P$ we decrease $f(v, u)$ by b, as $b \leq f(v, u), f'(v, u) = f(u, v) - b \geq 0$.
- Conservation law, $\forall v \in P$ given edges e_1, e_2 in P and incident to v, it is easy to check the 4 cases based whether e_1, e_2 are forward or backward edges. $\square$

Max-Flow Min-Cut theorem

Theorem

For any (G, s, t, c) the value of the max flow f^* is equal to the capacity of the min (S, T)-cut (over all $s - t$ cuts in G)

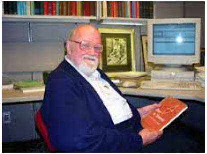
$$f^* = \max\{|f|\} = \min_{\forall(S,T)} \{c(S, T)\}.$$

Proof:

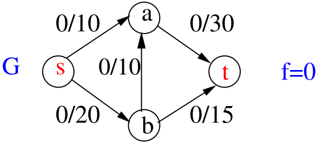
- For any $s - t$ cut (S, T) in $G \Rightarrow f^*(S) \leq c(S, T)$.
- If f^* in G is a max flow then G_{f^*} has no augmenting path $s \leadsto t$ so it is disconnected.
  Let $S_s = \{v \in V | \exists s \leadsto v \text{ in } G_{f^*}\}$, then $(S_s, V - \{S_s\})$ is a $s - t$ cut in $G_{f^*} \Rightarrow \forall v \in S_s, u \in V - \{S_s\}$, $(\vec{v, u})$ is not a residual edges, so in G $f^*(v, u) = c(v, u)$, i.e. $c(S_s, V - \{S_s\}) = f^*(S_s, V - \{S_s\})$ in G. In particular $(S_s, V - \{S_s\})$ is a min-cut in G and $=$ max-flow f^*. $\square$

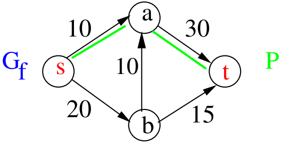
Ford-Fulkerson algorithm

L.R. Ford, D.R. Fulkerson: *Maximal flow through a network*. Canadian J. of Math. 1956.



Ford-Fulkerson(G, s, t, c)
for all $(u, v) \in E$ let $f(u, v) = 0$
$G_f = G$
while there is an $s - t$ path in G_f **do**
  find a simple path P in G_f (use DFS)
  $f' = \text{Augment}(f, P)$
  Update f to f'
  Update G_f to $G_{f'}$
end while
return f





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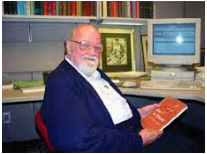

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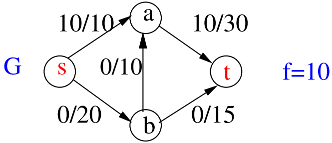

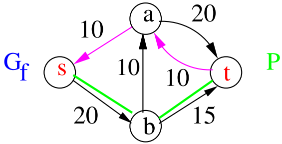

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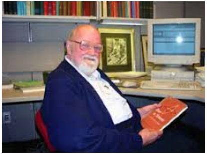

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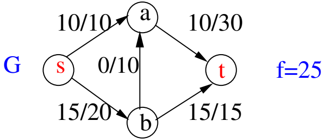

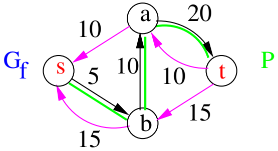

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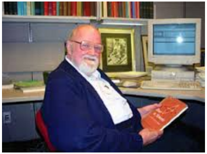

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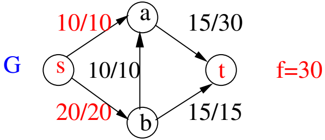
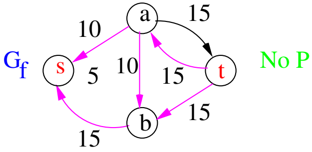




Analysis of Ford Fulkerson

We are considering networks that initial flow and capacities are integers,

Lemma (Integrality invariant)

At every iteration of the Ford-Fulkerson algorithm, the flow values $f(e)$ and the residual capacities in G_f are integers.

Proof: (induction)

- The statement is true before the **while** loop.
- Inductive Hypothesis: The statement is true after j iterations.
- iteration $j+1$: As all residual capacities in G_f are integers, then bottleneck $(P, f) \in \mathbb{Z}$, for the augmenting path found in iteration $j+1$. Thus the flow f' will have integer values $\Rightarrow$ so will the capacities in the new residual graph. $\square$

Corollary: Integrality theorem

Theorem (Integrality theorem)

There exists a max-flow f^* for which every flow value f^* is an integer.

Proof:

Since the algorithm terminates, the theorem follows from the integrality invariant lemma. $\square$

Analysis of Ford Fulkerson

Lemma

If f is a flow in G and f' is the flow after an augmentation, then $|f| < |f'|$.

Proof: Let P be the augmenting path in G_f. The first edge $e \in P$ leaves s, and as G has no incoming edges to s, e is a forward edge. Moreover P is simple $\Rightarrow$ never returns to s. Therefore, the value of the flow increases in edge e. $\square$

Correctness of Ford-Fulkerson

Consequence of the Max-flow min-cut theorem.

Theorem

The flow returned by Ford-Fulkerson f^* is the max-flow.

Proof:

- For any flow f and $s - t$ cut (S, T) we have $|f| \leq c(S, T)$.
- The flow f^* is such that $|f^*| = c(S^*, T^*)$, for some $s - t$ cut $(S^*, T^*) \Rightarrow f^*$ is the max-flow. $\square$

Therefore, for any (G, s, t, c) the value of the max $s - t$ flow is equal to the capacity of the minimum $s - t$ cut.

Analysis of Ford Fulkerson: Running time

Lemma

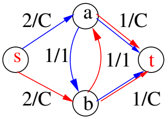
Let C be the min cut capacity (=max. flow value), Ford-Fulkerson terminates after finding at most C augmenting paths.

Proof: The value of the flow increases by ≥ 1 after each augmentation. □

- The number of iterations is $\leq C$. At each iteration:
- We have to modify G_f, with $E(G_f) \leq 2m$, to time $O(m)$.
- Using DFS, the time to find an augmenting path P is $O(n+m)$
- Total running time is $O(C(n+m)) = O(Cm)$
- Is that polynomic?

Running time of Ford-Fulkerson

The number of iterations of Ford-Fulkerson could be $\Omega(C)$
As it is described Ford-Fulkerson can alternate C times between the blue and red paths if the figure.



C=1000000000
2000 million iteractions in a G with 4 vertices!!

Recall a pseudopolynomial algorithm is an algorithm that is polynomial in the unary encoding of the input.

Is there a polynomial time algorithm for the max-flow problem?

Edmonds-Karp, Dinic algorithm

J.Edmonds, R. Karp: *Theoretical improvements in algorithmic efficiency for network flow problems*. Journal ACM 1972.

Y. Dinic: *Algorithm for solution of a problem of maximum flow in a network with power estimation*. Doklady Ak.N. 1970

Choosing a good augmenting path can lead to a faster algorithm.
Use BFS to find shorter augmenting paths in G_f.

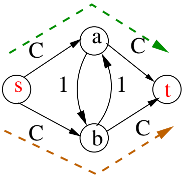


Using BFS on G_f we can find the shortest augmenting path P in $O(m)$, independently of max capacity C.

Edmonds-Karp algorithm

Uses BFS to find the augmenting path at each G_f with fewer number of edges.

Edmonds-Karp(G, s, t, c)
For all $e = (u, v) \in E$ let $f(u, v) = 0$
$G_0 = G$
while there is an $s \rightsquigarrow t$ path in G_f
do
  $P = \mathrm{BFS}(G_f, s, t)$
  $f' = \mathrm{Augment}(f, P)$
  Update $G_f = G_{f'}$ and $f = f'$
end while
return f



The BFS in EK will choose: or

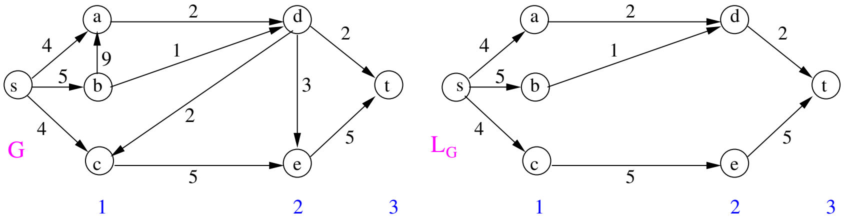
Level graph

Given $G = (V, E), s$, define $L_G = (V, E_G)$ to be its the level graph by:

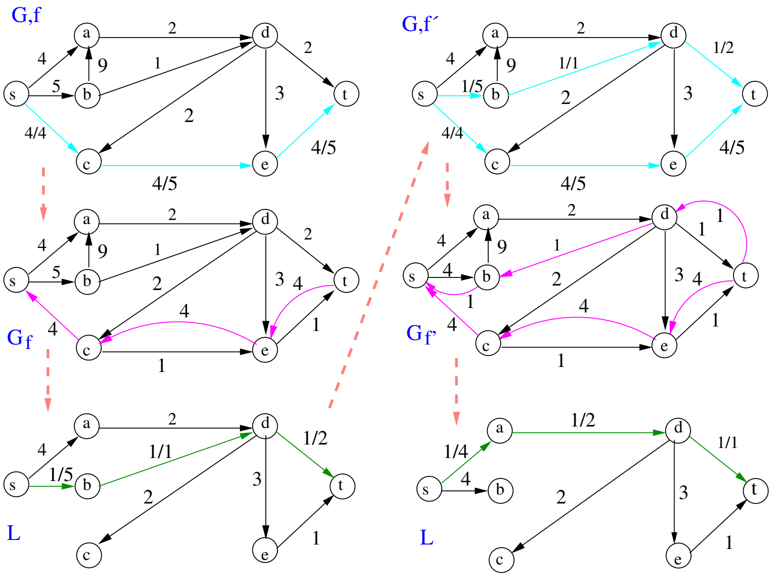
- $\ell(v)$ = number of edges in shortest path $s \rightsquigarrow v$ in G,
- $L_G = (V, E_G)$ is the subgraph of G that contains only edges $(v, w) \in E$ s.t. $\ell(w) = \ell(v) + 1$.

Notice:

- Using BFS we can compute L_G in $O(n + m)$
- Important property: P is a shortest $s \rightsquigarrow t$ in G iff P is an $s \rightsquigarrow t$ path in L_G.



The working of the EK algorithm



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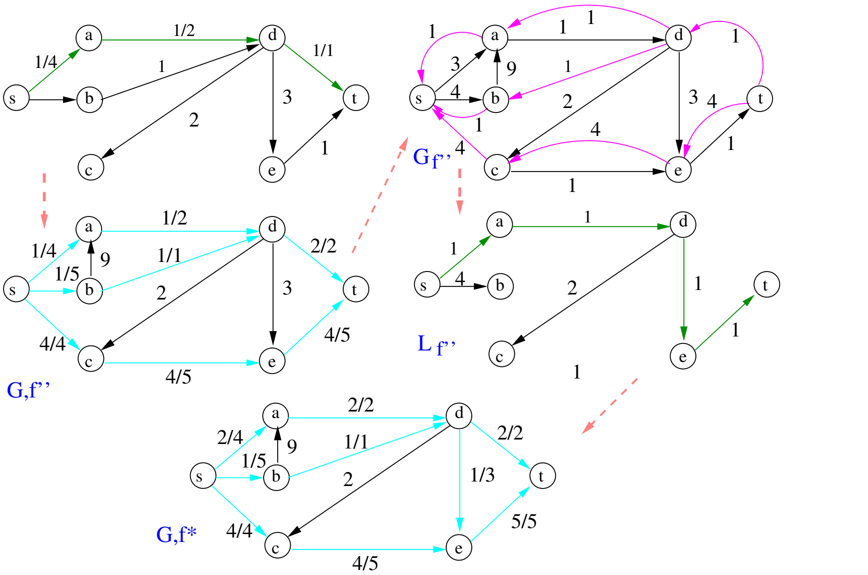


EK algorithm: Properties

Lemma

Throughout the algorithm, the length of the shortest path never decreases.

Proof:

- Let f and f' be the flow before and after a shortest path augmentation
- let L and L' be the levels graphs of G_f and $G_{f'}$.
- Only back edges added to $G_{f'}$. $\square$

Lemma

After at most m shortest path augmentations, the length of P is monotonically increasing.

Proof:

- The bottleneck edge is deleted from L after each augmentation.
- No new edge is added to L until length of shortest path strictly increases $\square$

Complexity of Edmonds-Karp algorithm

Using the the previous lemmas, we prove

Theorem

The EK algorithms runs in $O(m^2 n)$ steps. Therefore it is a polynomial time algorithm.

Proof:

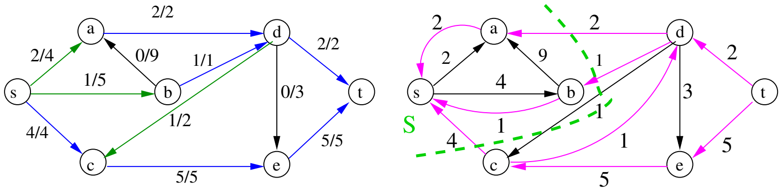
- Need time $O(m+n)$ to find the augmenting path using BFS.
- Need $O(m)$ augmentations for paths of length k.
- Every augmentation path is simple $\Rightarrow 1 \leq k \leq n \Rightarrow O(nm)$ augmentations $\square$

Finding a min-cut

Given (G, s, t, c) to find a min-cut:

1. Compute the max-flow f^* in G.
2. Obtain G_{f^*}.
3. Find the set $S = \{v \in V | s \rightsquigarrow v\}$ in G_{f^*}.
4. Output the cut $(S, V - \{S\}) = \{(v, u) | v \in S and u \in V - \{S\}\}$ in G.

The running time is the same than the algorithm to find the max-flow.



The max-flow problems: History

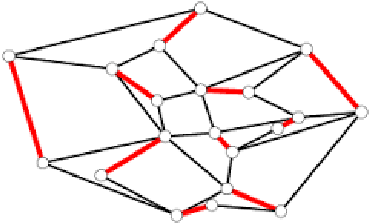
- Ford-Fulkerson (1956) $O(mC)$, where C is max capacity.
- Dinic (1970) (blocking flow) $O(n^2 m)$
- Edmond-Karp (1972) (shortest augmenting path) $O(nm^2)$
- Karzanov (1974), $O(n^2 m)$ Goldberg-Tarjant (1986) (push re-label preflow + dynamic trees) $O(nm \lg(n^2/m))$ (for this time uit uses parallel implementation)
- King-Rao-Tarjan (1998) $O(nm \log_{m/n \lg n} n)$.
- J. Orlin (2013) $O(nm)$ (clever follow up to KRT-98)

Maximum matching problem

Given an undirected graph $G = (V, E)$ a subset of edges $M \subseteq E$ is a matching if each node appears at most in one edge (a node may not appear at all).

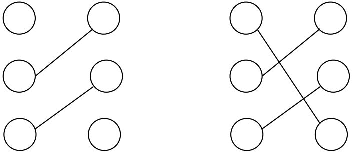
A perfect matching in G is a matching M such that $|M| = |V|/2$

The maximum matching problem given a graph G a matching with maximum cardinality.

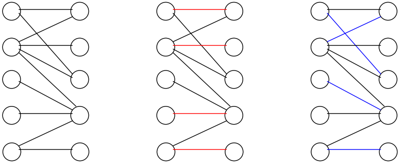


Maximum matching in graphs bipartite

A graph $G = (V, E)$ is said to be bipartite if V can be partite in L and R, $L \cup R = V$, $L \cap R = \emptyset$, such that every $e \in E$ connects L with R.



The max matching bipartite graph problem: given a bipartite $G = (L \cup R, E)$ with $2n$ vertices find a maximum matching.

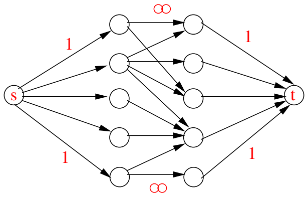


Max matchings = 4

Maximum matching: flow formulation

Given a bipartite graph $G = (L \cup R, E)$ construct $\hat{G} = (\hat{V}, \hat{E})$:

- Add vertices s and t: $\hat{V} = L \cup R \cup \{s, t\}$.
- Add directed edges $s \to L$ with capacity 1. Add directed edges $R \to t$ with capacity 1.
- Direct the edges E from L to R, and give them capacity ∞.
- $\hat{E} = \{s \to L\} \cup E \cup \{R \to t\}$.



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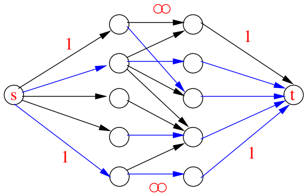
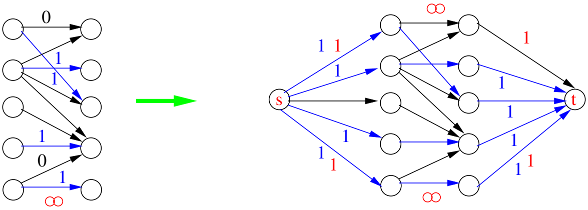


Maximum matching: Analysis

Theorem
Max flow in $\hat{G}$=Max bipartite matching in G.

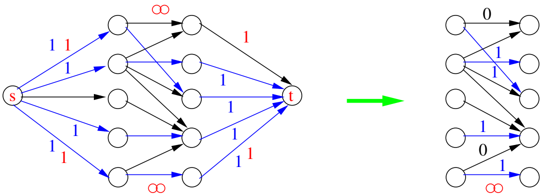
Proof $\leq$
Given a matching M in G with k-edges,
consider the flow F that sends 1 unit along each one of the k paths.
f is a flow and has value k.



Maximum matching: Analysis

Max flow $\leq$Max bipartite matching

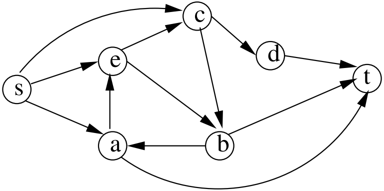
- If there is a flow f in $\hat{G}$, $|f| = k$, as capacities are $\mathbb{Z}^*$ $\Rightarrow$ an integral flow existsis.
- Consider the cut $C = (\{s\} \cup L, R \cup \{t\})$ in $\hat{G}$.
- Let F be the set of edges in C with flow=1, then $|F| = k$.
- Each node in L is in at most one $e \in F$ and every node in R is in at most one head of an $e \in F$
- Therefore, exists a bipartite matching F in G with $|F| \leq |f|$ $\square$



Disjoint path problem

Given a digraph $(G = (V, E), s, t)$, a set of paths is edge-disjoint if their edges are disjoint (although them may go through some of the same vertices)

The disjoint path problem given G, s, t find the max number of edge disjoint paths $s \rightsquigarrow t$



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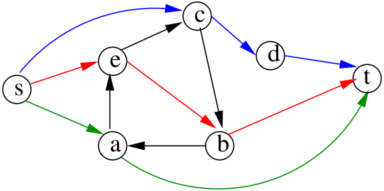
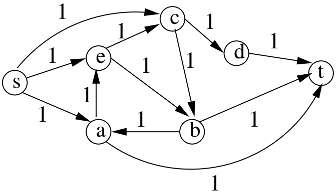


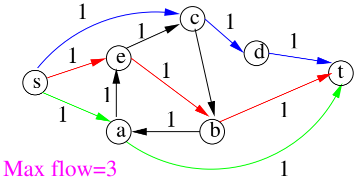
Disjoint path problem: Max flow formulation

Assign unit capacity to every edge

Theorem

The max number of edge disjoint paths $s \rightsquigarrow t$ is equal to the max flow value





Disjoint path problem: Proof of the Theorem

Number of disjoints paths $\leq$ max flow

If we have k edge-disjoints paths $s \rightsquigarrow t$ in G then making $f(e) = 1$ for each e in a path, we get a flow $= k$

Number of disjoints paths $\geq$ max flow

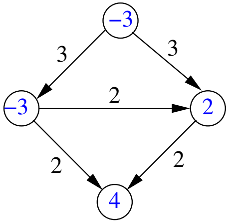
If max flow $|f^*| = k \Rightarrow \exists$ 0-1 flow f^* with value k
$\Rightarrow \exists k$ edges (s, v) s.t. $f(s, v) = 1$, by flow conservation we can extend to k paths $s \rightsquigarrow t$, where each edge is a path carries flow $= 1$. $\square$

If we have an undirected graph, with two distinguised nodes u, v, how would you apply the max flow formulation to solve the problem of finding the max number of disjoint paths between u and t?

Circulation with demands

Given a graph $G = (V, E)$ with capacities c in the edges, such that each $v \in V$ is associate with a demand $d(v)$, where

- If $d(v) > 0 \Rightarrow v$ is a sink, v can receive $d(v)$ units of flow more than it sends.
- If $d(v) < 0 \Rightarrow v$ is a source, v can send $d(v)$ units of flow more than it receives.
- If $d(v) = 0$ then v is neither a source or a sink.
- Let S be the set of sources and T the set of sinks.

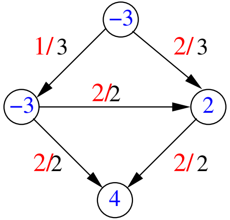


Circulation with demands problem

Given $G = (V, E)$ with $c \geq 0$ and $\{d(v)\}_{v \in V}$, define a circulation as a function $f : E \to \mathbb{R}^+$ s.t.

1. capacity: For each $e \in E$, $0 \leq f(e) \leq c(e)$,
2. conservation: For each $v \in V$,

$$\sum_{(u,v) \in E} f(u,v) - \sum_{(v,z) \in E} f(v,z) = d(v).$$



Circulation with demands feasibility problem: Given $G = (V, E)$ with $c \geq 0$ and $\{d(v)\}_{v \in V}$, does it exists a feasible circulation?
Feasible circulation: a function f on G with $c \geq 0$ and $\{d(v)\}_{v \in V}$, such that it satisfies (1) and (2)?

Circulation with demands problem

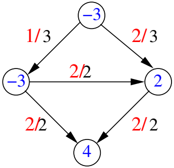
Notice that if f is a feasible circulation, then

$$\sum_{v \in V} d(v) = \sum_{v \in V} \left(\underbrace{\sum_{(u,v) \in E} f(u,v)}_{\text{edges to } v} - \underbrace{\sum_{(v,z) \in E} f(v,z)}_{\text{edges out of } v} \right).$$

Notice $\sum_{v \in V} d(v) = 0$, so we have,

So If there is a feasible circulation with demands $\{d(v)\}_{v \in V}$, then $\sum_{v \in V} d(v) = 0$.

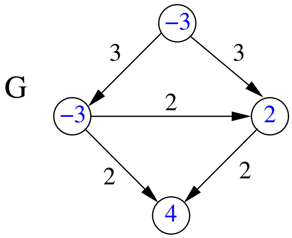
Therefore as $S = \{v \in V | d(v) > 0\}$ and $T = \{v \in V | d(v) < 0\}$, we can define $D = -\sum_{v \in S} d(v) = \sum_{v \in T} d(v)$.

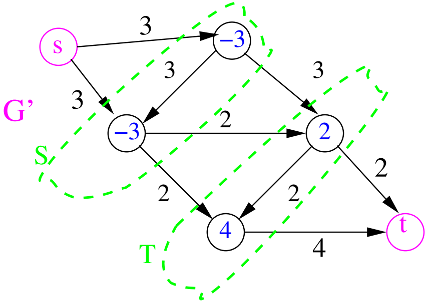


Circulation with demands: Max-flow formulation

Extend $G = (V, E)$ to $G' = (V', E')$ by

- Add new source s and sink s.
- For each $v \in S$ $(d(v) < 0)$ add (s, v) with capacity $-d(v)$.
- For each $v \in T$ $(d(v) > 0)$ add (v, s) with capacity $d(v)$.

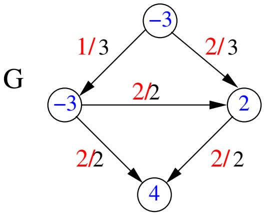


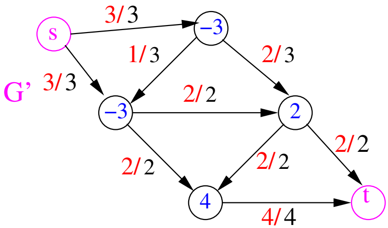


Analysis

1.- Every flow $f : s \rightsquigarrow t$ in G' must be $|f| \leq D$
The capacity $c(\{s\}, V) = D \Rightarrow$ by max-flow min-cut Thm. any max-flow f in G', $|f| \leq D$.

2.- If there is a feasible circulation f with $\{d(v)\}_{v \in V}$ in G, then we have a max-flow $f : s \rightsquigarrow t$ in G with $|f| = D$
$\forall (s, v) \in E'$, $f'(s, v) = -d(v)$ and $\forall (u, t) \in E'$, $f'(u, t) = d(v)$.

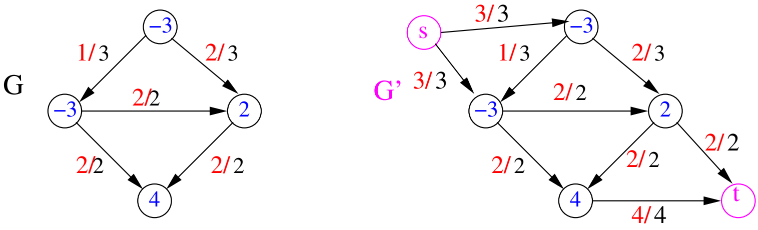




Analysis

3.- If there is a flow $f' : s \rightsquigarrow t$ in G' with $|f| = D$:

1. $\forall (s, v) \in E'$ and $\forall (u, t) \in E'$ must be saturated $\Rightarrow$ if we delete these edges in G' we obtain a circulation f in G.
2. f satisfies $d(v) = \underbrace{\sum_{(u,v) \in E} f(u, v)}_{\text{edges to } v} - \underbrace{\sum_{(v,z) \in E} f(v, z)}_{\text{edges out of } v}$.



Main results

Theorem (Circulation integrality theorem)

If all capacities and demands are integers, and there exists a circulation, then there exists an integer valued circulation.

Sketch Proof Max-flow formulation + integrality theorem for max-flow $\square$

From the previous discussion, we can conclude:

Theorem (Necessary and sufficient condition)

There is a feasible circulation with $\{d(v)\}_{v \in V}$ in G iff the max-flow in G' has value D.

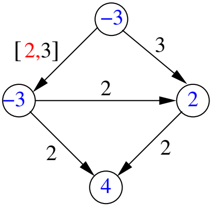
Circulations with demands and lower bounds: Max-flow formulation

Generalization of the previous problem: besides satisfy demands at nodes, we want to force the flow to use certain edges.
Introduce a new constrain $\ell(e)$ on each $e \in E$, indicating the min-value the flow must be on e.

Given $G = (V, E)$ with $c(e)$, $c(e) \geq \ell(e) \geq 0$, for each $e \in E$ and $\{d(v)\}_{v \in V}$, define a circulation as a function $f : E \to \mathbb{R}^+$ s.t.

1. capacity: For each $e \in E$, $\ell(e) \leq f(e) \leq c(e)$,
2. conservation: For each $v \in V$,

$$\sum_{(u,v) \in E} f(u,v) - \sum_{(v,z) \in E} f(v,z) = d(v).$$

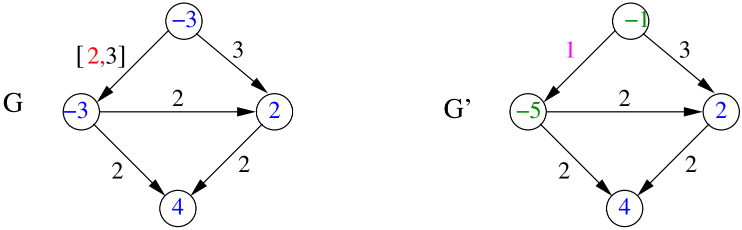


Circulation problems with lower bounds: Given $(G, c, \ell, \{d(v)\})$, does there exists a feasible circulation?

Circulations with demands and lower bounds: Max-flow formulation

Let $(G = (V, E), c, \ell, d(\cdot))$ be a graph, construct $G' = (V, E), c', d')$, where for each $e = (u, v) \in E$, with $\ell(e) > 0$:

- $c'(e) = c(e) - \ell(e)$ (sent $\ell(e)$ units along e).
- Update the demands on both ends of e ($d'(u) = d(u) + \ell(e)$ and $d'(v) = d(v) - \ell(e)$)



Main result

Theorem

There exists a circulation in G iff there exists a circulation in G'. Moreover, if all demands, capacities and lower bounds in G are integers, then there is a circulation in G that is integer-valued.

Sketch Proof Need to prove $f(e)$ is a circulation in G iff $f'(e) = f(e) - \ell(e)$ is a circulation in G'.
The integer-valued circulation part is a consequence of the integer-value circulation Theorem for f' in G'. $\square$

Survey design problem

Problem: Design a survey among customers of products

- Each customer will receive questions about some products.
- Each customer i can only be asked about a number of products between c_i and c'_i ($[c_i, c'_i]$) which he has purchased.
- For each product j we want to collect date for a minimum of p_j distinct customers and a maximum of p'_j ($[p_j, p'_j]$)

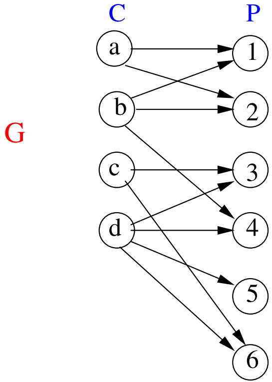
Survey design problem

Measuring customer satisfaction.

Formally we want to model the problem as:

- A bipartite graph $G = (C \cup P, E)$, where $C = \{i\}$ is the set of customers and $P = \{i\}$ is the set of products.
- There is an $(i, j) \in E$ is i has purchased product j.
- For each $i \in \{1, \ldots, n\}$ we we have bounds $[c_i, c_i']$ on the number of products i can be asked about.
- For each $j \in \{1, \ldots, n\}$ we we have bounds $([p_j, p_j'])$ on the number of customers that can be asked about it.

Survey design problem: Bipartite graph G



Customers C={a,b,c,d}

Products P={1,2,3,4,5,6}

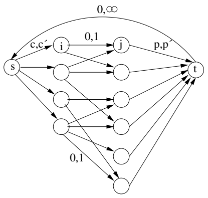
Customer	Buys
a	1,2
b	1,2,4
c	3,6
d	3,4,5,6

a:[1,2]
b:[1,3]
c:[1,2]
d:[2,4]

1: [1,2]
2: [1,2]
3: [1,2]
4: [1,2]
5: [0,1]
6: [1,2]

Survey design problem: Max flow formulation

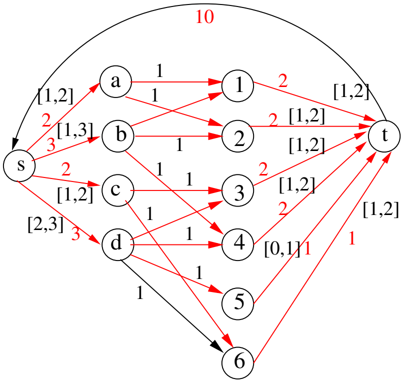
We construct G' from G, by adding:
Edges: $s \to \{C\}$, $\{P\} \to t$, and (t, s).
Capacities: $c(t, s) = \infty$
$c(i, j) = 1$,
$c(s, i) = [c_i, c'_i]$,
$c(j, t) = [p_j, p'_j]$.



Notice if f is the flow:

- $f(i, j) = 1 \Rightarrow$ customer i is asked about product j,
- $f(s, i)$ # products to ask customer i for opinion,
- $f(j, t) =$ # customers to be asked to review product j,
- $f(t, s)$ is the number of questions asked.

Max flow formulation: Example



a ⟶	1,2	a:[1,2]	1: [1,2]
b ⟶	1,2,4	b:[1,3]	2: [1,2]
c ⟶	3,6	c:[1,2]	3: [1,2]
d ⟶	3,4,5,6	d:[2,3]	4: [1,2]
			5: [0,1]
			6: [1,2]

Main result

Theorem G' has a feasible circulation iff there is a feasible way to design the survey.

Proof if there is a feasible way to design the survey:

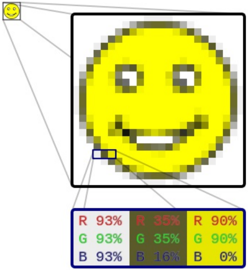
- if i is asked about j then $f(i,j) = 1$,
- $f(s,i) =$ number questions asked to i,
- $f(j,t) =$ number of customers who were asked about j,
- $f(t,s) =$ total number of questions.
- easy to verify that f is feasible in G'

If there is an integral, feasible circulation in G':

- if $f(i,j) = 1$ then i will be asked about j,
- the constrains (c_i, c'_i, p_j, p'_j) will be satisfied. $\square$

Pixels and digital image

- In digital imaging, a pixel is the smallest controllable element of a picture represented on the screen.
- Digital images are represented by a raster graphics image, a dot matrix data structure representing rectangular grid of pixels, or points of color
- The address of a pixel corresponds to its physical coordinates.



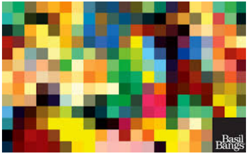
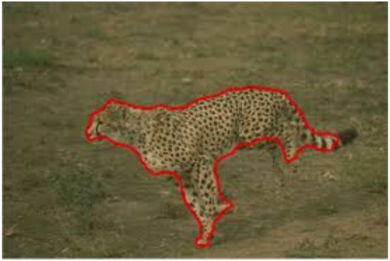
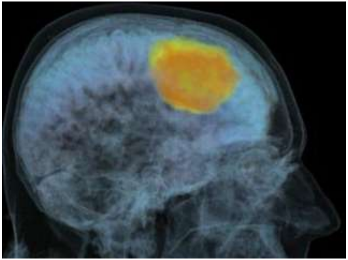


Image segmentation

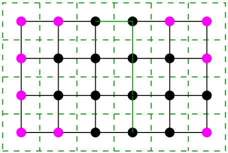
Given a set of pixels classify each pixel as part of the main object or as part of the background.
Important problem in different techniques for image processing.





Foreground/background segmentation

- We aim to label each pixel as belonging to the foreground or the background
- Picture pixels as a grid of dots.
- Define the undirected graph $G = (V, E)$, where, $V =$ set pixels in image, $E =$ pairs of neighbors of pixels (in the grid)
- For each pixel i, $a_i \geq 0$ is likelihood that i is in the foreground and $b_i \geq 0$ is likelihood that i is in the background.
- For each (i, j) of neighboring pixels, there is a separation penalty $p_{ij} \geq 0$ for placing one in the foreground and the other in the background.



Foreground/background segmentation

Goals:

- For i isolated, if $a_i > b_i$ we prefer to label i as foreground (otherwise we label i as background)
- If many neighbors of i are labeled foreground we prefer to label i as foreground. This makes the labeling smoother by minimizing the amount of foreground/background
- We want to partition V into A (set of foreground pixels) and B (set of background pixels), such that we maximize the objective function:

$$\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{(i,j) \in E, |A \cap \{i,j\}| = 1} p_{ij}$$

Formulate as a min-cut problem

Segmentation has the flavor of a cut problem, but

- it is a maximization different than the min-cut,
- G is undirected,
- it does not have sink s and source t.

From maximization to minimization

Recall we want to $\max\underbrace{(\sum_{i\in A} a_i + \sum_{j\in B} b_j - \sum_{(i,j)\in E,|A\cap\{i,j\}|=1} p_{ij})}_{(*)}$

- Let $Q = \sum_{i\in V}(a_i + b_i) = \underbrace{\sum_{i\in V} b_i + \sum_{j\in V} a_j}_{\Theta(1)}$, then

  $\sum_{i\in A} a_i + \sum_{j\in B} b_j = Q - (\sum_{i\in A} b_i + \sum_{j\in B} a_j)$
- So, $(*) \equiv Q - (\sum_{i\in A} b_i - \sum_{j\in B} a_j) - \sum_{(i,j)\in E,|A\cap\{i,j\}|=1} p_{ij}$
- Therefore,

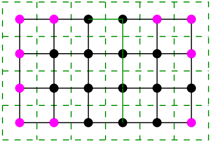
$$\max(*) = \min \sum_{i\in A} b_i + \sum_{j\in B} a_j + \sum_{(i,j)\in E,|A\cap\{i,j\}|=1} p_{ij}.$$

Transforming G into an $s \to t$ network G'

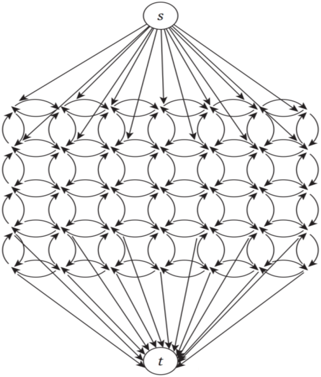
We transform $G = (V, E)$ to $G' = (V', E')$ by

- Add a upper node s representing the foreground
- Add a lower node t representing the background
- $V' = V \cup \{s, t\}$
- For each $(v, u) \in E$ create antiparallel directed edges (u, v) and (v, u) in E'
- For each pixel i create directed edges (s, i) and (i, t)
- $E' = \{(s, v) \cup (v, t)\}_{v \in E} \cup \{(u, v) \cup (v, u)\}_{(u,v) \in E}$

The undirected pixel graph G and the digraph G'



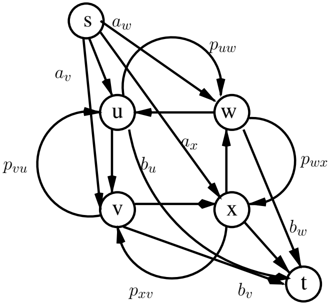
G



G'

Adding capacities to the edges of G'

- For each $i \in V$, $c(s, i) = a_i$, $c(i, t) = b_i$
- For each $(i, j) \in E$, $c(i, j) = c(j, i) = p_{ij}$



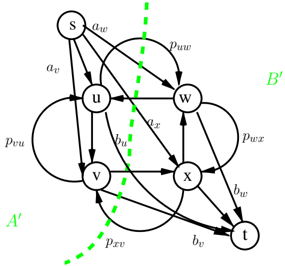
Min-cut in G'

An $s - t$ cut (A', B') corresponds to a partition of the pixels into $(A' - \{s\}, B' - \{t\})$, where

- Edges (s, j) with $j \in B$ contributes with a_j to $c(A', B')$,
- edges (i, t) where $i \in A$ contributes with b_j to $c(A', B')$,
- edges (i, j) where $i \in A$ and $j \in B$ contributes with p_{ij} to $c(A', B')$

Therefore,

$$c(A', B') = \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{(i,j) \in E, |A \cap \{i,j\}| = 1} p_{ij}$$



We want to find the min value of the above quantity, which is equivalent to find the min-cut (A, B) in G' $\Rightarrow$ find the max-flow $s - t$ in G'.

Conclusions

Max-Flow/ Min-Cut problem is an intuitively easy problem with lots of applications.
We just presented a few ones.
An alternative point of view can be obtained from duality in LP

The material in this talk has been basically obtained from two textbooks:

- Chapter 26 of Cormen, Leiserson, Rivest, Stein: Introduction to Algorithms, and
- chapter 7 of kleinberg, Tardos: Algorithm Design.