Pairing functions

A *pairing function* is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ that is also strictly monotone in each of its arguments. If we let $p : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ be a pairing function, then we require:

- p is a bijection,
- p is strictly monotone in each argument: for all $x, y \in \mathbb{N}$ we have both $p(x, y) < p(x + 1, y)$ and $p(x, y) < p(x, y + 1)$.

We shall denote an arbitrary pairing function $p(x, y)$ with pointed brackets as $< x, y >$.

Given some pairing function, we need a way to reverse and to recover x and y from $< x, y >$, thus we need two functions, one to recover each argument. We call this two functions *projections* and write them as $\Pi_1(z)$ and $\Pi_2(z)$. If $z =< x, y >$ then we have that $\Pi_1(z) = x$ and $\Pi_2(z) = y$.

An example of pairing function can be obtained by considering a *dovetailing process* to enumerate all the elements of $\mathbb{N} \times \mathbb{N}$. Let us consider the infinite table of natural pairs (i, j) . Dovetailing process consists of enumerating the first element of the first row, followed by the second element of the first row and the first of the second row, followed by the third element of the first row, the second of the second row, and the first of the third row, and so on. We use the backwards diagonals in the table, one after the other, $(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), \ldots$ This process to enumerate is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$. Moreover, the item in row i, column j of our infinite table is enumerated before item in row $i + 1$, column j and before item in row i, column $j + 1$ so that the bijection is monotone in each argument. Conversely, if we know the index k of an element, we can determine the corresponding pair (i, j) as follows: Let l be the least integer with $l(l - 1) > 2k - 2$; then we have $l = i + j$ and $2j = l(l - 1) - (2k - 2)$.

In computability we are often forced to resort to dovetailing along 3, 4 or even more dimensions. We can define "higher pairing"functions recursively, by using two-dimensional pairing functions as a base case. Formally, we define the function $< ., ., \ldots, . >_n$, which pairs n natural numbers recursively as follows:

$<>_0 = 0$ and $< x >_1 = x$,

$< x, y >_2 =< x, y >$,

$< x_1, \ldots, x_{n-1}, x_n >_n =< x_1, \ldots, < x_{n-1}, x_n >>_{n-1}$.

For this general pairing functions we need matching general projections, we would like to define a function $\Pi(i, n, z)$ which takes z to be the result of a pairing of n natural numbers and then returns the i-th of these. Note that we need n as argument. We define $\Pi(i, n, z)$, which we shal normally write as $\Pi_i^n(z)$, recursively as follows:

$\Pi_i^0(z) = 0$ and $\Pi_i^1(z) = z$, for all i,

$\Pi_i^2(z) = \Pi_1(z)$ if $i \leqslant 1$ and $\Pi_i^2(z) = \Pi_2(z)$ if $i > 1$,

$\Pi_i^{n+1}(z) = \Pi_i^n(z)$ if $i < n$,

$\Pi_i^{n+1}(z) = \Pi_1(\Pi_n^n(z))$ if $i = n$, and

$\Pi_i^{n+1}(z) = \Pi_2(\Pi_n^n(z))$ if $i > n$.

When we need to enumerate all possible k-tuples of natural numbers or arguments from countable infinite sets, we simply enumerate the natural numbers, $1, 2, 3, \ldots, \ldots$, and consider i to be the pairing of k natural numbers, $i =< x_1, x_2, \ldots, x_k >_k$ hence the i-th number gives us the k arguments allowing us to dovetail. Conversely, whenever we need to handle a finite number of k arguments, each taken from a countably infinite set, we can just pair the k arguments. If we need to pair together k values $x_1, x_2, \ldots, x_k$ where k may vary, we can encode the arity k as part of the pairing: $z =< k, < x_1, x_2, \ldots, x_k >_k >$, $k = \Pi_1(z)$ and $x_i = \Pi_i^{\Pi_1(z)}(\Pi_2(z))$

Exercise 1. An interesting pairing function is defined as $z = < x, y > = 2^x(2y+1) - 1$ Verify that it is bijective and monotone in each argument. And it is reversed simply by factoring $z+1$ into a power of two and an odd factor.

Exercise 2. Consider the new pairing function given by

$$< x, y > = x + (y + \lfloor \frac{(x+1)}{2} \rfloor)^2$$

Verify that it is a pairing function and can be reversed with $\Pi_1(z) = z - \lfloor \sqrt{z} \rfloor^2$ and $\Pi_2(z) = \lfloor \sqrt{z} \rfloor - \frac{(\Pi_1(z)+1)}{2}$.

Exercise 3. Verify that our definition of projection functions is correct. Note that the notation $\Pi^n_i(z)$ allows argument triple that do not correspond to any valid pairing such us $\Pi^5_{10}(z)$. In this case we can define the result in any convenient way.

Exercise 4 Verify that function below is a bijection between the positive, nonzero rational numbers and the nonzero natural numbers, and define a procedure to reverse it:

$$f(1) = 1$$

$$f(2n) = f(n) + 1$$

$$f(2n+1) = \frac{1}{f(2n)}$$

Show that it is not monotonic in each argument and hence not a pairing function, but it can be used for dovetailing.

Exercise 5 Would diagonalization work with a finite set?

Exercise 6 Show that the union, intersection, and Cartesian product of two countable sets are themselves countable.

Exercise 7 Let S a finite set and T a countable set. Is the set of all functions from S to T countable?

Exercise 8 Is the set of all polynomials with any finite number of variables with integer coefficients countable?