Linear Sorting

Curs: Spring 2019

Upper and lower bounds on time complexity of a problem.

A problem has a time upper bound $T_U(n)$ if there is an algorithm A such that for any input e of size n:
$A(e)$ gives the correct answer in $\leq T_U(n)$ steps.

A problem has a time lower bound $T_L(n)$ if there is NO algorithm which solves the problem if time $< T_L(n)$, for any input $e, |e| = n$.

It may be that an algorithm solves the problem faster that $T_L(n)$ for a specific input.

Lower bounds are hard to prove, as we have to consider every possible algorithm.

Upper and lower bounds on time complexity of a problem.

- Upper bound: $\exists A, \forall e$ time $A(e) \leq T_U(|e|)$,
- Lower bound: $\forall A, \exists e$ time $A(e) \geq T_L(|e|)$,

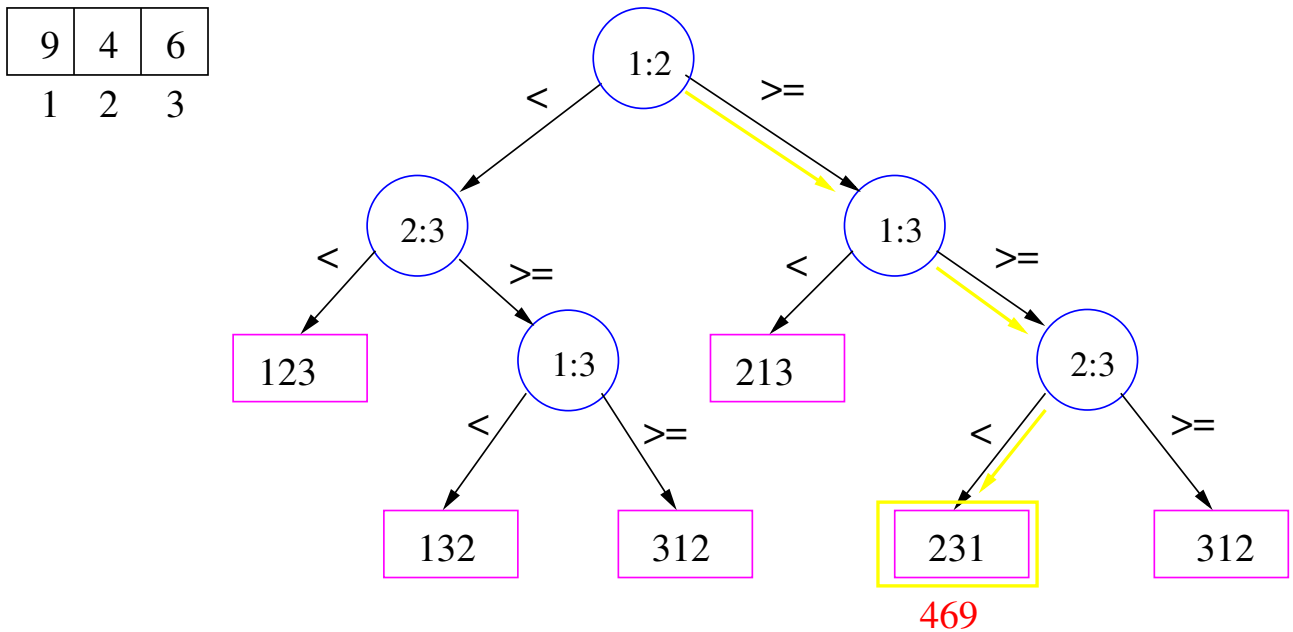
To prove an upper bound: produce an A which works for any e, $|e| = n$.

To prove a lower bound , show that for any possible algorithm, the time on an input is greater than the lower bound.

Lower bound for **comparison based** sorting algorithm.

Use a decision tree: A binary tree where,

- each internal node represents a comparison $a_i : a_j$, the left subtree represents the case $a_i \leq a_j$ and the right subtree represents the case $a_i > a_j$
- each leaf represents one of the $n!$ possible permutations $(a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(n)})$. Each of the n permutations must appear as one of the leaves of the tree



Theorem

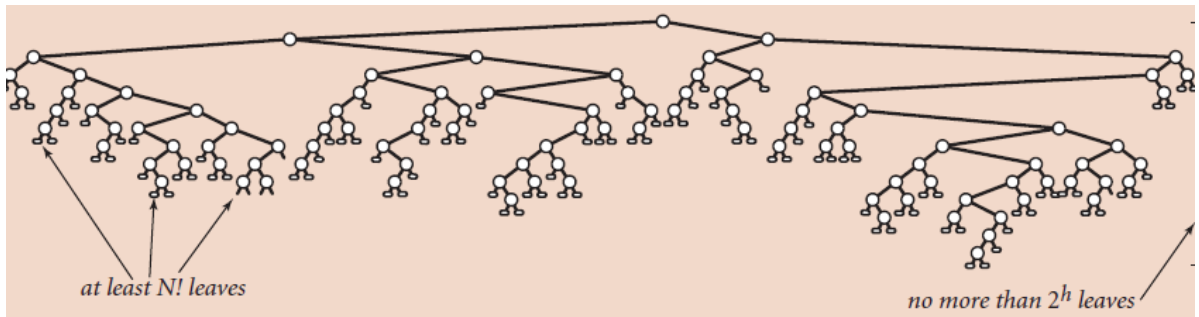
Any comparison sort that sorts n elements must perform $\Omega(n \lg n)$ comparisons.

Proof.

Equivalent to prove: Any decision tree that sorts n elements must have height $\Omega(n \lg n)$.

Let h the height of a decision tree with $n!$ leaves,

$n! \leq 2^h \Rightarrow h \geq \lg(n!) > \lg(\frac{n}{e})^n = \Omega(n \lg n)$. □



Linear sorting: Counting sort

CLRS Ch.8

Assume the input $A[1 \ldots n]$, is an array of integers in $[0, b]$.
Need: $B[1 \ldots n]$ as the output and $C[0 \ldots b]$ as scratch.

Counting (A, b)
for $i = 0$ to b **do**
  do $C[i] := 0$
end for
for $i = 1$ to n **do**
  do $C[A[i]] := C[A[i]] + 1$
end for
for $i = 0$ to b **do**
  do $C[i] := C[i] + C[i-1]$
end for
for $i = n$ downto 1 **do**
  do $B[C[A[i]]] := A[i]$
  $C[A[i]] := C[A[i]] - 1$
end for

Counting (A, b)
for $i = 0$ to b **do**
  do $C[i] := 0$ { $O(b)$}
end for
for $i = 1$ to n **do**
  do $C[A[i]] := C[A[i]] + 1$ { $O(n)$}
end for
for $i = 0$ to b **do**
  do $C[i] := C[i] + C[i-1]$ { $O(b)$}
end for
for $i = n$ down to 1 **do**
  $B[C[A[i]]] := A[i]$ { $O(n)$}
  $C[A[i]] := C[A[i]] - 1$
end for

Complexity: $T(n) = O(n + b)$ if $b = O(n)$, then $T(n) = O(n)$.
When using counting on short arrays, $b = 10$.

What does it mean radix?

Radix means the base in which we express an integer

Radix 10=Decimal; Radix 2= Binary; Radix 16=Hexadecimal;

Radix 20 (The Maya numerical system)

Binary	Hex	Decimal
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	8
1001	9	9
1010	A	10
1011	B	11
1100	C	12
1101	D	13
1110	E	14
1111	F	15

Radix Change: Example

- To convert an enter from binary $\rightarrow$ decimal:
  $1011 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 12 \Rightarrow 11$
- To convert an enter from decimal $\rightarrow$ binary: Repeatedly dividing the enter by 2 will give a result plus a remainder:
  $19 \Rightarrow \underbrace{19/2}_{1}\ \underbrace{9/2}_{1}\ \underbrace{4/2}_{0}\ \underbrace{2/2}_{01} = 10011$
- To transform an integer radix 16 to decimal:
  $(4CF5)_{16} = (4 \times 16^3 + 12 \times 16^2 + 15 \times 16^1 + 5 \times 16^0) = 19701$

radix 10		radix 2	radix 16
7134785012		11010100101000100001011011110100	1a9442df4
4561343780		10000111111100000100101010100100100	10fe09524
0051889437	.	00000001100010111110001010001110 1	0317c51d

Example

Is it true that if n integer a in radix 512 has d digits, and we change it to be expressed in radix 2, the number of bits that n would have is $\Theta(d^2)$?

NO. When converting radix-256 to binary, we increase d to $\lg 512 \times d = 9d = \Theta(d)$.

Linear sorting: RADIX sort

An important property of counting sort is that it is stable, numbers with the same value, appear in the output in the same order as they do in the input.
For instance Heap sort or Quicksort is not stable.

Given an array A with n keys, each one with d digits, the Radix (Least Significant Digit),

RADIX LSD (A, d)
for $i = 1$ to d **do**
  Use stable sorting to sort the i-th digit of A.
end for

Example

329		720		720		329
475		475		329		355
657		355		436		436
839	$\Rightarrow$	436	$\Rightarrow$	839	$\Rightarrow$	457
436		657		355		657
720		329		657		720
355		839		475		839

Theorem (Correctness of RADIX)

The previous algorithm sort correctly n keys.

Induction on d.

If $d = 1$ the stable sorting works. Assume it is true for $d - 1$, to sort the d-th digit,
if $a_d < b_d$ then a will be placed before b,
if $b_d < a_d$ then b will be placed before a,
if $b_d = a_d$ then as we are using a stable sorting a and b will remain in the same order, which by hypothesis was already the correct one. □

Complexity of RADIX

Given n integers ≥ 0, each integer with at most d digits, and each digit in the range 0 to 9, if we use counting sorting:
$T(n, d) = \Theta(d(n + 9))$.

- Consider that each integer has a value up to $f(n)$.
- Then the number of digits is $d = \log f(n)$, so $T(n, d) = \Theta((\log f(n))(n + 9))$,
- if $f(n) = \omega(1)$ then $T(n) = \omega(n)$. So in this case RADIX is not lineal.

Can we do it faster?: Yes, change the radix to express the integers.

RADIX

Given n integers, each integer with a value up to $f(n)$ then the decimal expression had at most $d = \lfloor \log f(n) \rfloor + 1 \sim \log f(n)$ digits.
Consider each integers in radix b, then the number of "digits" is $d = \log_b f(n)$, and apply RADIX to the columns of new digits in base b.

Complexity: Given n positive integers, each integer with a maximal value of $f(n)$ and with at most d digits needed to represent each using radix b. Then the complexity $T(n)$ of using RADIX to sort the n integers is: $T(n) = O((n+b)d) = O(n+b)\log_b f(n)$.

RADIX Change: Example

RADIX 10		RADIX 2	RADIX 16
7134785012		1101010010100010000101101111110100	1a9442df4
4561343780		1000011111111000001001010100100100	10fe09524
0051889437	.	0000000110001011111000101000011101	0317c51d

	n	d	b
RADIX 10	3	10	9
RADIX 2	3	33	1
RADIX 16	3	8	15

Implementation for bit integers

Given n integers given as binary strings with length d,
we want to choose an integer $1 < r < d$ such minimizes $(d/r)(n + 2^r)$.
Then use RADIX on each of the new $\hat{d} = \lceil d/r \rceil$ digits.
For ex., if we have words of $b = 32$ bits, which we split in $d = 4$ $r = 8$-bit digits:

1 1 0 0 1 0 1 0 0 0 1 1 0 1 0 0 1 1 1 0 1 0 0 1 1 1 0 0 1 0 0 0
Each new digit is an integer with $b = 2^r - 1$.
So we can use counting sort with $k = 2^r - 1$.

Each pass of counting sort takes $\Theta(n + k) = \Theta(n + 2^r)$,
as there are d passes $\Rightarrow T(n) = \Theta(d(n + 2^r)) = \Theta((b/r)(n + 2^r))$.

Comparing radix and counting:

For n integers, each integer with at most d digits, where each digit is in the range $[0, 9]$:

- Counting sort is $\Theta(9dn)$,
- Radix with the choice of $r = \log_9 n$-digits can sort n d-digit numbers in $\frac{d}{\log_9 n}\Theta(9n)$.

Consider 2000 integers of 32 bits each:

- Quicksort needs to do $\lg 2000 = 11$ passes over the data,
- Radix sort with digits of 11-bits, takes 3 passes (at each one counting sort makes 2 passes).

Empirically, when dealing with natural numbers, radix is better than other sorting methods for values of $n > 2000$.

A bit of history.

Radix and all counting sort are due to Herman Hollerith.
In 1890 he invented the card sorter that allowed to shorten the US census to 5 weeks, using punching cards.