Automatic Generation of Polynomial Invariants for System Verification

Enric Rodríguez-Carbonell

Plan of the Talk

- Introduction
  - Need for program verification
  - Invariants and abstract interpretation
  - Polynomial invariants

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- Introduction
- Generation of Invariant Polynomial Equalities
  (with D. Kapur: ISSAC'04, SAS'04)
  - Related work
  - Abstract domain of ideals
  - Particular case: loops without nesting

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- Introduction
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
  - Imperative programs
    (with D. Kapur: ICTAC'04)
  - Petri nets
    (with R. Clarisó, J. Cortadella: ATPN'05)
  - Hybrid systems
    (with A. Tiwari: HSCC'05)

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- Introduction
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
- Generation of Invariant Polynomial Inequalities (with R. Bagnara, E. Zaffanella: SAS'05)
  - Abstract domain of polynomial cones

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- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
- Generation of Invariant Polynomial Inequalities
- Conclusions and Future Work

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Need for Software Verification

- Critical systems
  - safety
  - security
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Failure of the Ariane 5 launcher in 1996

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- Fundamental finding errors asap. See Microsoft's Software Productivity Tools group: verification pays off
- Invariants are crucial for program verification!

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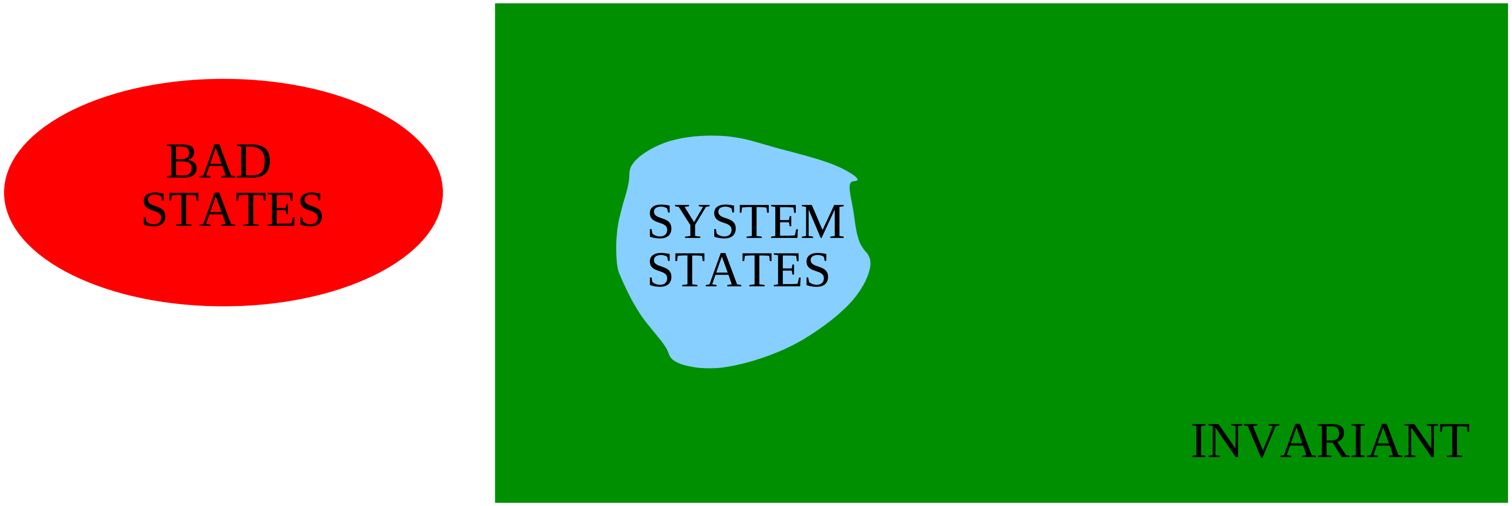
Invariants in Verification

CORRECTNESS OF THE SYSTEM:

$$\text{SYSTEM STATES} \;\cap\; \text{BAD STATES} \;=\; \emptyset$$

Invariants in Verification



CORRECTNESS OF THE SYSTEM:

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SUFFICIENT CONDITION:

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Overview of Abstract Interpretation

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- linear inequalities (Cousot & Halbwachs 1978, Colón & Sankaranarayanan & Sipma 2003)

$$x + 2y - 3z \leq 3$$

Overview of Abstract Interpretation

- octagonal inequalities (Mine 2001)

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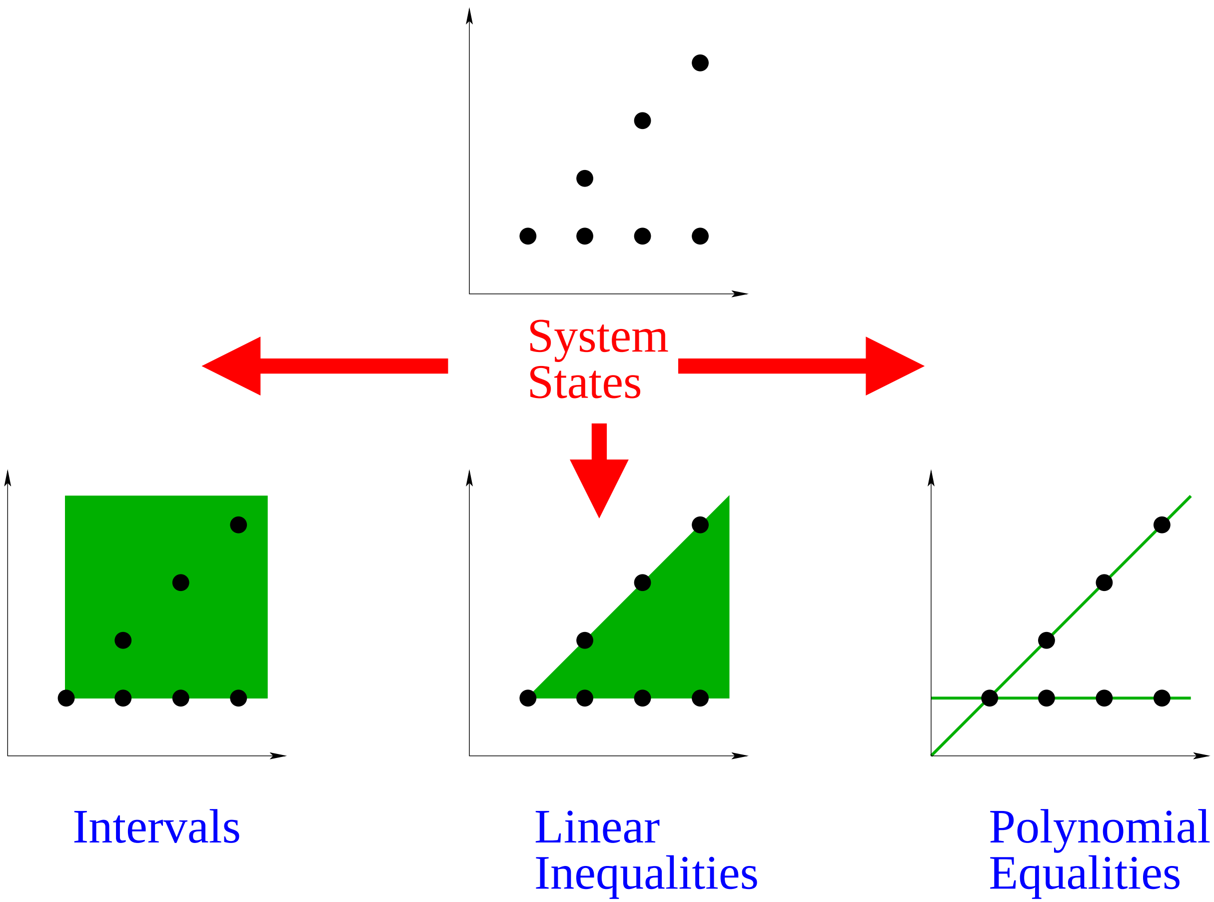
- ...
- polynomial equalities and inequalities

$$x = y^2$$

$$(a+1)^2 > b^2 \geq a^2$$

Abstract Interpretation: Overapproximation

Sets of variable values overapproximated by abstract values



Abstract Interpretation: Operations

- Invariants generated by symbolic execution of system using abstract values
- Symbolic execution requires abstracting concrete operations on states:

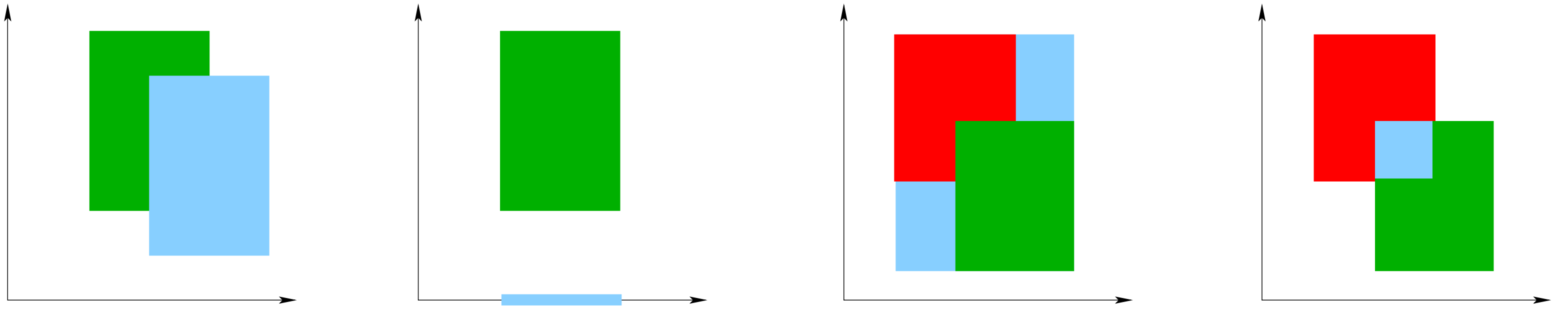


Image	Projection	Union	Intersection
↓	↓	↓	↓
assignments	*assignments*	*merging in loops and conditionals*	*guards in loops and conditionals*

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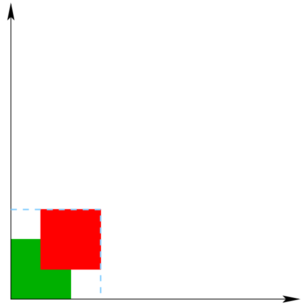

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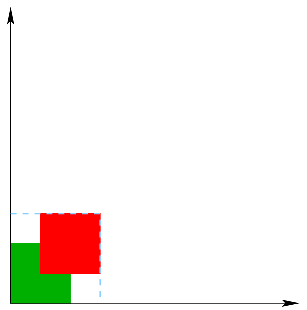

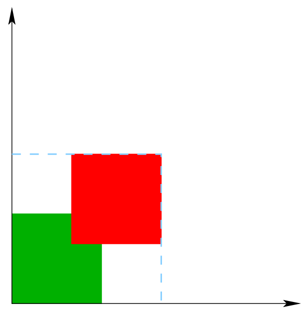

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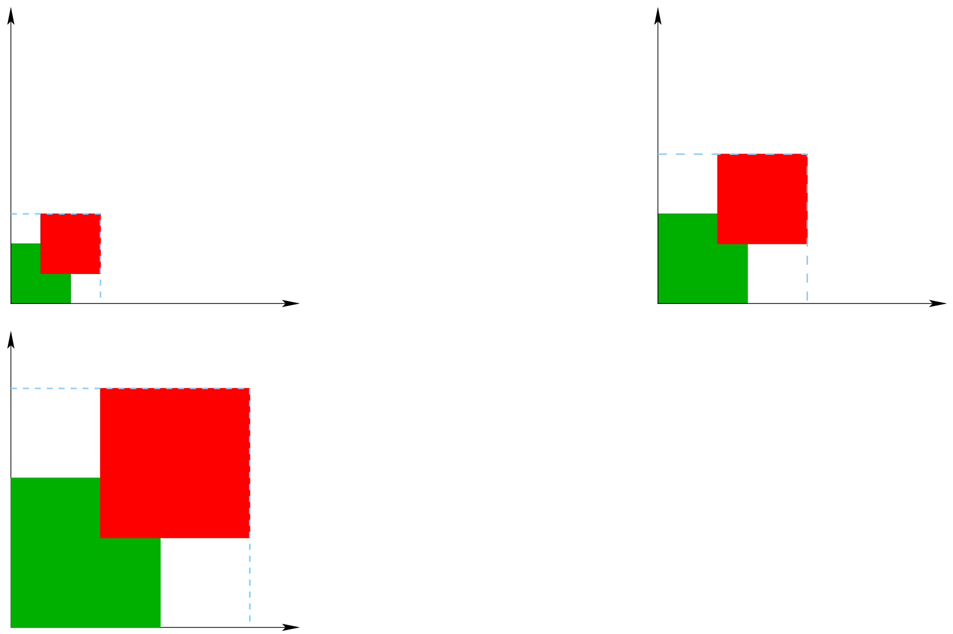

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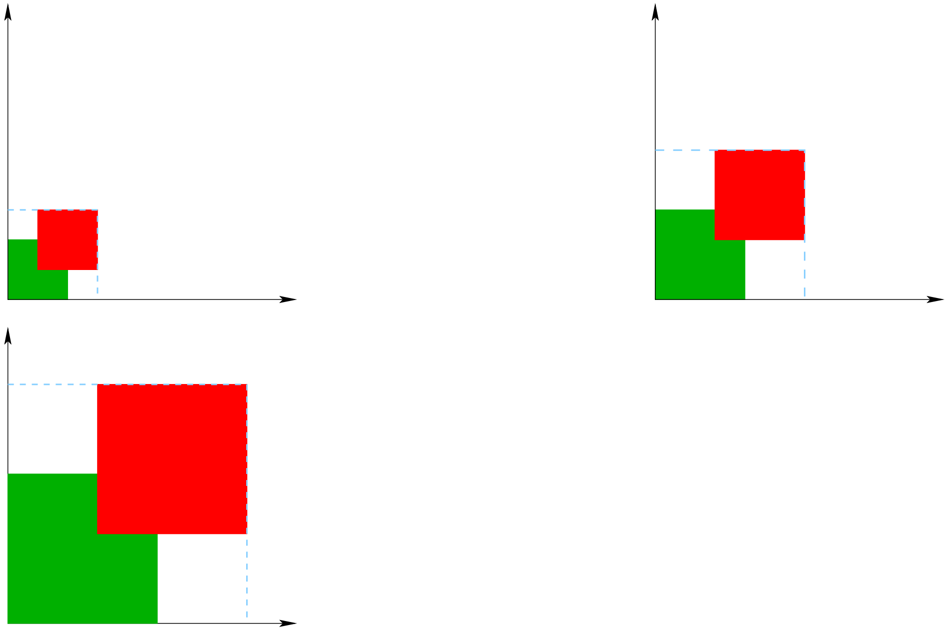

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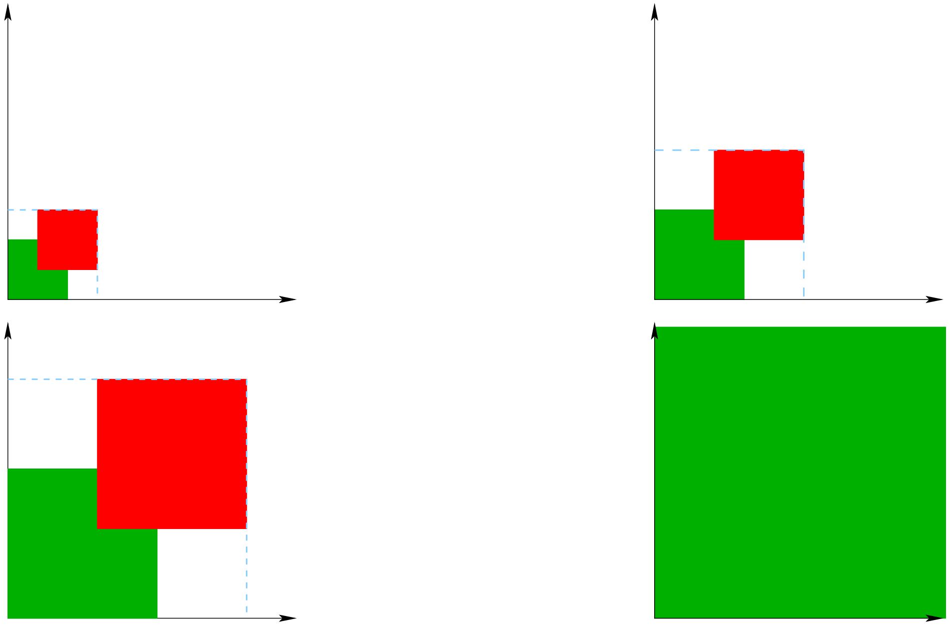

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Why Care about Polynomial Invariants?

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- Linear invariants used to verify many classes of systems:
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- But some applications require polynomial invariants:

The abstract interpreter ASTRÉE employs polynomial invariants to verify absence of run-time errors in flight control software

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Related Work (1)

- Iterative fixpoint approaches
  - Forward propagation
    - Rodríguez-Carbonell & Kapur 2004
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- Constraint-based approaches
  - Sankaranarayanan & Sipma & Manna 2004

Related Work (2)

Work	Restrictions	Conds $=$	Conds $\neq$	Complete
MOS, POPL'04	bounded deg	no	no	yes
SSM, POPL'04	fi xed form	yes	no	no
MOS, IPL'04	fi xed form	no	yes	yes
COL, SAS'04	bounded deg	yes	no	no
RCK, SAS'04	bounded deg	yes	yes	yes*
RCK, ISSAC'04	no restriction	no	no	yes

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- Implementation successfully applied to many programs
- Ideals of polynomials represented by special finite bases of generators: Gröbner bases
- Many tools available manipulating ideals, Gröbner bases, e.g. Macaulay 2, Maple

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- An ideal is a set of polynomials I such that
  - $0 \in I$
  - If $p, q \in I$, then $p + q \in I$
  - If $p \in I$ and q any polynomial, $pq \in I$

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  - If p evaluates to 0 on S, then pq evaluates to 0 on S

$$\forall \omega \in S, \quad p(\omega) = 0 \Longrightarrow p(\omega) \cdot q(\omega) = 0$$

Ideals of Polynomials (3)

- E.g. multiples of a polynomial p, $\langle p \rangle$
  - $0 = 0 \cdot p \in \langle p \rangle$
  - $q_1 \cdot p + q_2 \cdot p = (q_1 + q_2)p \in \langle p \rangle$
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- Hilbert's basis theorem: all ideals are finitely generated $\longrightarrow$ there is finite representation for ideals

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  - intersection: $I \cap J$
- All operations implemented using Gröbner bases
- These operations are used in abstract semantics

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- Termination guaranteed

Example

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while $b \neq c$ **do**

$\quad a := a + 2b + 1; b := b + 1;$

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$$F_0(I) = \langle 0 \rangle$$
$$F_1(I) = (\langle a \rangle + \langle I_0(a \leftarrow a') \rangle) \cap \mathbb{C}[a, b, c]$$
$$F_2(I) = (\langle b \rangle + \langle I_1(b \leftarrow b') \rangle) \cap \mathbb{C}[a, b, c]$$
$$F_3(I) = I_3 \nabla_2 (I_2 \cap I_6)$$
$$F_4(I) = \langle I_3 \rangle : \langle b - c \rangle$$
$$F_5(I) = I_4(a \leftarrow a - 2b - 1)$$
$$F_6(I) = I_5(b \leftarrow b - 1)$$
$$F_7(I) = \mathrm{I}(\mathrm{V}(I_3 + \langle b - c \rangle))$$

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In 6 steps found loop invariant:

$$a = b^2$$

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while $b \neq c$ **do**
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Overview of the Method

- $(a_n, b_n, c) \equiv$ program state after n loop iterations

$$\begin{cases} a_{n+1} & = & a_n + 2b_n + 1 \\ b_{n+1} & = & b_n + 1 \end{cases} , \quad \begin{cases} a_0 & = & 0 \\ s_0 & = & 0 \end{cases}$$

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- Quantifi er elimination: $b = n \Longrightarrow a = b^2$ is loop invariant
- Gröbner bases can be used to eliminate loop counters

Our Handling of Conditional Statements (1)

$x := R;$

$y := 0;$

$r := R^2 - N;$

while $?$ **do**

    if $?$ **then**

        $r := r + 2x + 1;$

        $x := x + 1;$

    else

        $r := r - 2y - 1;$

        $y := y + 1;$

    end if

end while

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1. Compute invariants for two distinct loops:

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$x := x + 1;$	$y := y + 1;$
end while	**end while**

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- Finding common invariants $\equiv$
  Finding intersection of invariant ideals

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- Finding common invariants $\equiv$
  Finding intersection of invariant ideals
- But this is not sound!

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- 2nd idea: take intersection as initial condition and repeat

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Program	Algorithm
$\bar{x} := \bar{\alpha};$	$I' := \langle 1 \rangle; I := \langle x_1 - \alpha_1, \cdots, x_m - \alpha_m \rangle;$
while $?$ **do**	**while** $I' \neq I$ **do**
$\quad \bar{x} := f(\bar{x});$	$\quad I' := I;$
$\quad$ **or**	$\quad I := \bigcap_{n=0}^{\infty} [\, I(\bar{x} \leftarrow f^{-n}(\bar{x}))$
$\quad \bar{x} := g(\bar{x});$	$\qquad \bigcap I(\bar{x} \leftarrow g^{-n}(\bar{x}))\,];$
end while	**end while**

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- Correct and complete:
  finds all polynomial equality invariants
- Implemented in Maple:

  1. Solving recurrences

  2. Eliminating variables
  3. Intersecting ideals

  $\left.\begin{array}{l}\text{2. Eliminating variables}\\ \text{3. Intersecting ideals}\end{array}\right\}$ Gröbner bases

Proof of Termination (1)

$x := 0; y := 0;$

while $?$ **do**

$x := x + 1;$

or

$y := y + 1;$

end while

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$$x := 0; y := 0;$$

while $?$ **do**

$$x := x + 1;$$

or

$$y := y + 1;$$

end while

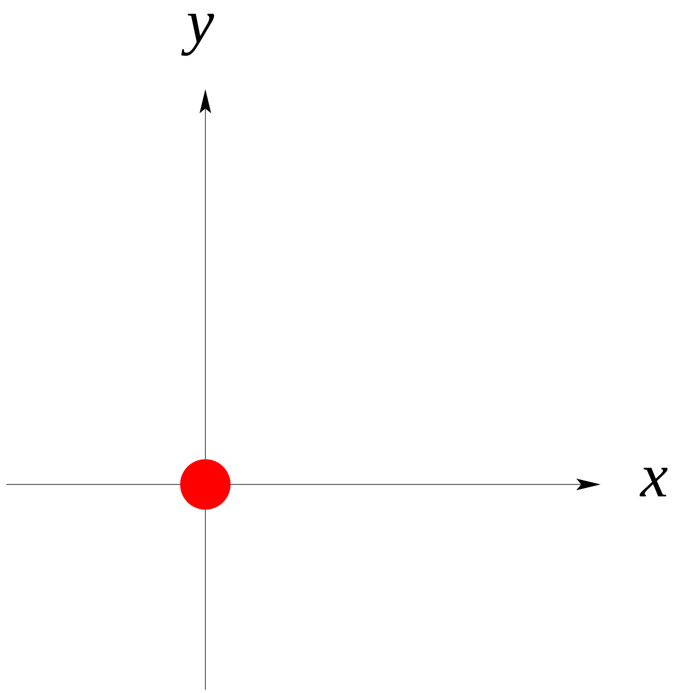
- Program states $\equiv \mathbb{N} \times \mathbb{N}$

Proof of Termination (1)

$x := 0; y := 0;$
while $?$ **do**
    $x := x + 1;$
    or
    $y := y + 1;$
end while

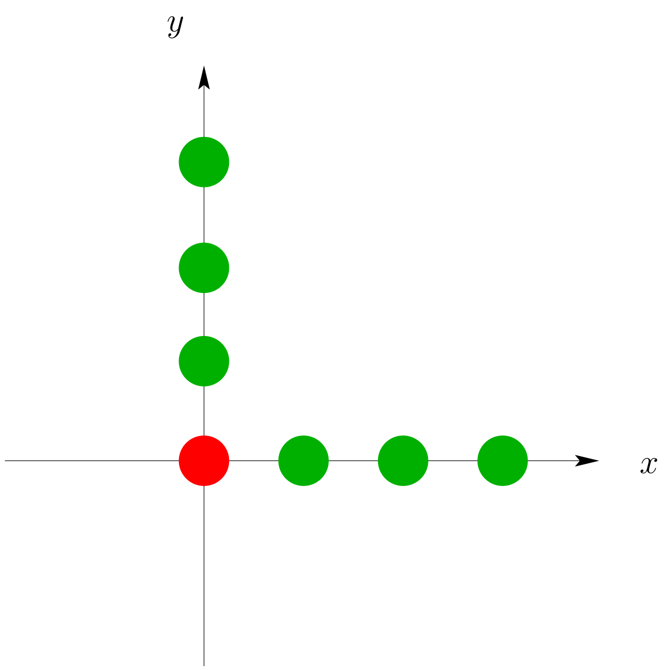
- Program states $\equiv \mathbb{N} \times \mathbb{N}$
- Initial state $(x, y) = (0, 0) \longrightarrow$ initial ideal $\langle x, y \rangle$

Proof of Termination (2)



Step 0: $\langle x, y \rangle \quad \longrightarrow \quad \{(0,0)\}$ dimension 0

Proof of Termination (2)



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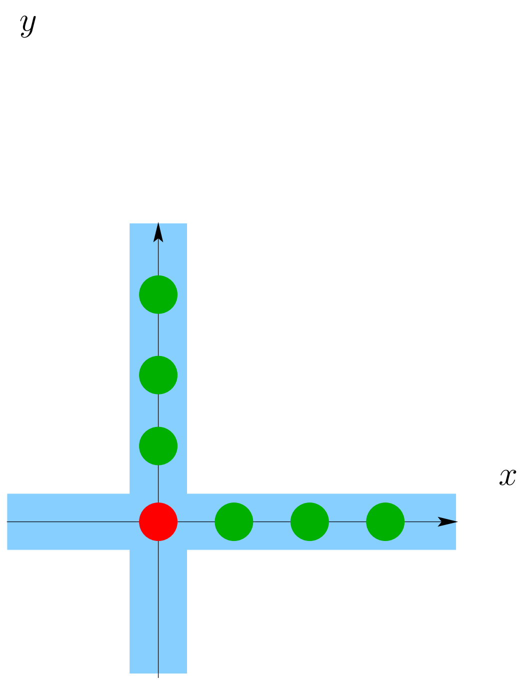

Step 0:	$\langle x, y \rangle$	$\longrightarrow$	$\{(0,0)\}$	dimension 0
Step 1:	$\langle xy \rangle$	$\longrightarrow$	$\{(\alpha,0)\} \cup \{(0,\alpha)\}$	dimension 1

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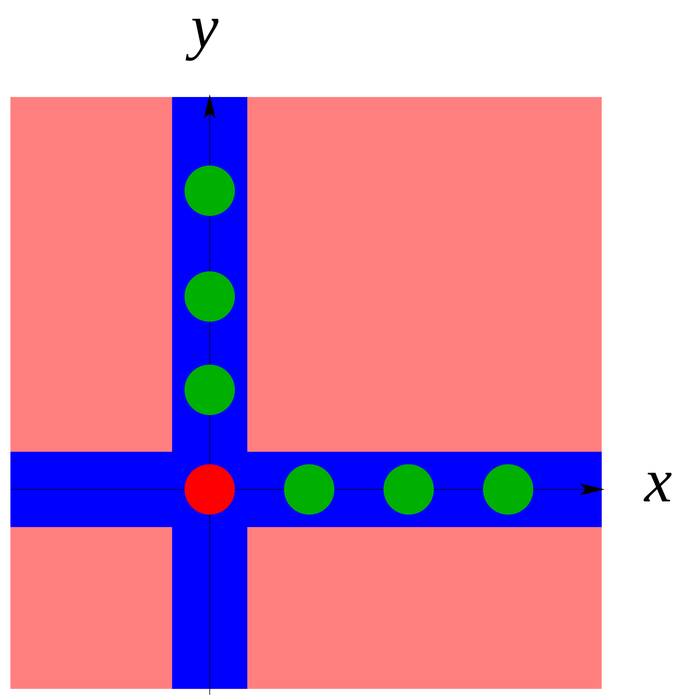
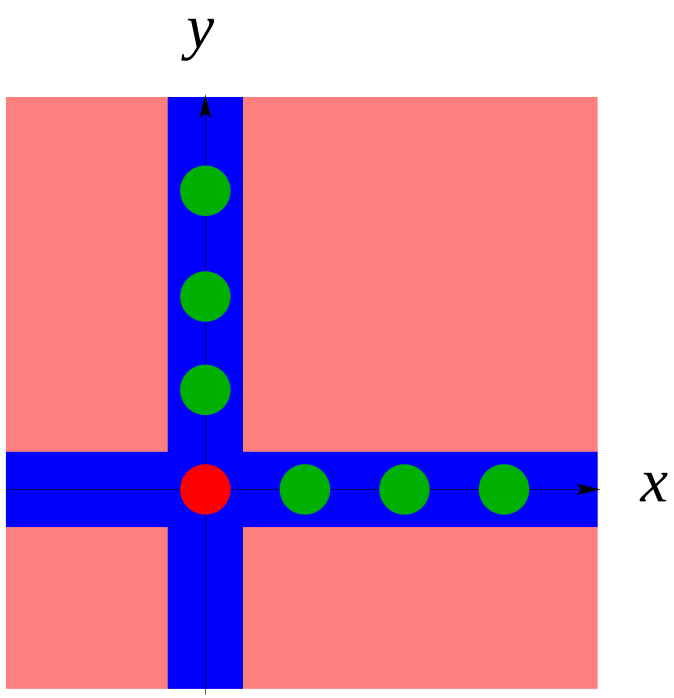


Step 0:	$\langle x, y \rangle$	$\longrightarrow$	$\{(0,0)\}$	dimension 0
Step 1:	$\langle xy \rangle$	$\longrightarrow$	$\{(\alpha,0)\} \cup \{(0,\alpha)\}$	dimension 1
Step 2:	$\langle 0 \rangle$	$\longrightarrow$	$\mathbb{R}^2$	dimension 2

Proof of Termination (2)



The dimension has increased at every step,
and there is a fi nite number of variables!

Example

$x := R;$

$y := 0;$

$r := R^2 - N;$

while $?$ **do**

    if $?$ **then**

        $r := r + 2x + 1;$

        $x := x + 1;$

    else

        $r := r - 2y - 1;$

        $y := y + 1;$

    end if

end while

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Invariant polynomial equality:

$$x^2 - y^2 = r + N$$

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Imperative Programs

Pre: $\{\ N \geq 1\}$

$x := R;\ y := 0;\ r := R^2 - N;$

Inv: $\{\ N \geq 1\ \wedge\ x^2 - y^2 = r + N\ \}$

while $r \neq 0$ **do**

    if $r < 0$ **then**

        $r := r + 2x + 1;$

        $x := x + 1;$

    else

        $r := r - 2y - 1;$

        $y := y + 1;$

    end if

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Post: $\{x \neq y \wedge N \bmod (x - y) = 0\}$

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- $N \geq 1 \implies R^2 - 0^2 = (R^2 - N) + N$

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- $N \geq 1 \implies$ $R^2 - 0^2 = (R^2 - N) + N$
- $x^2 - y^2 = r + N\ \wedge\ r < 0 \implies$ $(x+1)^2 - y^2 = (r+2x+1) + N$

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  $x^2 - (y+1)^2 = (r - 2y - 1) + N$
- $N \geq 1 \ \wedge \ x^2 - y^2 = r + N \implies$
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  - concurrency
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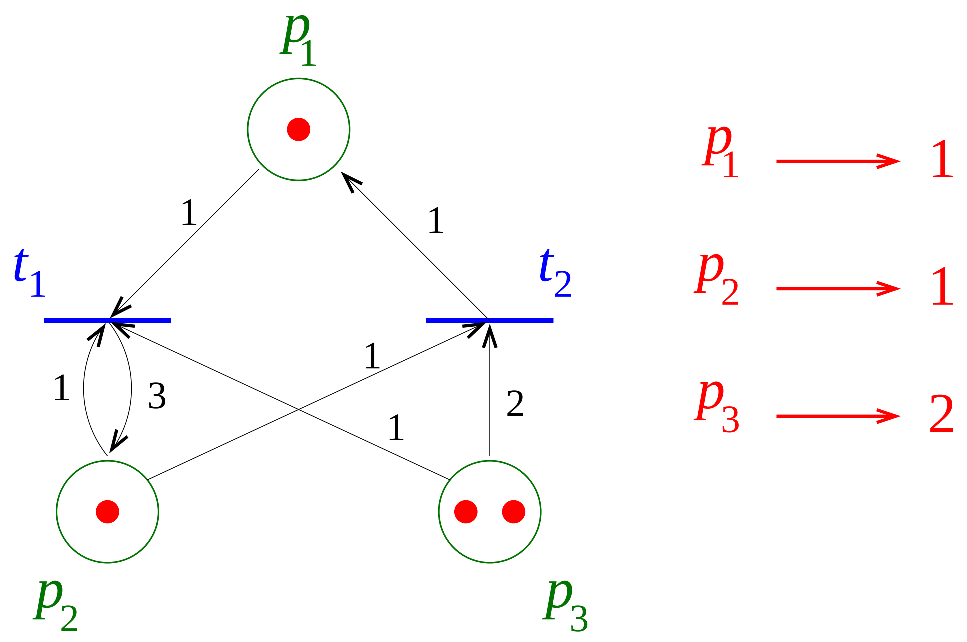
  - non-determinism
- Applications:
  - Manufacturing and Task Planning
  - Communication Networks
  - Hardware Design

Definitions

- A Petri net is a bipartite directed graph where:
  - Nodes partitioned into places (○) and transitions (|)
  - Arcs are labelled with a natural number
- A marking maps a number of tokens to each place



Dynamics (1)

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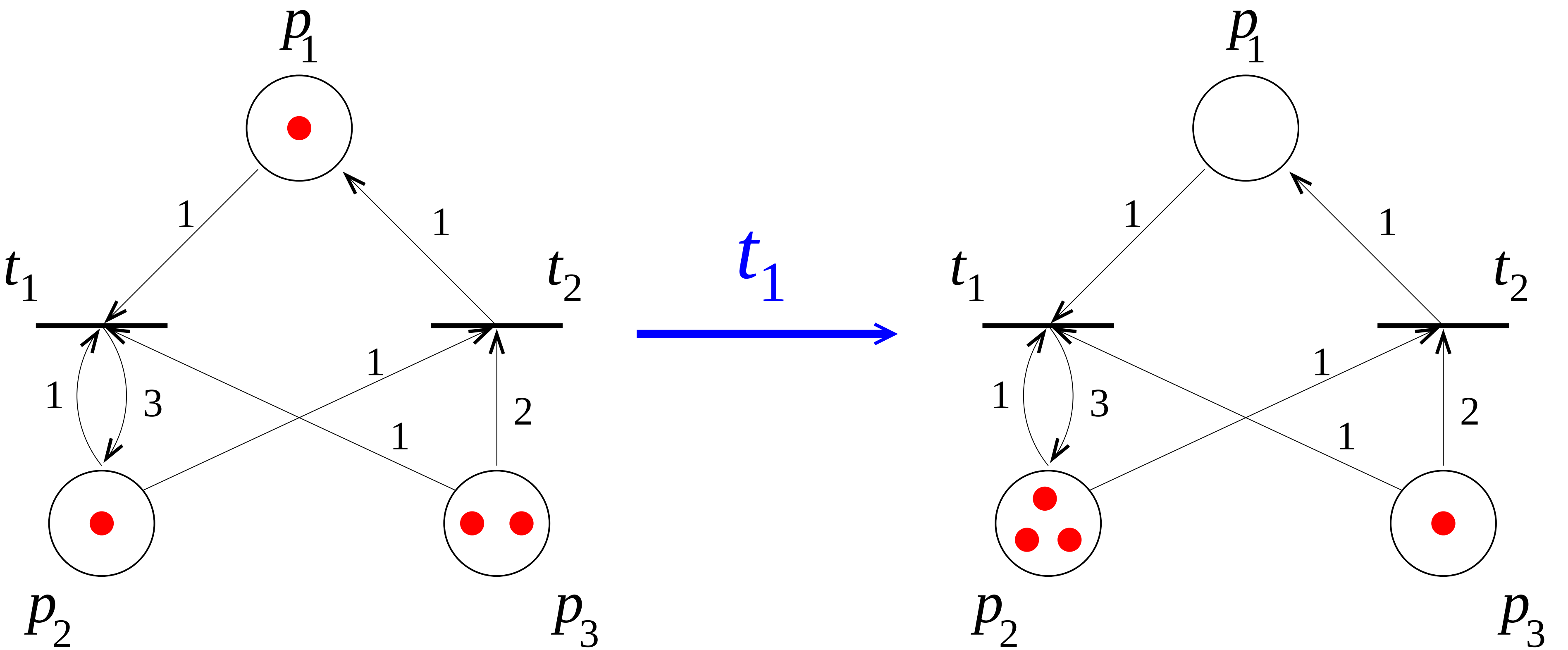
- A transition is enabled if there are $\geq$ tokens in each input place than indicated in the arcs
- When a transition is enabled, it can fire:
  1. the number of tokens indicated in the arcs is removed from input places
  2. tokens are produced in output places according to arcs

Dynamics (2)



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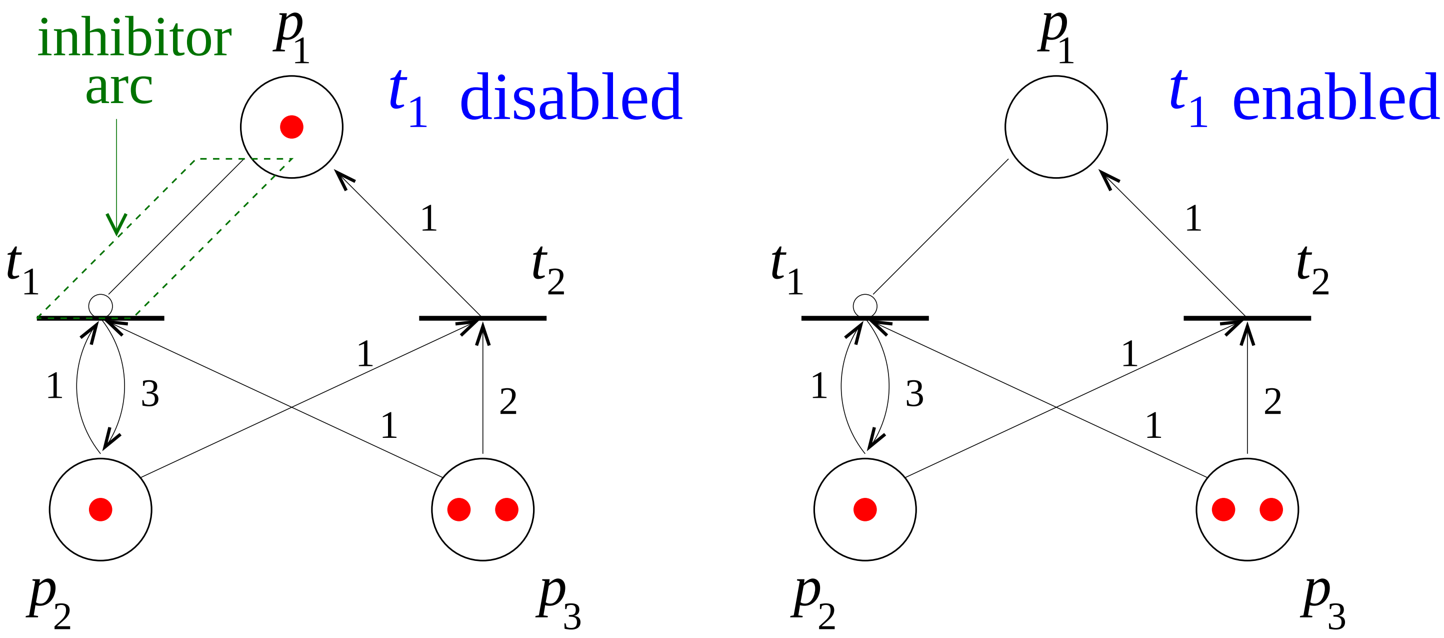
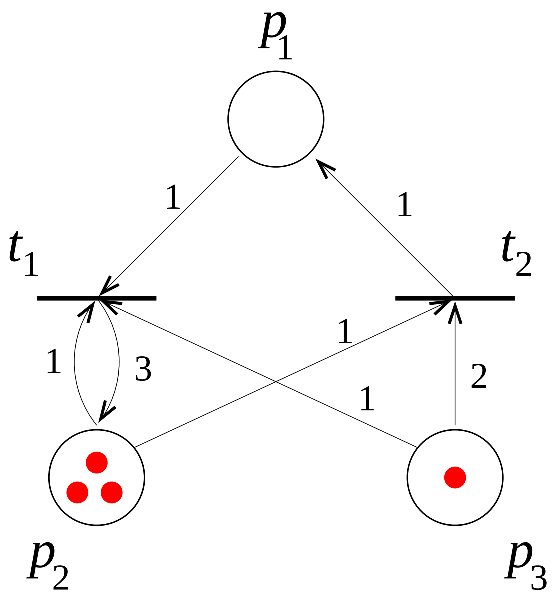


Dynamics (4)

- Deadlocks are markings for which all transitions are disabled



t_1 disabled

t_2 disabled

$\longrightarrow$ DEADLOCK !!

Dynamics (4)

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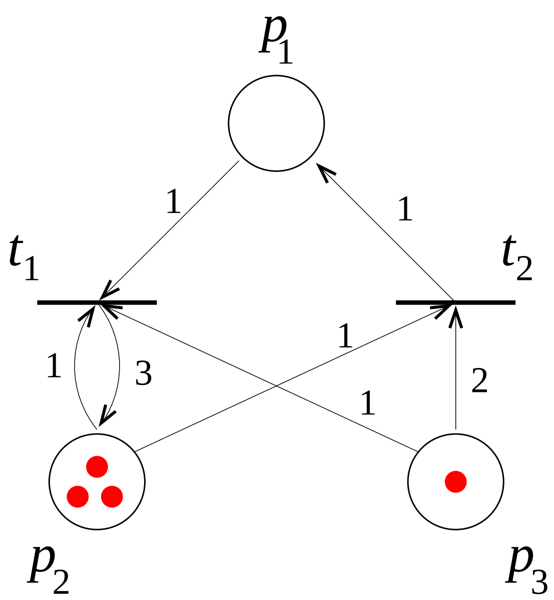

t_1 disabled

t_2 disabled → DEADLOCK !!

- Given a Petri net with an initial marking:
  - Invariant properties of reachable states ?
  - Any deadlocks ?

Translation into Loop Programs

- Define variable x_i meaning number of tokens at place p_i

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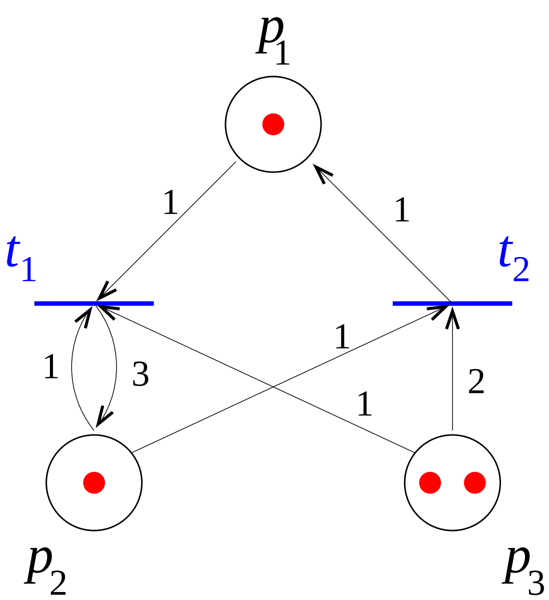
- Firing of a transition
  - with input place p_i and label c_i: $x_i := x_i - c_i;$
  - with output place p_i and label c_i: $x_i := x_i + c_i;$

Translation into Programs (2)



$x_1 := 1; x_2 := 1; x_3 := 2;$

while $?$ **do**

t_1 : **if** $x_1 \neq 0 \wedge x_2 \neq 0 \wedge x_3 \neq 0 \rightarrow$

$x_1 := x_1 - 1;$

$x_2 := x_2 + 2;$

$x_3 := x_3 - 1;$

t_2 : $[]\ x_2 \neq 0 \wedge x_3 \neq 0 \wedge x_3 \neq 1 \rightarrow$

$x_1 := x_1 + 1;$

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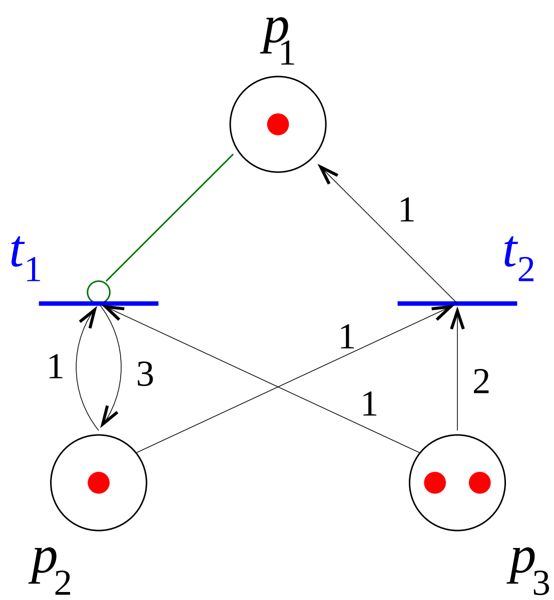
Translation into Programs (3)



$$x_1 := 1; x_2 := 1; x_3 := 2;$$

while $?$ **do**

t_1 : **if** $x_1 = 0 \wedge x_2 \neq 0 \wedge x_3 \neq 0 \rightarrow$

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Generating Polynomial Invariants (1)

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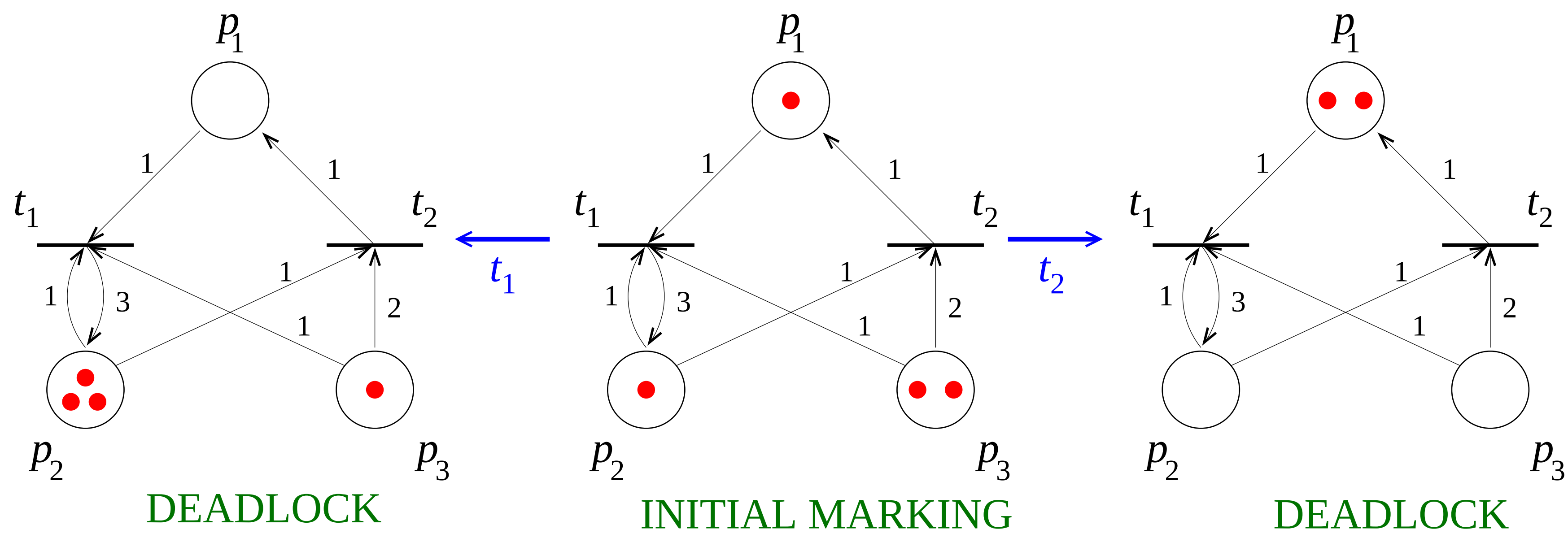

Generating Polynomial Invariants (2)

- Polynomial invariants obtained:

$$Inv = \begin{cases} 5x_1 + 3x_2 + x_3 - 10 & = 0 \\ 5x_3^2 + 2x_2 - 11x_3 & = 0 \\ x_2x_3 + 2x_3^2 - 5x_3 & = 0 \\ 5x_2^2 - 17x_2 + 6x_3 & = 0 \end{cases}$$

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- In this example invariants characterize reachability set

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- In this example invariants characterize reachability set

$$Inv \Leftrightarrow (x_1, x_2, x_3) \in \{(0,3,1), (1,1,2), (2,0,0)\}$$

- In general overapproximation of reach set is obtained

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- Hybrid System: discrete system in analog environment

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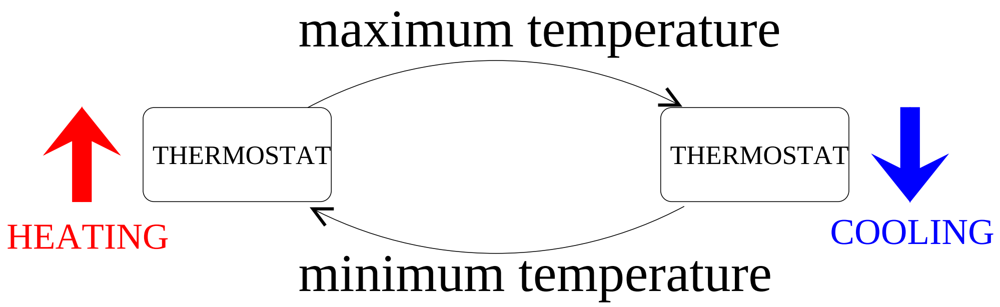

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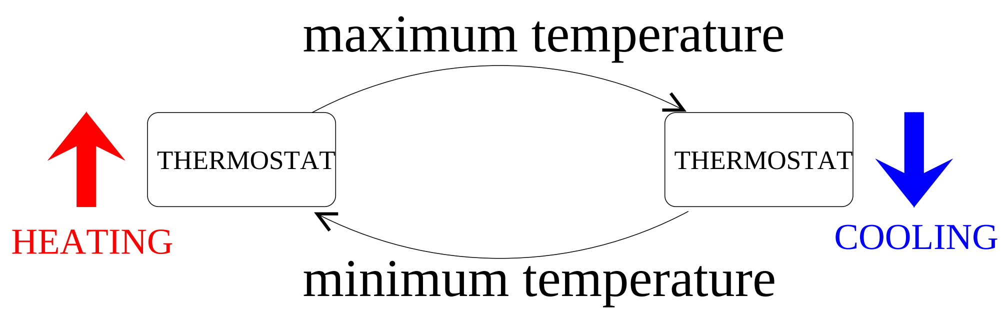

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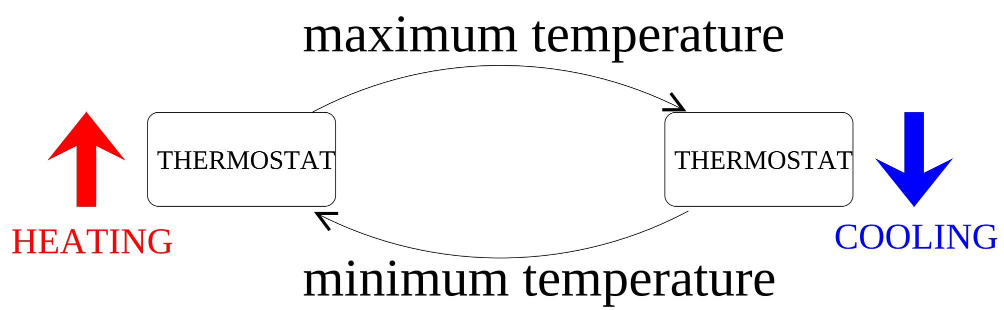

  - A robot controller that changes the direction of movement if the robot is too close to a wall.
  - A biochemical reaction whose behaviour depends on concentration of substances in environment

Definition

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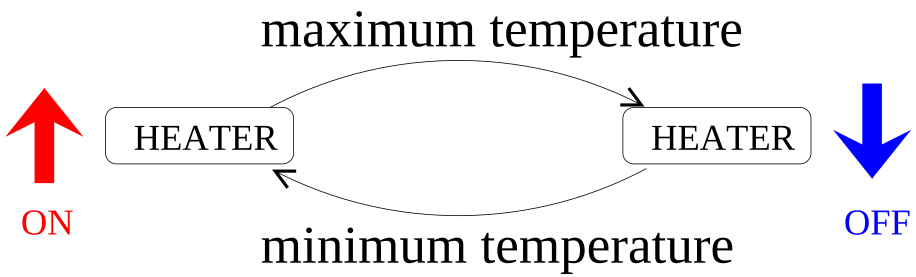

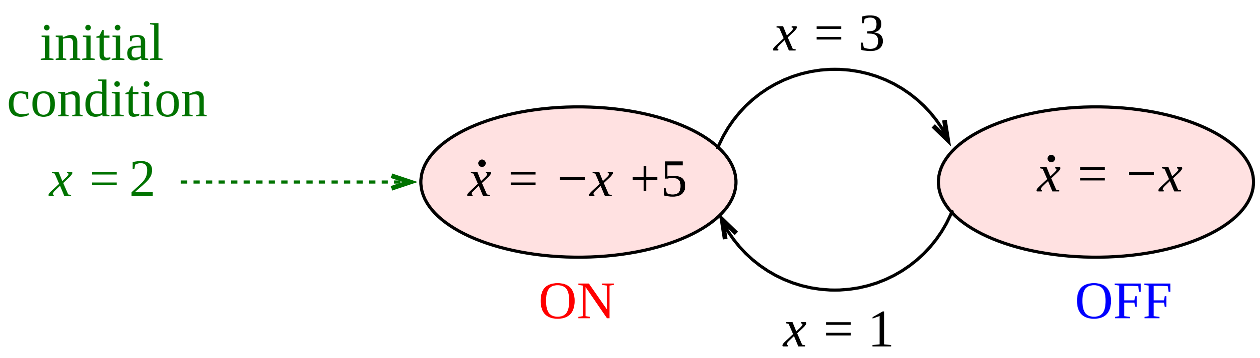

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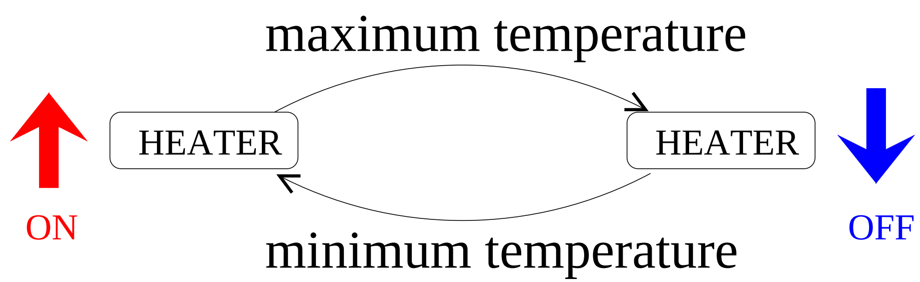

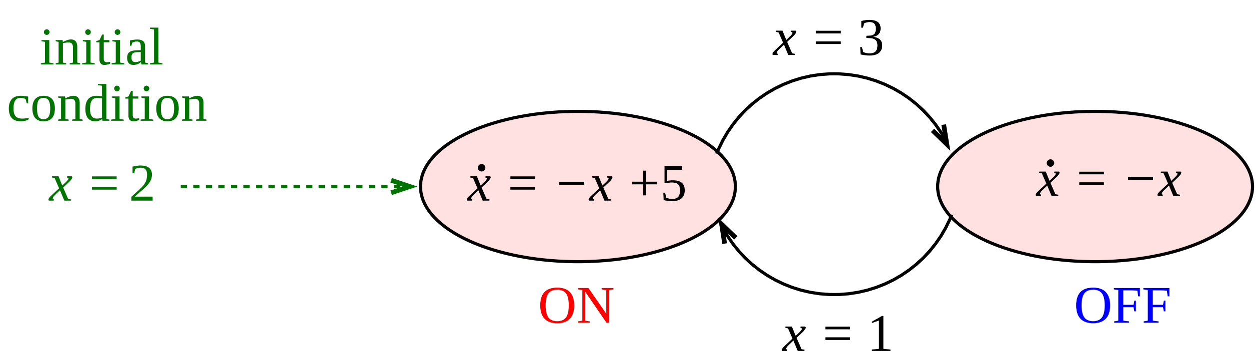

- Restricted to linear differential equations at locations

Dynamics (1)

- A computation is a sequence of states (discrete location, valuation of variables)

$$(l_0, x_0), (l_1, x_1), (l_2, x_2), \ldots$$

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  2. For each consecutive pair of states $(l_i, x_i), (l_{i+1}, x_{i+1})$:
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     - Discrete transition: there is a transition of the automaton (l_i, l_{i+1}, ρ) such that $(x_i, x_{i+1}) \models \rho$
     - Continuous evolution: there is a trajectory going from x_i to x_{i+1} along the flow determined by the differential equation $\dot{x} = Ax + B$ at location $l_i = l_{i+1}$

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  - We know how to deal with discrete systems
  - How to handle continuous evolution?
- Problem:
  computing polynomial invariants of linear systems of differential equations

Form of the Solution

Solution to $\dot{x} = Ax + B$ can be expressed as polynomials in t, $e^{\pm at}$, $\cos(bt)$, $\sin(bt)$, where $\lambda = a + bi$ are eigenvalues of matrix A.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{v_x} \\ \dot{v_y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix}$$

$$\left\{ \begin{array}{rcl} x & = & x^* + 2\sin(t/2)\, v_x^* + (2\cos(t/2) - 2)\, v_y^* \\ y & = & y^* + (-2\cos(t/2) + 2)\, v_x^* + 2\sin(t/2)\, v_y^* \\ v_x & = & \cos(t/2)\, v_x^* - \sin(t/2)\, v_y^* \\ v_y & = & \sin(t/2)\, v_x^* + \cos(t/2)\, v_y^* \end{array} \right.$$

Elimination of Time

Idea: eliminate terms depending on t from solution:

- transform solution into polynomials using new variables
- eliminate by means of Gröbner bases using auxiliary equations

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SOLUTION

$$\begin{cases} x & = & x^* + 2zv_x^* + (2w-2)v_y^* \\ y & = & y^* + (-2w+2)\,v_x^* + 2zv_y^* \\ v_x & = & wv_x^* - zv_y^* \\ v_y & = & zv_x^* + wv_y^* \end{cases}$$

INITIAL CONDITIONS

$$\begin{cases} v_x^* & = & 2 \\ v_y^* & = & -2 \end{cases}$$

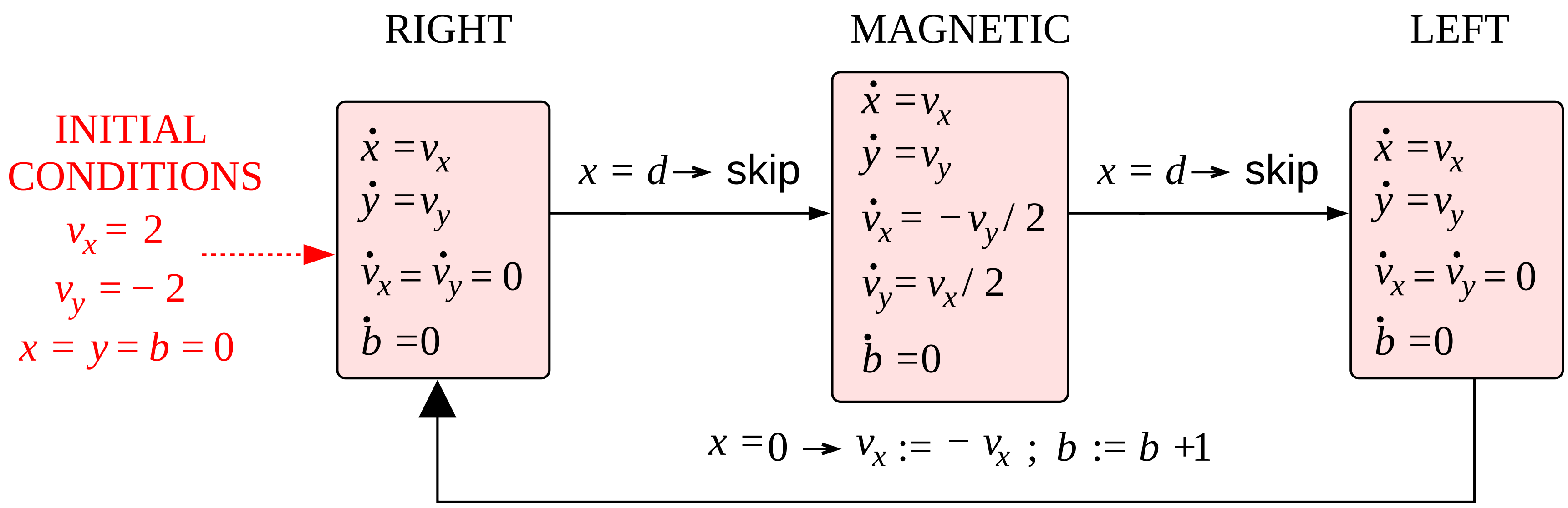
AUXILIARY EQUATIONS

$$\{\ w^2 + z^2 \ = \ 1$$

$$\Downarrow$$

$v_x^2 + v_y^2 = 8$ (conservation of energy)

Example



RIGHT $\rightarrow$ $v_y = -2 \wedge v_x = 2 \wedge 2db - 8b + y + x = 0$

MAGNETIC $\rightarrow$ $x - 2v_y - d = 4 \wedge v_x^2 + v_y^2 = 8 \wedge 2v_x + y + 2db - 8b + d = 4$

LEFT $\rightarrow$ $v_y = -2 \wedge v_x = -2 \wedge 2db - 8b + y - x = 8$

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Drawing a Parallel from Equalities

Linear equalities
[Karr'76]

↓

Polynomial equalities
[Colon'04]

Drawing a Parallel from Equalities

Linear equalities	Linear inequalities
[Karr'76]	[Cousot & Halbwachs'78]
↓	↓
Polynomial equalities	Polynomial inequalities
[Colon'04]	[Bagnara & Rodríguez-Carbonell & Zaffanella'05]

From Linear to Polynomial Equalities

$a := 0\ ;$

$b := 0\ ;$

$c := 1\ ;$

while $?$ **do**

$a := a + 1\ ;$

$b := b + c\ ;$

$c := c + 2\ ;$

end while

From Linear to Polynomial Equalities

$a := 0\ ;$
$b := 0\ ;$
$c := 1\ ;$

$\{\ a = 0 \wedge b = 0 \wedge c = 1\ \}$
while $?$ **do**

$a := a + 1\ ;$
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end while

From Linear to Polynomial Equalities

$a := 0\ ;$
$b := 0\ ;$
$c := 1\ ;$

$\{\ a = b \wedge c = 2a + 1\ \}$
while $?$ **do**

$a := a + 1\ ;$
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From Linear to Polynomial Equalities

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Loop invariant

$\{\ c = 2a + 1\ \}$

From Linear to Polynomial Equalities

$a := 0\ ;$
$b := 0\ ;$
$c := 1\ ;$

Introduce new variable s
standing for a^2

while $?$ **do**

$a := a + 1\ ;$
$b := b + c\ ;$
$c := c + 2\ ;$

end while

From Linear to Polynomial Equalities

$a := 0$;
$b := 0$;
$c := 1$;
$s := 0$; ⟵

while $?$ **do**

$a := a + 1$;
$b := b + c$;
$c := c + 2$;
$s := s + 2a + 1$; ⟵

end while

Introduce new variable s standing for a^2

Extend program with new variable s

$$a := 0 \quad \to \quad s := 0$$

$$a := a + 1 \quad \to \quad s := s + 2a + 1$$

From Linear to Polynomial Equalities

$a := 0$;
$b := 0$;
$c := 1$;
$s := 0$;
$\{\ a = 0 \wedge b = 0 \wedge c = 1 \wedge s = 0\ \}$
while $?$ **do**

$a := a + 1$;
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From Linear to Polynomial Equalities

$a := 0\ ;$
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$\{\ a = b \wedge b = s \wedge c = 2a + 1\ \}$
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Loop invariant

$\{\ b = a^2 \wedge c = 2a + 1\ \}$

is more precise

From Linear to Polynomial Inequalities

$\{ \text{ Pre : } \; b \geq 0 \; \}$

$a := 0 \; ;$

while $(a+1)^2 \leq b$ **do**

$\quad a := a + 1 \; ;$

end while

$\{ \text{ Post : } \; (a+1)^2 > b \wedge b \geq a^2 \; \}$

From Linear to Polynomial Inequalities

$\{\ \mathrm{Pre}:\ \ b \geq 0\ \}$

$a := 0\ ;$

while $(a+1)^2 \leq b$ **do**

$\quad a := a + 1\ ;$

end while

$\{\ \mathrm{Post}:\ \ (a+1)^2 > b \wedge b \geq a^2\ \}$

Linear analysis cannot deal with the quadratic condition

$$(a+1)^2 \leq b$$

From Linear to Polynomial Inequalities

$\{ \text{ Pre}: \ b \geq 0 \ \}$

$a := 0 \ ;$

$\{ \ a \geq 0 \wedge b \geq 0 \ \}$
while $(a+1)^2 \leq b$ **do**

$a := a + 1 \ ;$

end while

$\{ \text{ Post}: \ (a+1)^2 > b \wedge b \geq a^2 \ \}$

Loop invariant $\{ \ a \geq 0 \wedge b \geq 0 \ \}$
not precise enough

From Linear to Polynomial Inequalities

$\{ \text{ Pre : } \; b \geq 0 \; \}$

$a := 0 \; ;$

$s := 0 \; ;$ ⟵

while $(a+1)^2 \leq b$ **do**

$\quad a := a + 1 \; ;$

$\quad s := s + 2a + 1 \; ;$ ⟵

end while

$\{ \text{ Post : } \; (a+1)^2 > b \wedge b \geq a^2 \; \}$

Introduce new variable s standing for a^2

Extend program with new variable s

$$a := 0 \quad \rightarrow \quad s := 0$$

$$a := a + 1 \quad \rightarrow \quad s := s + 2a + 1$$

From Linear to Polynomial Inequalities

$\{ \text{ Pre : } b \geq 0 \}$

$a := 0 ;$
$s := 0 ;$
$\{ b \geq s \wedge \cdots \}$
while $(a+1)^2 \leq b$ **do**

$a := a + 1 ;$
$s := s + 2a + 1 ;$

end while

$\{ \text{ Post : } (a+1)^2 > b \wedge b \geq a^2 \}$

Loop invariant
$\{ b \geq a^2 \wedge \cdots \}$
enough to prove partial correctness

Linearization of Polynomial Constraints

- Abstract values = sets of constraints
- Given a degree bound d, all terms x^α with $\deg(x^\alpha) \leq d$ are considered as different and independent variables

Vector Spaces $\leftrightarrow$ Polynomial Cones

polynomial $= 0$

- $\forall$ polynomial p, $p \sim p = 0$
- Vector space = set of polynomials V s.t.
  - $0 \in V$
  - $\forall p, q \in V$ and $\lambda, \mu \in \mathbb{R}$, $\lambda p + \mu q \in V$

$$\frac{}{0 = 0}$$

$$\frac{p = 0 \quad q = 0 \quad \lambda, \mu \in \mathbb{R}}{\lambda p + \mu q = 0}$$

Vector Spaces $\leftrightarrow$ Polynomial Cones

polynomial $= 0$	polynomial ≥ 0
• $\forall$ polynomial p, $p \sim p = 0$	• $\forall$ polynomial p, $p \sim p \geq 0$
• Vector space = set of polynomials V s.t.	• Polynomial cone = set of polynomials C s.t.
◦ $0 \in V$	◦ $1 \in C$
◦ $\forall p, q \in V$ and $\lambda, \mu \in \mathbb{R}$, $\lambda p + \mu q \in V$	◦ $\forall p, q \in C$ and $\lambda, \mu \in \mathbb{R}_+$, $\lambda p + \mu q \in C$
$\dfrac{}{0 = 0}$	$\dfrac{}{1 \geq 0}$
$\dfrac{p = 0 \quad q = 0 \quad \lambda, \mu \in \mathbb{R}}{\lambda p + \mu q = 0}$	$\dfrac{p \geq 0 \quad q \geq 0 \quad \lambda, \mu \in \mathbb{R}_+}{\lambda p + \mu q \geq 0}$

Explicitly Adding Other Inference Rules

polynomial $= 0$

$$\frac{p = 0 \quad \deg(pq) \le d}{pq = 0}$$

Explicitly Adding Other Inference Rules

polynomial $= 0$

$$\frac{p = 0 \quad \deg(pq) \leq d}{pq = 0}$$

polynomial ≥ 0

$$\frac{p \geq 0 \quad p \leq 0 \quad \deg(pq) \leq d}{pq = 0}$$

$$\frac{p \geq 0 \quad q \geq 0 \quad \deg(pq) \leq d}{pq \geq 0}$$

- Introduction
- Generation of Invariant Polynomial Equalities
- Applications of Polynomial Equality Invariants
- Generation of Invariant Polynomial Inequalities
- Conclusions and Future Work

Conclusions

- Designed a new abstract domain for generating invariant polynomial equalities based on ideals of polynomials

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- Identified a class of programs for which all polynomial equality invariants can be generated
- Applied polynomial equality invariants to verifying imperative programs, Petri nets and hybrid systems
- Designed a new abstract domain for generating invariant polynomial inequalities based on polynomial cones

Future Work

- Extend the techniques to interprocedural analyses

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- Find new areas of application for polynomial invariants
- Design specific widening operators for the context of polynomial invariants

Publications

- E. Rodríguez-Carbonell, D. Kapur. An abstract interpretation approach for automatic generation of polynomial invariants. SAS'04.
- E. Rodríguez-Carbonell, D. Kapur. Automatic generation of polynomial loop invariants: algebraic foundations. ISSAC'04.
- E. Rodríguez-Carbonell, D. Kapur. Program verifi cation using automatic generation of polynomial invariants. ICTAC'04.
- E. Rodríguez-Carbonell, A. Tiwari. Generating polynomial invariants for hybrid systems. HSCC'05.
- R. Clarisó, E. Rodríguez-Carbonell, J. Cortadella. Derivation of non-structural invariants of Petri nets using abstract interpretation. ATPN'05.
- R. Bagnara, E. Rodríguez-Carbonell, E. Zaffanella. Generation of basic semi-algebraic invariants using convex polyhedra. SAS'05.
- E. Rodríguez-Carbonell, D. Kapur. Automatic generation of polynomial invariants of bounded degree using abstract interpretation. Science of Computer Programming, 2006. Accepted for publication.